

Forward Discretely Self-Similar Solutions of the Navier-Stokes Equations

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Abstract

Denote $\mathbb{R}_+^4 = \mathbb{R}^3 \times (0, \infty)$. Consider the 3D incompressible Navier-Stokes equations for velocity $u : \mathbb{R}_+^4 \rightarrow \mathbb{R}^3$ and pressure $p : \mathbb{R}_+^4 \rightarrow \mathbb{R}$,

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad (1)$$

in \mathbb{R}_+^4 , coupled with the initial condition $u|_{t=0} = u_0$ with $\operatorname{div} u_0 = 0$. The system (1) enjoys a scaling property: If $u(x, t)$ is a solution, then so is

$$u^{(\lambda)}(x, t) := \lambda u(\lambda x, \lambda^2 t) \quad (2)$$

for any $\lambda > 0$. We say $u(x, t)$ is **self-similar** (SS) if $u = u^{(\lambda)}$ for every $\lambda > 0$. In that case, the value of $u(x, t)$ is decided by its value at $t = 1$. On the other hand, if $u = u^{(\lambda)}$ only for one particular $\lambda > 1$, we say u is **discretely self-similar** (DSS) with **factor** λ . Its value in \mathbb{R}_+^4 is decided by its value in the strip $x \in \mathbb{R}^3$ and $1 \leq t < \lambda^2$. They are called **forward** because $0 < t < \infty$. We can also consider (1) for $-\infty < t < 0$ or for time independent u . For both cases the scaling law (2) still holds, and we define **backward** and **stationary** SS and DSS solutions in the same manner.

When $u(x, t)$ is either SS or DSS, then so is $u_0(x)$. Thus it is natural to assume $|u_0(x)| \leq \frac{C_*}{|x|}$ for some constant $C_* > 0$ and look for solutions satisfying

$$|u(x, t)| \leq \frac{C(C_*)}{|x|}, \quad \text{or} \quad \|u(\cdot, t)\|_{L^{3,\infty}} \leq C(C_*). \quad (3)$$

Here by $L^{q,r}$, $1 \leq q, r \leq \infty$, we denote the Lorentz spaces. In such classes, with sufficiently small C_* , the unique existence of mild solutions – solutions of the integral equation version of (1) via contraction mapping argument, was obtained by Giga-Miyakawa and refined by Cannone-Meyer-Planchon. As a consequence of the uniqueness, if $u_0(x)$ is SS or DSS with small C_* , the corresponding small mild solution is also SS or DSS. For large C_* , the existence theory for mild solutions is not available, and one may extend the concept of weak solutions and consider local-Leray solutions constructed by Lemarié-Rieusset. However, there is no uniqueness theorem for them to guarantee self-similarity.

In a surprising recent preprint [1], Jia and Šverák constructed SS solutions for every SS u_0 which is locally Hölder continuous. Their main tool is a local Hölder estimate of the solution near $t = 0$, assuming minimal control of u_0 in the large. This estimate enables them to prove a priori estimates of SS solutions, and then show the existence by applying the Leray-Schauder degree theorem. Note that this existence theorem does not assert uniqueness.

In this talk, I will present results asserting the existence of DSS solutions for DSS initial data u_0 , assuming either the DSS factor λ is sufficiently close to 1 according to C_* , or if u_0 is axisymmetric with no swirl. This extends the result of [1] to DSS setting. I will also discuss their relevance to the uniqueness problem.

References

- [1] Jia, H, Šverák, V.: Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions, preprint: arXiv:1204.0529
- [2] Tsai, T.-P.: Forward discretely self-similar solutions of the Navier-Stokes equations, preprint: arXiv:1210.2783