Quantum symmetry in homological representations of braid groups and hypergeometric integrals

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The notion of braid groups was introduced by E. Artin in 1920's. Especially after the discovery of the Jones polynomial in the middle of 1980's braid groups have appeared in various areas of mathematics such as quantum groups, number theory and conformal field theory. In the beginning of 2000's S. Bigelow and D. Krammer investigated homological representations of braid groups and independently showed that these representations are faithful.

In this talk I will focus on the following developments concerning braid groups.

- correspondence between homological representations and monodromy of KZ connection
- quantum group symmetry and hypergeometric integrals
- description of KZ connection as Gauss-Manin connection
- the image and the kernel of the action of mapping class groups on the space of conformal blocks

For a space X we we define the configuration space of ordered distinct n points in X as

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in \mathcal{F}^n \mid x_i \neq x_j,\ i \neq j\}.$$

We define the configuration space of unordered distinct n points in X as $\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$ where the symmetric group $\mathfrak{S}_n$ acts as the permutation of n points.

In the case X is the complex plane $\mathbf{C}$ the fundamental group $\pi_1(\mathcal{C}_n(\mathbf{C}))$ is by definition the braid group with n strands denoted by B_n. The fundamental group $\pi_1(\mathcal{F}_n(\mathbf{C}))$ is called the pure braid group with n strands and is denoted by P_n.

Let us first explain the construction of homological representations of braid groups. We fix a set of distinct n points in $\mathbf{C}$ and take a 2-dimensional disk D in $\mathbf{C}$ containing Q in the interior. We fix a positive integer m and consider the configuration space $\mathcal{F}_{n,m}(D) = \mathcal{F}_m(D \setminus Q)$. We set $\mathcal{C}_{n,m}(D) = \mathcal{F}_{n,m}(D)/\mathfrak{S}_m$. We have $H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z}$ and consider the abelian covering $\widetilde{\mathcal{C}}_{n,m}(D)$ corresponding to the map

$$\alpha : H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$. The group of deck transformation is $\mathbf{Z} \oplus \mathbf{Z}$ and the homology group $H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ is considered as the module over the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. It can be shown that $H_{m,n}$ is a free R-module. We obtain a representation of the braid group

$$\rho_{n,m} : B_n \longrightarrow \operatorname{Aut}_R H_{n,m}$$

which is called the homological representation of the braid group.

On the other hand, we have the following construction of flat connections on the configuration space $X_n = \mathcal{F}_n(\mathbf{C})$ associated with a complex semi-simple Lie algebra $\mathfrak{g}$ and its representations. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i : \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leq i \leq n$, be representations of the Lie algebra $\mathfrak{g}$. We denote by Ω_{ij} the action of Ω on the i-th and j-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$. We define the Knizhnik-Zamolodchikov (KZ) connection as the 1-form

$$\omega = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} \Omega_{ij} d \log(z_i - z_j)$$

with values in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ for a non-zero complex parameter κ. A horizontal section of the above flat bundle is a solution of the total differential equation $d\varphi = \omega\varphi$ for a function $\varphi(z_1, \cdots, z_n)$ with values in $V_1 \otimes \cdots \otimes V_n$. As the above holonomy of the connection ω we have a one-parameter family of linear representations of the pure braid group $\theta : P_n \to \mathrm{GL}(V_1 \otimes \cdots \otimes V_n)$.

For a complex number λ we denote by M_λ the Verma module of $sl_2(\mathbf{C})$ with highest weight λ. For $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ we put $|\Lambda| = \lambda_1 + \cdots + \lambda_n$ and consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$. For a non-negative integer m we define the space of weight vectors with weight $|\Lambda| - 2m$ by

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \mid Hx = (|\Lambda| - 2m)x\}$$

and consider the space of null vectors defined by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m] \mid Ex = 0\}.$$

We have the following comparison theorem.

Theorem 0.1. *There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the homological representation $\rho_{n,m}$ with the specialization*

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

For the proof we use the description of the horizontal sections of the KZ connection by hypergeometric integrals due to V. Schechtman and A. Varchenko. By looking at the action of the quantum group on the homology of local systems we recover the quantum symmetry on the monodromy of the KZ connection due to V. G. Drinfel'd and myself. In the case of conformal field theory the parameters κ and λ are special. We can define the action of the braid group on the space of conformal blocks, which is a quotient space of the above space of null vectors and the KZ connection has a description as a Gauss-Mannin connection. This corresponds to representations of quantum groups at roots of unity. We study the structure of the image and the kernel of the action of braid groups and mapping class groups and get interesting series of finite index subgroups of mapping class groups. This part is a joint work with L. Funar.