Strong turbulence in nonlinear Schrödinger equation

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The order parameter $\psi(\boldsymbol{x},t)$ for the condensed phase of a Bose gas satisfies a nonlinear-Schrödinger (NLS) equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi - \mu\psi + g|\psi|^2\psi, \tag{1}$$

which is also called the Gross-Pitaevskii equation, under a certain approximation. Here m is the mass of the particles, μ the chemical potential, g the coupling constant and the unit of $\hbar = 1$, $\hbar$ is the Planck constant divided by 2π, is used.

When the nonlinear term, the last term in the r.h.s. of (1), is small enough compared to the first term in the r.h.s. of (1), the statistical properties of the turbulent solutions of NLS equation is well described by the weak wave turbulence (WWT) theory[1,2]. Within the WWT theory, the spectrum $F(k)$ [see (4) for the definition] obeys k^{-1} law for the energy-transfer range and $k^{-1/3}$ law for the particle-number-transfer range. The former is also observed in a numerical simulation of the NLS equation accompanied with external forcing and dissipation[3].

In the present study, we attempt to derive theoretically the spectrum $F(k)$ of the turbulence obeying the NLS equation not only in the WWT range but also in the strong turbulence (ST) range where the nonlinear term becomes dominant in the r.h.s. of (1), by means of a spectral closure approximation, or in other words, a two-point closure approximation.

Let $\psi_{\boldsymbol{k}}(t)$ be the Fourier transform of $\psi(\boldsymbol{x},t)$ with respect to the coordinate variable $\boldsymbol{x}$. It is convenient to introduce a doublet

$$\begin{pmatrix}\psi^+_{\boldsymbol{k}}(t)\\ \psi^-_{\boldsymbol{k}}(t)\end{pmatrix} := \begin{pmatrix}\mathrm{e}^{\mathrm{i}(k^2/2m-\mu)t}\psi_{\boldsymbol{k}}(t)\\ \mathrm{e}^{-\mathrm{i}(k^2/2m-\mu)t}\psi^*_{-\boldsymbol{k}}(t)\end{pmatrix}. \tag{2}$$

By assuming statistical homogeneity in space, the two-point correlation function Q and the two-point response function G can be defined by

$$\langle \psi^\alpha_{\boldsymbol{k}}(t)\psi^\beta_{-\boldsymbol{k}'}(t')\rangle = Q^{\alpha\beta}_{\boldsymbol{k}}(t,t')(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}'), \qquad \left\langle \frac{\delta\psi^\alpha_{\boldsymbol{k}}(t)}{\delta\psi^\beta_{\boldsymbol{k}'}(t')}\right\rangle = G^{\alpha\beta}_{\boldsymbol{k}}(t,t')(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{3}$$

where $\langle\cdot\rangle$ denotes an ensemble average and upper Greece indices denote $\{+,-\}$. The spectrum $F(k)$ is defined by

$$F(k,t) = \frac{1}{2}\int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\delta(|\boldsymbol{k}'|-k)\left[Q^{+-}_{\boldsymbol{k}'}(t,t) + Q^{-+}_{\boldsymbol{k}'}(t,t)\right]. \tag{4}$$

Closed equations for Q and G can be obtained by the method of renormalized expansion and truncation. The closure approximation is essentially NLS equation equivalent of the direct interaction approximation (DIA)[4] of the Navier-Stokes equation.

It is found that, for the energy-transfer range, the time scale $T_{\mathrm{NL}}(k)$ associated with $Q_k(t,t')$ and $G_k(t,t')$ is given by $T_{\mathrm{NL}}(k) = g^{-1}n^{-1}$. The time scale associated to the linear wave is given by $T_{\mathrm{L}}(k) = 2mk^{-2}$. In the WWT range where $T_{\mathrm{NL}}(k) \gg T_{\mathrm{L}}(k)$, the spectral closure reduce to the WWT theory. In the ST range where $T_{\mathrm{NL}}(k) \ll T_{\mathrm{L}}(k)$, we obtained a new scaling law $F(k) \propto k^{-2}$. Similar analysis is also done for the particle-number-transfer range.

References

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