ON THE MATHEMATICAL THEORY OF BLACK HOLES

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Outline

1. ON THE REALITY OF BLACK HOLES
2. RIGIDITY
3. STABILITY
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1. ON THE REALITY OF BLACK HOLES
2. RIGIDITY
3. STABILITY
KERR FAMILY $\mathcal{K}(a, m)$

Boyer-Lindquist $\{t, r, \theta, \varphi\}$ coordinates.

$$-\frac{\rho^2 \Delta}{\Sigma^2} (dt)^2 + \frac{\Sigma^2 (\sin \theta)^2}{\rho^2} \left( d\varphi - \frac{2amr}{\Sigma^2} dt \right)^2 + \frac{\rho^2}{\Delta} (dr)^2 + \rho^2 (d\theta)^2,$$

$$\begin{align*}
\Delta &= r^2 + a^2 - 2mr; \\
\rho^2 &= r^2 + a^2 (\cos \theta)^2; \\
\Sigma^2 &= (r^2 + a^2)^2 - a^2 (\sin \theta)^2 \Delta.
\end{align*}$$

Stationary. $T = \partial_t$

Axisymmetric. $Z = \partial_\varphi$

Schwarzschild. $a = 0, m > 0$, static, spherically symmetric.

$$-\frac{\Delta}{r^2} (dt)^2 + \frac{r^2}{\Delta} (dr)^2 + r^2 d\sigma_{\mathbb{S}^2}, \quad \frac{\Delta}{r^2} = 1 - \frac{2m}{r}$$
ON THE REALITY OF BLACK HOLES

\[ -(1 - \frac{2m}{r}) dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\sigma^2_{S^2} \]

- **Event horizon** \( r = 2m \).
  Metric can be extended past it. Kruskal coordinates
- **Black and white holes** \( r < 2m \)
- **Curvature singularity** \( r = 0 \).
- **Exterior domains** \( r > 2m \).
- **Photon sphere** \( r = 3m \).
- **Null infinity** \((I^+ \cup I^-)\) \( r = \infty \).
**KERR SPACETIME** $\mathcal{K}(a, m), \ |a| \leq m$

**MAXIMAL EXT.** \[ \Delta(r_-) = \Delta(r_+) = 0, \ \Delta = r^2 + a^2 - 2mr \]

- **External region** $r > r_+$
- **Event horizon** $r = r_+$
- **Black Hole** $r < r_+$
- **Null Infinity** $r = \infty$
ON THE REALITY OF BLACK HOLES

EXTERNAL KERR

- Stationary, axisymmetric.
- Nonempty ergoregion. Non-positive energy.
- Region of trapped null geodesics.
PRINCIPAL NULL DIRECTIONS

\[ e_3 = \frac{r^2 + a^2}{q \sqrt{\Delta}} \partial_t - \frac{\sqrt{\Delta}}{q} \partial_r + \frac{a}{q \sqrt{\Delta}} \partial_\varphi \]
\[ e_4 = \frac{r^2 + a^2}{q \sqrt{\Delta}} \partial_t + \frac{\sqrt{\Delta}}{q} \partial_r + \frac{a}{q \sqrt{\Delta}} \partial_\varphi. \]

NULL FRAME

\[ S = \{ e_3, e_4 \}^\perp, \quad (e_a)_{a=1,2} \in S \quad \text{orthonormal} \]

\[ \alpha_{ab} := R_{a4b4}, \quad \alpha_{ab} := R_{a3b3}, \quad \beta_a := \frac{1}{2} R_{a434}, \quad \underline{\beta}_a := \frac{1}{2} R_{a334} \]
\[ \rho = \frac{1}{4} R_{3434}, \quad \star \rho = \frac{1}{4} \star R_{3434} \]

FACT

\[ \alpha = \beta = \underline{\beta} = \alpha = 0, \quad \rho + i \star \rho = -\frac{2m}{r + i \cos \theta}. \]
ON THE REALITY OF BLACK HOLES

