ON STABILITY OF ONE-DIMENSIONAL DYNAMICS

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In this talk, we shall discuss stability of one-dimensional dynamical systems, from both topological and measure-theoretical point of view.

A time-discrete dynamical system is represented by a self map $f : M \to M$. For $n \geq 0$, let f^n denote the n-th iterate of f. For each $x \in M$, the orbit of x is defined to be the sequence $\text{orb}(x) = \{f^n(x)\}_{n=0}^{\infty}$. The theory of dynamical system studies the orbit structure. Stability, roughly speaking, means that the global dynamical property remains unchanged under a small perturbation of the system. It is generally believed that "most" dynamical systems have certain kind of *hyperbolicity* which in turn implies stability in a suitable sense.

Structural stability. Let M denote a compact manifold (possibly with boundary). For each $r = 1, 2, \ldots$, let $C^r(M)$ denote the collection of C^r maps from M to itself, endowed with the C^r topology. A map $f \in C^r(M)$ is C^r-*structurally stable* if there is a neighborhood $\mathcal{U}$ of f in $C^r(M)$ such that each $g \in \mathcal{U}$ is topologically conjugate to f, i.e. there is a homeomorphism $h = h_g : M \to M$ such that $h \circ f = g \circ h$.

Conjecture. (Smale [12]) *If a map $f \in C^r(M)$ is C^r structurally stable, then it satisfies Axiom A.*

When $\dim(M) \geq 2$, the conjecture was proved in the case $r = 1$ (for diffeomorphisms and flows). In the case $\dim(M) = 1$, it is completely proved for all $r = 1, 2, \ldots$ by Kozlovski, Shen and van Strien ([7, 8]). Indeed, when $r \geq 2$, Axiom A maps are dense in $C^r([0, 1])$ and $C^r(S^1)$. It should be noted that this result was proved by considering iteration of real polynomials on the complex plane. Recall that a one-dimensional map $f : M \to M$ satisfies Axiom A if f is uniformly expanding outside the attracting basin of periodic attractors.

The complex one-dimensional case (the *Fatou conjecture*) is still open.

Stochastic stability. Given a map $f : M \to M$, a probability Borel measure μ on M is called *invariant* if for any Borel subset $B \subset M$, we have $\mu(f^{-1}(B)) = B$. The basin of μ, denote by $B(\mu)$, is defined as the set of all $x \in M$ for which

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \to \mu \text{ as } n \to \infty,$$

in the weak star topology, where δ_a denote the Dirac measure at the point a. An invariant probability measure μ is called *physical* if $B(\mu)$ has positive Lebesgue measure.

Let $M = [0,1]$ or S^1. We say that $f \in C^r(M)$ is *weakly stochastic*, if there exist finitely many physical measures μ_i, $i = 1, 2, \dots, m$, such that

$$\text{Leb}\left(M \setminus \left(\bigcup_{i=1}^{m} B(\mu_i)\right)\right) = 0.$$

Furthermore, if for each i, μ_i is either supported on a periodic attractor or absolutely continuous, then we say that f is *strongly stochastic*. It was known (Jakobson [6]) that strongly stochastic but non-Axiom A maps are abundant in the measure-theoretical sense.

Conjecture.(Palis [10]) *"Most" dynamical systems are strongly stochastic and stochastically stable in a suitable sense.*

This conjecture was verified for the family of quadratic polynomials (and for more general unimodal maps with non-degenerate critical points) by Lyubich [9] and Avila-Moreira [3]. Recent works of Avila-Lyubich-Shen [2] and Avila-Lyubich [1] enable us to extend the result to unimodal maps with higher order criticality.

For multimodal interval maps, the Palis conjecture is still open. Nevertheless, it was shown by Bruin, Rivera-Letelier, Shen and van Strien [4] that a very weak *large derivatives* condition implies the strongly stochastic property and by Shen [11] that a summability condition implies strongly stochastic stability. In [5], a strengthened version of the Jakobson's theorem was proved.

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