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Pushing Polynomial Reproducing Kernels to their Non-polynomial Limit

Let μ be a positive Borel measure on the real line, all of whose moments $\int x^j d\mu(x)$, $j = 0, 1, 2, 3, ...$ are finite. Then we may form orthonormal polynomials $p_n(x) = \gamma_n x^n$+lower powers, $\gamma_n > 0$, satisfying the orthonormality relation

$$\int p_n p_m d\mu = \delta_{mn}.$$

The nth reproducing kernel is

$$K_n(x,t) = \sum_{j=0}^{n-1} p_j(x) p_j(t).$$

Its reproducing property is

$$P(x) = \int K_n(x,t) P(t) d\mu(t),$$

for polynomials of degree $\leq n-1$. "Along the diagonal" it has an extremal property that is common to reproducing kernels in all inner product spaces:

$$K_n(x,x) = \sup \left\{ \frac{|P(x)|^2}{\int |P|^2 d\mu} : P \text{ is a polynomial of degree } \leq n-1 \right\}.$$

In the case of orthogonal polynomials, it is especially useful, as it shows that $K_n(x,x)$ decreases as μ increases. This allows us to compare $K_n(x,x)$ for different measures, and establish asymptotics.

Indeed, it played in a crucial role in a breakthrough 1991 result of Maté, Nevai, and Totik. They proved that if μ is a measure with support $[-1,1]$, and if, for example, $\int_{-1}^{1} \log \mu'(x) dx$ is finite, then for a.e. x in $(-1,1)$,

$$\lim_{n\to\infty} \mu'(x) \frac{K_n(x,x)}{n} = \frac{1}{\pi\sqrt{1-x^2}}.$$

In mathematical physics, this limit is loosely described as the *density of the states.* The asymptotic was subsequently generalized to measures μ with arbitrary compact support on the real line by Totik. In this case, $\frac{1}{\pi\sqrt{1-x^2}}$ is replaced by the (potential theoretic) equilibrium density for the support of the measure. There are analogues for measures with non-compact support, and also for sequences of measures, where we have an nth measure μ_n at the nth stage.

How do reproducing kernels connect to random matrices? It was the physicist Eugene Wigner who had the idea to model scattering theory for neutrons off heavy nuclei, using random Hermitian matrices. He placed a probability distribution $\mathcal{P}^{(n)}$ on the eigenvalues of $n \times n$ Hermitian matrices. A key statistic is the $m-$point correlation function

$$R_m(x_1, x_2, ..., x_m) = \frac{n!}{(n-m)!} \int ... \int P^{(n)}(x_1, x_2 ..., x_n) dx_{m+1}\ dx_{m+2}\ ...dx_n.$$

It can be used to measure the number of $m-$tuples of eigenvalues lying in a given set. Under appropriate assumptions on the underlying measure, there is the remarkable identity

$$R_m\left(x_1, x_{2,}..., x_m\right) = \det\left(\sqrt{\mu'\left(x_i\right)\mu'\left(x_j\right)}K_n\left(x_i, x_j\right)\right)_{1\leq i,j\leq m}.$$

The *universality limit in the bulk* asserts that for x in the interior of the support of μ, and real $a_1, a_2, ..., a_m$, we have

$$\lim_{n\to\infty}\frac{1}{\mu'\left(x\right)K_n\left(x,x\right)^m}R_m\left(x+\frac{a_1}{\mu'\left(x\right)K_n\left(x,x\right)}, ..., x+\frac{a_m}{\mu'\left(x\right)K_n\left(x,x\right)}\right)$$
$$= \det\left(\frac{\sin\pi\left(a_i-a_j\right)}{\pi\left(a_i-a_j\right)}\right)_{1\leq i,j\leq m}.$$

Because R_m is the determinant of an $m\times m$ matrix, and m is fixed in this limit, this reduces to the limit for each entry, namely, for real a, b,

$$\lim_{n\to\infty}\frac{K_n\left(x+\frac{a}{\mu'(x)K_n(x,x)}, x+\frac{b}{\mu'(x)K_n(x,x)}\right)}{K_n\left(x,x\right)} = \frac{\sin\pi\left(a-b\right)}{\pi\left(a-b\right)}. \qquad (1)$$

Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a technical limit involving orthogonal polynomials. The right-hand side is independent of x, and the underlying measure μ, which well justifies the name *universality limit.*

This limit has been established for a very wide range of measures μ, with compact and non-compact support, and for varying measures, where μ changes as n does. In Wigner's original case, the so-called Gaussian unitary ensemble, $\mu'\left(x\right)$ was a scaled Hermite weight e^{-nx^2} at the nth stage. Techniques of proof include asymptotics for classical orthogonal polynomials, and the Christoffel-Darboux formula, Riemann-Hilbert methods, and more recently, classical tools from orthogonal polynomials. The limit itself has also given new insight into spacing of zeros of orthogonal polynomials. It is still unresolved how *universal is universality*, that is what is the full range of measures μ for which is is true? We'll discuss this.

Is it an accident that the famous sinc kernel

$$S\left(t\right) = \frac{\sin\pi t}{\pi t}$$

arises in (1)? It appears in so many contexts, most notably in the sampling theorem of signal processing. Of course it is scarcely surprising that the scaled limit of reproducing kernels for polynomials is a reproducing kernel for some space. We'll explain why it is $S\left(a-b\right)$, the reproducing kernel for *Paley-Wiener space.* We'll also discuss L_p analogues of $K_n\left(x,x\right)$, and their universality limits.