RIGIDITY

STABILITY

OTHER PROPERTIES OF KERR

\[ F_{\alpha\beta} := D_{\alpha} T_{\beta} \text{ is a 2-form.} \]

\[ \mathcal{F} = F + i^* F, \quad \mathcal{R} = R + i^* R, \]

\[ D_{\alpha} \mathcal{F}_{\beta\gamma} = T^{\lambda} \mathcal{R}_{\lambda\alpha\beta\gamma} \quad \Rightarrow \quad d(i_T \mathcal{F}) = 0, \quad 2i_T \mathcal{F} = d\sigma, \]

KERR

\[ \sigma = 1 - \frac{2m}{r + ia \cos \theta}, \quad \mathcal{F}^2 = -\frac{4m^2}{(r + ia \cos \theta)^4}. \]

MARS-SIMON

\[ S = \mathcal{R} + 6(1 - \sigma)^{-1} Q, \]

\[ Q_{\alpha\beta\mu\nu} = \mathcal{F}_{\alpha\beta} \mathcal{F}_{\mu\nu} - \frac{1}{3} \mathcal{F}^2 \mathcal{I}_{\alpha\beta\mu\nu}, \]

\[ \mathcal{I}_{\alpha\beta\mu\nu} = (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} + i \epsilon_{\alpha\beta\mu\nu})/4. \]

THEOREM (Mars 1999)

\[ S = 0 \quad \text{characterizes Kerr (locally)} \]
I. RIGIDITY

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m), 0 \leq a \leq m$, exhaust all stationary, asymptotically flat, regular vacuum black holes.

- True in the axially symmetric case [Carter-Robinson]
- True in general, under an analyticity assumption [Hawking]
- True close to a Kerr space-time [Alexakis-Ionescu-Kl]
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I. RIGIDITY  MAIN NEW IDEAS

- There exists a second Killing v-field along $\mathcal{N} \cup \mathcal{N}$.
- Its natural extension leads to an **ill posed problem**.

**NEW APPROACH.** Extend using a geometric continuation argument based on Carleman estimates.
- Local extension based on Null Convexity.
- Global extension based on $T$-Null Convexity,

**MAIN OBSTRUCTION.** Presence of $T$-trapped null geodesics.

No such objects in Kerr or close to Kerr!
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ON THE REALITY OF BLACK HOLES

LOCAL EXTENSION

**DEFINITION.**  \( O \subset M \) is \((T)\) strongly null-convex at \( p \in \partial O \) if, for any defining function \( f \),

\[
D^2 f(X, X)(p) < 0, \quad \forall X \in T_p(O), \quad g(X, X) = 0, \quad (g(X, T) = 0)
\]

**THEOREM** (Ionescu-Kl)  \((M, g)\) Ricci flat, pdo-riemannian manifold. \((O \subset M, Z)\) verify:

- \( Z \) Killing v-field in \( O \),
- \( \partial O \) is strongly null-convex at \( p \in \partial O \)

\( \Rightarrow \) \( Z \) extends as a Killing vector-field to a neighborhood of \( p \).
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FACT. STRICT NULL CONVEXITY IS ESSENTIAL.

THEOREM (Ionescu-Kl) \( \exists \) smooth, stationary, extensions of \( \mathcal{K}(a, m), 0 < a < m \), locally defined in a neighb. of a point on the horizon \( p \in \mathcal{N} \cup \overline{\mathcal{N}} \setminus \mathcal{N} \cap \overline{\mathcal{N}} \), which possesses no additional Killing v-fields.
There exist no other \textit{explicit} stationary solutions.

There exist no other stationary solutions \textit{close} to a non extremal Kerr, (or Kerr-Newman).

The full problem is far from being solved. \textit{Surprises} cannot be ruled out for large perturbations.

\textbf{Conjecture.} [Alexakis-Ionescu-Kl]. Rigidity conjecture holds true provided that there are \textbf{no T-trapped} null geodesics.
RIGIDITY IN DEPTH

1. GENERAL LOCAL EXTENSION THEOREM
2. A COUNTEREXAMPLE
3. GLOBAL RIGIDITY
4. CARLEMAN ESTIMATES
THEOREM I.  \((M, g)\) Ricci flat, pseudo-riemannian manifold; \((O, Z)\) verify:

- \(Z\) Killing v-field in \(O\),
- \(\partial O\) is strongly null-convex at \(p \in \partial O\)

\(\Rightarrow\) \(Z\) extends as a Killing vector-field to a neighborhood of \(p\).

