Point-condensation phenomenon in a reaction-diffusion system: geometry of domain vs. heterogeneity of media

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Abstract:
Turing regarded the Diffusion-Driven-Instability as the cause of pattern formation in developmental biology. The diffusion-driven instability is the destabilization of a spatially uniform steady state in a system consisting of chemicals with different diffusion rates. Following Turing's idea, Gierer and Meinhardt proposed an activator-inhibitor system in which the activator concentrates sharply around finitely many points in the closure of the domain. We say that such a solution exhibits a point-condensation phenomenon.

In the limit of infinite diffusion rate for the inhibitor, the activator-inhibitor system reduces to the so-called shadow system, which is essentially a single equation for the activator alone. In this talk we consider a (single) semilinear elliptic equation with power nonlinearity under homogeneous Neumann boundary conditions, which is a prototype problem for the activator-inhibitor system.

If the medium is homogeneous and the diffusion is isotropic, we have an equation with constant coefficients. It is known that when the diffusion coefficient is sufficiently small, the ground-state solution concentrates around a point on the boundary of the domain and decays exponentially away from the concentration point. By the ground-state solution, we mean the solution which has the least energy among positive solutions. As the diffusion coefficient tends to zero, the concentration point approaches the maximum point of the mean curvature function of the boundary. Here, the mean curvature is with respect to the inner normal, so that it is positive when the domain is convex. In addition to the ground state, solutions concentrating around multiple points in the interior of the domain or on the boundary of domain are found. The location of the concentration points are determined in terms of the mean curvature functions or solutions of the sphere-packing problem.

On the other hand, if the medium is heterogeneus, the coefficients in the equation depend on the spatical variable. Recently the effect of spatial heterogeneity on the patterns in reaction-diffusion systems has attracted much attention, but not so many results have appeared on the point-condensation phenomenon in heterogeneous media. In this talk, we introduce a locator function to find out concentration points and apply it to the ground state. Interestingly, the ground state can concentrate at an interior point but it occurs only when the global minimum of the locator function is significantly smaller than its minimum over the boundary.

This talk is based mainly on the joint work with Hiroko Yamamoto.