Incompressible fluids on foliated manifolds

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In order to understand the topology and geometry of manifolds or geometric structures on them, it might be a natural idea to look at how fluids flow on them. Unfortunately, the analytical foundations for the fluid mechanics is very hard to establish and not yet enough to employ this idea. However, it is still tempting the author, even to investigate foliations on manifolds.

We have already a difficulty in writing down the proper Euler equation, namely, the equation of motion for incompressible fluids without viscosity whose flow lines are contained in leaves. If the time allows, we will discuss on this difficulty in the talk.

The main topic of this talk is concerning a more primitive stage than the genuine fluid mechanics. We try to understand in the case of codimension 1 foliations on closed oriented 3-manifolds the space of velocity fields of such fluids by using an interpretation of the notion of *helicity* as a symmetric bi-linear form on the space (not of velocity fields but) of volticity fields, which is so called the *asymptotic linking*.

Fix a volume form *dvol* on a closed oriented manifold M. Let $\mathcal{X}$ denote the set of smooth vector fields on M and $\mathcal{X}_d$ the set of divergence free vector fields. By taking so called the *asymptotic cycle* $\mathcal{X}_d$ surjects to $H_1(M;\mathbb{R})$ and its kernel $\mathcal{X}_h$ is called homology free vector fields. $\mathcal{X}_h$ coincides with the span of locally supported ones in $\mathcal{X}_d$. The asymptotic linking lk is a symmetric bi-linear form on $\mathcal{X}_h$.

Now take a codimension 1 foliation $\mathcal{F}$ on a closed orinted 3-manifold M and set $\mathcal{X}(M,\mathcal{F}) = \{X \in \mathcal{X}\,;\, X /\!\!/ \mathcal{F}\}$. The velocity fields of our foliated incompressible fluids are those who belong to $\mathcal{X}_d(M;\mathcal{F}) = \mathcal{X}(M,\mathcal{F}) \cap \mathcal{X}_d$. Also consider holomogy free ones $\mathcal{X}_h(M;\mathcal{F}) = \mathcal{X}(M,\mathcal{F}) \cap \mathcal{X}_h$ and locally supported ones $\mathcal{X}_{loc}(M;\mathcal{F})$. As in a reasonable sense $\mathcal{X}_{loc}(M;\mathcal{F})$ is understandable, the understanding of $\mathcal{X}_d(M;\mathcal{F})$ may reduces to seeing the structures of $\mathcal{X}_d(M;\mathcal{F})/\mathcal{X}_h(M;\mathcal{F})$ and of $\mathcal{X}_h(M;\mathcal{F})/\mathcal{X}_{loc}(M;\mathcal{F})$.

Main Result 1 0) $\mathcal{X}_d(M;\mathcal{F})/\mathcal{X}_h(M;\mathcal{F})$ naturally injects to $H_1(M;\mathbb{R})$ and its image depends on each case. (This is almost trivial.)
1) $\mathcal{X}_{loc}(M;\mathcal{F})$ is a null subspace of $(\mathcal{X}_h, lk)$.
2) The orthogonal complement of $\mathcal{X}_{loc}(M;\mathcal{F})$ with respect to lk is exactly $\mathcal{X}_h(M;\mathcal{F})$.
3) On the quotient $\mathcal{X}_h(M;\mathcal{F})/\mathcal{X}_{loc}(M;\mathcal{F})$ a symmetic bi-liniar form (which is also denoted by lk) is naturally induced from lk.

Here is one more ingredient to analyse $(\mathcal{X}_h(M;\mathcal{F})/\mathcal{X}_{loc}(M;\mathcal{F}), lk)$, that is, the foliated (de Rham) cohomolgy $H^*(M,\mathcal{F})$, which is not necessarily of finite dimension and is quite often very hard to compute, and the characteristic pairing $CJ : H^1(M;\mathcal{F}) \otimes H^1(M;\mathcal{F}) \to H^3(M)$ ($\cong \mathbb{R}$ in our case).

Main Result 2 1) There exists a natural surjection $\Phi : H^1(M;\mathcal{F}) \to \mathcal{X}_h(M;\mathcal{F})/\mathcal{X}_{loc}(M;\mathcal{F})$ which intertwines the pairings CJ and lk.
2) On some particular cases we can compute $H^1(M;\mathcal{F})$ and Φ explicitely as well as the quotient $\mathcal{X}_d(M;\mathcal{F})/\mathcal{X}_h(M;\mathcal{F})$.

All these are computed in terms of differentital forms, volume dual to vector fields. The conservation laws should be studied.