DEFINITION  \(O \subset M\) is strongly null-convex at \(p \in \partial O\) if it admits defining function \(f\) at \(p\), s.t. for any \(X \neq 0 \in T_p(M)\), \(X(f)(p) = 0\) and \(g(X, X) = 0\), we have

\[D^2f(X, X)(p) < 0.\]
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\[D^2 f(X, X)(p) < 0.\]
THEOREM 1 (MAIN IDEAS)

- Consider $L, D_L L = 0$, in a neighborhood of $p \in \partial O$.
- Extend $Z$ by the Jacobi equation
  \[
  D_L D_L Z = R(L, Z)L
  \]  
  (1)
- Define $\pi = \mathcal{L}_Z g$
  \[
  (1) \implies L^\beta \pi_{\alpha\beta} = 0
  \]
- Define $W = \mathcal{L}_Z R - (B \ast R)$
  
  $B = \frac{1}{2}(\pi + \omega)$ appropriately defined s.t. $W$ is Weyl.
THEOREM I (MAIN IDEAS)

- Define

\[ P_{\alpha\beta\mu} = \frac{1}{2} (D_\alpha \pi_{\beta\mu} - D_\beta \pi_{\alpha\mu} - D_\mu \omega_{\alpha\beta}) \], \quad P_\rho : g^{\mu\nu} P_{\mu\rho\nu} 

where

\[ D_L \omega_{\alpha\beta} = \pi_{\alpha\rho} D_\beta L^\rho - \pi_{\beta\rho} D_\alpha L^\rho, \quad \omega|_O = 0. \]

- We have,

\[ L^\mu P_{\alpha\beta\mu} = 0, \quad L^\beta \omega_{\alpha\beta} = 0 \quad \text{in} \quad M \]

- We have,

\[ D^\alpha W_{\alpha\beta\gamma\delta} = \frac{1}{2} (B^{\mu\nu} D_\nu R_{\mu\beta\gamma\delta} + P_\rho R^{\rho}_{\beta\gamma\delta} + P \ast R) \]
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THEOREM I (MAIN IDEAS)

We have

\[ D_L B_{\alpha\beta} = L^\rho P_{\rho\beta\alpha} - D_\alpha L^\rho B_{\rho\beta}, \]

\[ D_L P_{\alpha\beta\mu} = L^\nu W_{\alpha\beta\mu\nu} + L^\nu B_{\mu}^\rho R_{\alpha\beta\rho\nu} - D_\mu L^\rho P_{\alpha\beta\rho}. \]

The last identity follows from

\[ D_\nu P'_{\alpha\beta\mu} - D_\mu P'_{\alpha\beta\nu} = (\mathcal{L}_Z R)_{\alpha\beta\mu\nu} - (1/2)\pi_\alpha^\rho R_{\rho\beta\mu\nu} - (1/2)\pi_\beta^\rho R_{\alpha\rho\mu\nu}. \]

where

\[ P'_{\alpha\beta\mu} := P_{\alpha\beta\mu} + (1/2)D_\mu \omega_{\alpha\beta}. \]
THEOREM I (MAIN IDEAS)

- Have derived a system of wave/transport equations
  \[ D_L B = \mathcal{M}(B, \dot{B}, P, W), \]
  \[ D_L \dot{B} = \mathcal{M}(B, \dot{B}, P, W), \]
  \[ D_L P = \mathcal{M}(B, \dot{B}, P, W), \]
  \[ \Box_g W = \mathcal{M}(B, DB, \dot{B}, D\dot{B}, P, DP, W, DW) \]

- Use unique continuation argument to conclude that \( W, \pi \) vanish in a neighborhood of \( Z \).
- Role of strong null-convexity.
THEOREM I (MAIN IDEAS)

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\[
\begin{align*}
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THEOREM II. \( \exists \) smooth, stationary, extensions of \( \mathcal{K}(a, m) \), \( 0 < a < m \), locally defined in a neighb. of a point on the horizon \( p \in \mathcal{N} \cup \mathcal{N} \setminus \mathcal{N} \cap \mathcal{N} \), which fail to be axi-symmetric.

