Derived categories in algebraic geometry

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A derived category is derived from an abelian category. An abelian category is closely attached to the original situation, but a derived category is not in the sense that derived categories of different origins are sometimes equivalent. This kind of rather unexpected equivalences of derived categories usually reveal deep mathematical facts. We review some of these phenomena in old and new examples.

(1) *Fourier-Mukai transform.*

Let X be an abelian variety and let $Y = \mathrm{Pic}^0(X)$ be its dual abelian variety. Y is the moduli space of line bundles on X which are algebraically equivalent to the trivial line bundle $\mathcal{O}_X$. Then their bounded derived categories of coherent sheaves $D^b(X)$ and $D^b(Y)$ are equivalent. The equivalence is given by the integration functor in the same way as the Fourier transform in analysis. The Fourier transform is given by $\Phi(h)(y) = \int_X e^{2\pi ixy}h(x)dx$, where $X = Y = \mathbf{R}/\mathbf{Z}$ and x, y are coordinates. The Fourie-Mukai transform is given by $\Phi(E) = Rp_{2*}(p_1^*E \otimes P)$, where p_1, p_2 are projections from the direct product $X \times Y$ and P is the universal line bundle.

(2) *Beilinson's decomposition.*

The bounded derived category of coherent sheaves on a projective space $D^b(\mathbf{P}^n)$ is generated by the line bundles $\mathcal{O}_X(-i)$ for $0 \le i \le n$. Let $A = \mathrm{End}(\bigoplus_{0\le i\le n} \mathcal{O}_X(-i))$. It is a non-commutative ring which is finite dimensional as a vector space over the base field k. Then $D^b(\mathbf{P}^n)$ is equivalent to the bounded derived category of finite right A-modules $D^b(\text{mod-}A)$.

$D^b(\mathbf{P}^n)$ is also generated by the locally free sheaves $\Omega^i_X(i)$ for $0 \le i \le n$, and is equivalent to the bounded derived category of finite right B-modules $D^b(\text{mod-}B)$ for $B = \mathrm{End}(\bigoplus_{0\le i\le n} \Omega^i_X(i))$.

(3) *BGG correspondence.*

Let V be a vector space of dimension n over a field k and W its dual space. Let x_i $(1 \le i \le n)$ be a basis of V and e_i $(1 \le i \le n)$ the dual basis. We put degree 1 on the x_i and -1 on the e_i.

Let $S = \mathrm{S}^*(V) \cong k[x_1, \dots, x_n]$ be the symmetric algebra. It is a commutative ring and is infinite dimensional as a vector space over k. Let $E = \bigwedge^* W$ be the exterior algebra. It is a non-commutative ring and is finite dimensional as a vector space over k.

There is an equivalence $D(\text{Gr-Mod-}S) \cong D(\text{Gr-Mod-}E)$ of derived categories of graded modules. This equivalence is a refinement of the equivalence in the previous section.

(4) *McKay correspondence.*

Let G be a finite subgroup of $SL(n, \mathbf{C})$ which acts on $\mathbf{C}^n$ naturally. The quotient space $X = \mathbf{C}^n/G$ has normal singularities, but the quotient stack $\mathcal{X} = [\mathbf{C}^n/G]$ is a smooth Deligne-Mumford stack. A sheaf on $\mathcal{X}$ is by definition a sheaf on $\mathbf{C}^n$ with an equivariant action of G. A resolution of singularities $f : Y \to X$ is said to be *crepant* if $K_Y = f^*K_X$, i.e., the G-invariant holomorphic volume form on $\mathbf{C}^n$ lifts to that on Y. The McKay correspondence theorem says that, if $n \le 3$, then there exists a crepant resolution $f : Y \to X$ and that there is an equivalence $D^b(Y) \cong D^b(\mathcal{X})$.

(5) *flops.*

This is a variant of the previous section. Let $f : Y \to X$ be a birational morphism from a smooth projective variety to a normal variety. Assume that $K_Y = f^*K_X$. Then in some cases we expect that there is a coherent sheaf of non-commutative $\mathcal{O}_X$-algebras $\mathcal{A}$ such that $D^b(Y) \cong D^b(\mathcal{A})$. For example, if $\dim Y = 3$ and f is small in the sense that f does not contract any divisor, then the assertion is true. This is generalized to the "K-equivalence and D-equivalence" conjecture.

(6) *Derived VGIT.*

I will also discuss more recent development on the relationship with the variation of GIT quotients.