PROOF. Modify the Kerr metric \( g \) smoothly across the horizon s.t. \( \text{Ric}(g) = 0 \), \( T = d/dt \) remains Killing, yet \( g \) admits no Killing extension for \( Z \).

MORAL. No hope to construct an additional symmetry, as in Hawking's Rigidity Theorem, by relying only on local informations.
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A COUNTEREXAMPLE

**THEOREM II.** \( \exists \) smooth, stationary, extensions of \( \mathcal{K}(a, m) \), \( 0 < a < m \), locally defined in a neighb. of a point on the horizon \( p \in \mathcal{N} \cup \mathcal{N} \setminus \mathcal{N} \cap \mathcal{N} \), which fail to be axi-symmetric.

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**MORAL.** No hope to construct an additional symmetry, as in Hawking’s Rigidity Theorem, by relying only on local informations.
**RECALL.** \[ D_\mu D_\nu T_\sigma = R^\lambda_{\mu\nu\sigma} T_\lambda, \quad [\mathcal{L}_T, D] = 0, \]

**DEFINITION.** Given a Killing v-field \( T \),

\[
\mathcal{F}_{\alpha\beta} : = F_{\alpha\beta} + i^* F_{\alpha\beta}, \quad F_{\alpha\beta} = D_\alpha T_\beta
\]

\[
\sigma_\alpha := 2T^\lambda \mathcal{F}_{\lambda\alpha}
\]

**LEMMA.** \[ D_\alpha \mathcal{F}_{\beta\gamma} = T^\lambda R_{\lambda\alpha\beta\gamma} \] and, in vacuum,

\[
\left\{
\begin{array}{l}
D_\mu \sigma_\nu - D_\nu \sigma_\mu = 0; \\
D^\mu \sigma_\mu = -G^2; \\
\sigma_\mu \sigma^\mu = g(T, T)G^2.
\end{array}
\right.
\]

**ERNST POTENTIAL.** \[ \sigma = -(X + iY), \quad X = g(T, T). \]
THEOREM II (MAIN IDEAS)

STATIONARITY. Ernst potentials \( X = g(T, T), \ Y = \ldots \)

SYMMETRY REDUCTION. \( g = (\Phi, h), \ \Phi = (X + iY) = -\sigma \)

\[
\begin{align*}
    h_{\alpha\beta} : & = X g_{\alpha\beta} - T_\alpha T_\beta \\
    \text{Ric}(h)_{\alpha\beta} & = \langle D_\alpha \Phi, D_\beta \Phi \rangle \\
    \Box_h \Phi & = X^{-1} D^\mu \Phi D_\mu \Phi
\end{align*}
\]

WAVE MAP. \( \Phi : (N, h) \longrightarrow \mathbb{H}^2. \)

RECONSTRUCTION

- \( \mathbb{M} = \mathbb{R} \times \mathbb{N}^{1+2}. \)
- \( g = (Xd\varphi + \omega_\alpha dx^\alpha)^2 + X^{-1} h_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha, \beta = 0, 1, 2 \)

\[
\begin{align*}
    \text{curl} \ \omega & = X^{-2} dY, \\
    \text{div} \ \omega & = 0.
\end{align*}
\]
THEOREM II (MAIN IDEAS)

Reduce the construction to solving a local characteristic Cauchy problem for a wave map equation \( \Phi = (X + iY) : (\mathcal{N}^{2+1}, h) \rightarrow \mathbb{H} \) coupled to a \( 2+1 \) dimensional Einstein equations.

\[
\begin{align*}
R_{\alpha\beta}(h) &= \frac{1}{X^2} (\nabla_{\alpha}X \nabla_{b}X + \nabla_{a}Y \nabla_{b}Y) \\
\Box_{h}(X + iY) &= \frac{1}{X} h^{ab} \nabla_{a}(X + iY) \nabla_{b}(X + iY)
\end{align*}
\]

where \( X = g(T, T) \), \( Y \) Ernst potential and,

\[
h_{\alpha\beta} := X g_{\alpha\beta} - T_{\alpha} T_{\beta}
\]

CHARACTERISTIC PROBLEM. Prescribe \( X, Y \). Solve locally the equations in a neighborhood of a point \( p \in \mathcal{N} \cup \overline{\mathcal{N}} \) away from the bifurcate sphere.

IMPORTANT. Vectorfield \( T \) must be strictly spacelike at \( p \)!
RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a,m)$, $0 \leq a \leq m$, exhaust all stationary, asymptotically flat, regular vacuum black holes.

- True if coincides with Kerr on $\mathcal{N} \cap \mathcal{N}$ [Ionescu-Kl]
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- True if close to a Kerr space-time [Alexakis-Ionescu-Kl]
There exists a complex valued 4-tensor $S$, defined in terms of the curvature $R$ and $DT$.

$$S \equiv 0 \iff (M, g) \text{ locally isometric to Kerr}$$

**Theorem.** $(M, g)$ regular, stationary, non-extremal with Mars tensor $S$ sufficiently small $\Rightarrow M$ isometric to external Kerr.

**Main Ideas:**

1. Hawking argument provides a second Killing v-field $K\big|_{\mathcal{N} \cup \mathcal{N}}$.
2. Strict null convexity on $\mathcal{N} \cap \mathcal{N}$ allows one to extend $K$ to a small neighborhood.
3. Smallness of $S$ allows us to extend $K$ everywhere based on $T$-null convexity.
4. A linear combination of $T, K$ generates a rotational Killing v-field $Z$. 
MARS-SIMON TENSOR. There exists a complex valued 4-tensor \( S \), defined in terms of the curvature \( R \) and \( DT \).

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MARS-SIMON TENSOR. There exists a complex valued 4-tensor $S$, defined in terms of the curvature $R$ and $DT$.

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I. RIGIDITY  CONCLUSIONS

- There exist no other \textit{explicit} stationary solutions.
- There exist no other stationary solutions \textit{close} to a non extremal Kerr, (or Kerr-Newman).
- The full problem is far from being solved. \textit{Surprises} cannot be ruled out for large perturbations.

\textbf{Conjecture.} [Alexakis-Ionescu-Kl]. Rrigidity conjecture holds true provided that there are \textbf{no T-trapped} null geodesics.
**CONJECTURE** [Stability of (external) Kerr].

Small perturbations of a given exterior Kerr \( \mathcal{K}(a, m), |a| < m \) initial conditions have max. future developments converging to another Kerr solution \( \mathcal{K}(a_f, m_f) \).
GENERAL STABILITY PROBLEM \( \mathcal{N}[\phi] = 0 \).

NONLINEAR EQUATIONS. \( \mathcal{N}[\phi_0 + \psi] = 0, \mathcal{N}[\Phi_0] = 0 \).

1. ORBITAL STABILITY (OS). \( \psi \) small for all time.
2. ASYMPT STABILITY (AS). \( \psi \rightarrow 0 \) as \( t \rightarrow \infty \).

LINEARIZED EQUATIONS. \( \mathcal{N}'[\phi_0] \psi = 0 \).

1. MODE STABILITY (MS). No growing modes.
2. BOUNDEDNESS.
3. QUANTITATIVE DECAY.
STATIONARY CASE. Possible instabilities for $\mathcal{N}'[\phi_0] \psi = 0$:

- Family of stationary solutions $\phi_\lambda$, $\lambda \in (-\epsilon, \epsilon)$,
  $$\mathcal{N}[\phi_\lambda] = 0 \implies \mathcal{N}'[\phi_0](\frac{d}{d\lambda} \Phi_\lambda)|_{\lambda=0} = 0.$$

- Mappings $\psi_\lambda : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$, $\psi_0 = I$ taking solutions to solutions,
  $$\mathcal{N}[\phi_0 \circ \psi_\lambda] = 0 \implies \mathcal{N}'[\phi_0] \frac{d}{d\lambda} (\phi_0 \circ \psi_\lambda)|_{\lambda=0} = 0.$$

- Intrinsic instability of $\phi_0$. Negative eigenvalues of $\mathcal{N}'(\phi_0)$. 
ON THE REALITY OF BLACK HOLES

RIGIDITY

STABILITY

GENERAL STABILITY PROBLEM \( \mathcal{N}[\phi_0] = 0 \)

STATIONARY CASE. Expected linear instabilities due to non-decaying states in the kernel of \( \mathcal{N}'[\phi_0] \):

1. Presence of continuous family of stationary solutions \( \phi_\lambda \) implies that the final state \( \phi_f \) may differ from initial state \( \phi_0 \).

2. Presence of a continuous family of invariant diffeomorphism requires us to track dynamically the gauge condition to insure decay of solutions towards the final state.

QUANTITATIVE LINEAR STABILITY. After accounting for (1) and (2), all solutions of \( \mathcal{N}'[\phi_0] \psi = 0 \) decay sufficiently fast.

MODULATION. Method of constructing solutions to the nonlinear problem by tracking (1) and (2).
TO PROVE NONLINEAR STABILITY.

- **NEED**: Dynamically defined *gauge condition* and mechanism to track the *final state*
- **NEED**: Robust mechanism for deriving “sufficient” decay for the main linearized quantities
  - *with respect to the gauge*
- **NEED**: A version of the null condition
  - *with respect to the gauge*
- **NEED**: A strategy to disentangle the nonlinear interdependence of the above.
STABILITY OF KERR

\[
\text{Ric}[g_{m,a}] = 0
\]

- Kerr family depends on two parameters \(a, m\).

\[
\delta \text{Ric} \left[ \frac{\partial g_{m,a}}{\partial m} \right] = \delta \text{Ric} \left[ \frac{\partial g_{m,a}}{\partial a} \right] = 0
\]

- Einstein vacuum equations are diffeomorphism invariant

\[
\delta \text{Ric} [\mathcal{L}_X g_{m,a}] = 0.
\]

\[
\text{Dim}(\text{Ker} \ \delta \text{Ric}) = 4 \times \infty + 2
\]
GEOMETRIC FRAMEWORK

1. Null Pair \( e_3, e_4 \).

2. Horizontal structures \( \{ e_3, e_4 \} \perp \).

3. Null Frames \( e_3, e_4, (e_a)_{a=1,2} \).

4. Null decompositions
   - Connection \( \Gamma = \{ \chi, \xi, \eta, \zeta, \eta, \omega, \xi, \omega \} \)
   - Curvature \( R = \{ \alpha, \beta, \rho, \ast \rho, \beta, \alpha \} \)

5. Main Equations

6. S-foliations
KERR FAMILY $\mathcal{K}(a, m)$

BOYER-LINDQUIST $(t, r, \theta, \varphi)$.

$$\begin{align*}
-\frac{\rho^2 \Delta}{\Sigma^2} (dt)^2 + \frac{\Sigma^2 (\sin \theta)^2}{\rho^2} \left( d\varphi - \frac{2amr}{\Sigma^2} dt \right)^2 + \frac{\rho^2}{\Delta} (dr)^2 + \rho^2 (d\theta)^2, \\
\Delta = r^2 + a^2 - 2mr; \\
q^2 = r^2 + a^2 (\cos \theta)^2; \\
\Sigma^2 = (r^2 + a^2)^2 - a^2 (\sin \theta)^2 \Delta.
\end{align*}$$

STATIONARY, AXISYMMETRIC. $T = \partial_t$, $Z = \partial_\varphi$

PRINCIPAL NULL DIRECTIONS.

$$\begin{align*}
e_3 &= \frac{r^2 + a^2}{q \sqrt{\Delta}} \partial_t - \frac{\sqrt{\Delta}}{q} \partial_r + \frac{a}{q \sqrt{\Delta}} \partial_\varphi \\
e_4 &= \frac{r^2 + a^2}{q \sqrt{\Delta}} \partial_t + \frac{\sqrt{\Delta}}{q} \partial_r + \frac{a}{q \sqrt{\Delta}} \partial_\varphi.
\end{align*}$$
ON THE REALITY OF BLACK HOLES

RIGIDITY

STABILITY

BASIC QUANTITIES

NULL FRAME

\[ e_3, e_4, (e_a)_{a=1,2}, \quad S = \text{span}\{e_1, e_2\} \]

CONNECTION COEFFICIENTS.

\[ \chi, \xi, \eta, \zeta, \eta, \omega, \xi, \omega \]

\[
\begin{align*}
\chi_{ab} &= g(D_a e_4, e_b), \\
\xi_a &= \frac{1}{2} g(D_4 e_4, e_a), \\
\eta_a &= \frac{1}{2} g(e_a, D_3 e_4), \\
\zeta_a &= \frac{1}{2} g(D_a e_4, e_3), \quad \omega = \frac{1}{4} g(D_4 e_4, e_3) \ldots
\end{align*}
\]

CURVATURE COMPONENTS.

\[ \alpha, \beta, \rho, \star \rho, \beta, \alpha \]

\[
\begin{align*}
\alpha_{ab} &= R(e_a, e_4, e_b, e_4), \quad \beta_a = \frac{1}{2} R(e_a, e_4, e_3, e_4), \\
\rho &= \frac{1}{4} R(e_4, e_3, e_4, e_3), \ldots
\end{align*}
\]
CRUCIAL FACT.

1. In Kerr relative to a principal null frame we have

\[ \alpha, \beta, \beta, \alpha = 0, \quad \rho + i^*\rho = -\frac{2m}{(r + ia \cos \theta)^3} \]

\[ \xi, \xi, \hat{\chi}, \hat{\chi} = 0. \]

2. In Schwarzschild we have, in addition,
   - \( \{e_3, e_4\}^\perp \) is integrable
   - \( ^*\rho = 0, \quad \eta, \eta, \zeta = 0 \)
   - The only nonvanishing components of \( \Gamma \) are
     \( tr \chi, tr \bar{\chi}, \omega, \bar{\omega} \)

3. In Minkowski we also have \( \omega, \bar{\omega}, \rho = 0 \).
**O(ε) - PERTURBATIONS**

**ASSUME.** There exists a null frame $e_3, e_4, e_1, e_2$ such that

$$\xi, \xi, \hat{\chi}, \hat{\chi}, \alpha, \alpha, \beta, \beta = O(\epsilon)$$

**FRAME TRANSFORMATIONS.** $(f, f)_{a=1,2}$, $\log \lambda = O(\epsilon)$

$$
e_4' = \lambda \left( e_4 + f_a e_a + O(\epsilon^2) \right)$$

$$e_3' = \lambda^{-1} \left( e_3 + f_a e_a + O(\epsilon^2) \right)$$

$$e_a' = e_a + \frac{1}{2} f_a e_4 + \frac{1}{2} f_a e_3 + O(\epsilon^2)$$

- Curvature components $\alpha, \alpha$ are $O(\epsilon^2)$ invariant.
- For perturbations of Minkowski all curvature components are $O(\epsilon^2)$-invariant.
\[ \square \phi = F(\phi, \partial \phi, \partial^2 \phi) \quad \text{in} \quad \mathbb{R}^{1+n}. \]

**FACT.** Vacuum state \( \Phi = 0 \),

- Unstable for most equations for \( n = 3 \).
- Stable if \( n = 3 \) and \( F \) verifies the null condition.
- Dimension \( n = 3 \) is critical

**FACT.** Geometric nonlinear wave equations verify some, *gauge dependent*, version of the null condition.
\[ \Box \phi = F(\phi, \partial \phi, \partial^2 \phi) \quad \text{in} \quad \mathbb{R}^{1+n}. \]

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VECTORFIELD METHOD

Use well adapted vectorfields, related to

1. Symmetries,
2. Approximate, symmetries,
3. Other geometric features

to derive generalized energy bounds and robust $L^\infty$-quantitative decay.

It applies to tensorfield equations such as Maxwell and Bianchi type equations,

$$dF = 0, \quad \delta F = 0.$$ 

and nonlinear versions such as Einstein field equations.
1. INITIAL DATA SETS. \((\Sigma^3, g, k)\)

- Constraint Eqts.
- Maximal \(\text{tr}_g k = 0\).
- Flat. \((\mathbb{R}^3, e, 0)\)
- A. F. \(g_{ij} - (1 + \frac{2m}{r})\delta_{ij} = O_{k+1}(r^{-2}), \quad k_{ij} = O_k(r^{-2})\)

- Smallness assumption

2. MGHD DEVELOPMENT.
THEOREM[Christodoulou-K 1993] Any A.F. initial data set, close to the flat one, has a complete, maximal development converging to Minkowski.

I. GAUGE CONDITION.

- optical function $u$ -properly initialized.
- time function $t$ -maximal

II. ROBUST DECAY.

- Bianchi identities $\delta R = \delta^* R = 0$
- Vectorfield method

Construct approximate Killing and conformal Killing fields adapted to the foliation

III. NULL CONDITION.
CHRISTODOULOU-K THEOREM (1990)

1. Maximal foliation $\Sigma_t$.
2. Null foliation $C_u$.
3. Adapted frame

$$L = e_4, \quad L = e_3, \quad (e_a)_{a=1,2} \in T(S(t,u))$$

$$S_{t,u} = \Sigma_t \cap C_u \quad 4\pi r^2 = \text{Area}(S_{t,u}).$$

4. Weak peeling decay, as $r \to \infty$ along $C_u$.

$$\alpha_{ab} = R(L, e_a, L, e_b) = O(r^{-7/2})$$
$$2\beta_a = R(L, L, L, e_a) = O(r^{-7/2})$$
$$4\rho = R(L, L, L, L) = O(r^{-3})$$
$$4^*\rho = *R(L, L, L, L) = O(r^{-7/2})$$
$$2\beta_{\phantom{a}a} = R(L, L, L, e_a) = O(r^{-2})$$
$$\alpha_{ab} = R(L, e_a, L, e_b) = O(r^{-1})$$
GLOBAL STABILITY OF MINKOWSKI
ROLE OF CURVATURE.

- All null components of $R$, relative to adapted null frames, stay within $O(\epsilon)$ of their initial values.

- All null components of $R$ are $O(\epsilon^2)$-invariant with respect to $O(\epsilon)$ frame transformations. Effectively gauge independent.

BIANCHI IDENTITIES

$$\delta R = \delta^* R = 0$$

- Bel-Robinson tensor $Q = R \cdot R + *R \cdot *R$.

$Q$ symmetric, traceless, $\delta Q = 0$.

Energy type quantities

1. Contractions
2. Commutations

Effective, *invariant*, way to treat the wave character of EVE.
ON THE REALITY OF BLACK HOLES

GLOBAL STABILITY OF MINKOWSKI

Proof is based on a huge bootstrap with three major steps,

- Assume the boundedness of the curvature norms and derive precise decay estimates for the connection coefficients of the $t, u$ foliations.

- Use the connection coefficients estimates to derive estimates for the deformation tensors of the approximate Killing and conformal Killing vectorfields.

- Use the latter to derive energy-like estimates for the curvature and thus close the bootstrap.
ASSUME:  \((\Sigma, g, k)\) A.F. maximal,

CONSTRUCT:

1. Double null foliation \(C(u), \overline{C(u)}\) with complete outgoing leaves \(C(u)\).

\[ g^{\alpha\beta} \partial_\alpha u \partial_\beta u = g^{\alpha\beta} \partial_\alpha \overline{u} \partial_\beta \overline{u} = 0. \]

2. Null frame \(L, \overline{L}; (e_a)_{a=1,2}\) adapted to foliation \(S_{u,\overline{u}} = C_u \cap \overline{C(u)}, \quad 4\pi r^2 = Area(S_{t,u}).\)

3. Weak peeling \(r \to \infty\) along \(C(u)\)

\[ \alpha, \beta, \rho - \overline{\rho}, \star \rho = O(r^{-7/2}), \quad \overline{\beta} = O(r^{-2}), \quad \overline{\alpha} = O(r^{-1}) \]

ASSUME

\[ g - g_s = O_{q+1}(r^{-\left(\frac{3}{2} + \gamma\right)}) , \quad k = O_q(r^{-\left(\frac{5}{2} + \gamma\right)}), \quad \gamma > \frac{3}{2}, \quad q \geq 5. \]

THEN

\[ \alpha = O(r^{-5}), \quad \beta = O(r^{-4}), \quad r \to \infty \quad \text{along} \quad C(u). \]

INTERPOLATED RESULTS
Outline

1. ON THE REALITY OF BLACK HOLES
2. RIGIDITY
3. STABILITY