The geography problems of 4-manifolds

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Abstract

Since M. Freedman classified simply connected topological 4-manifolds using intersection forms ([Fr]) and S. Donaldson introduced gauge theory ([DK]) to show that some of topological 4-manifolds do not admit a smooth structure in 1982, there has been a great progress in the study of 4-manifolds mainly due to Donaldson invariants, Seiberg-Witten invariants and Gromov-Witten invariants. But the complete understanding of 4-manifolds is far from reach.

One of the fundamental problems in 4-manifolds is to classify simply connected closed smooth (symplectic, complex) 4-manifolds. The classical invariants of a simply connected closed 4-manifold are encoded by its intersection form Q_X, a unimodular symmetric bilinear pairing on $H_2(X : \mathbf{Z})$. M. Freedman proved that a simply connected smooth 4-manifold is determined up to homeomorphism by Q_X ([Fr]). But it turned out that the situation is strikingly different in the smooth (symplectic, complex) category. That is, it has been known that only some unimodular symmetric bilinear integral forms are realized as the intersection form of a simply connected smooth (symplectic, complex) 4-manifold, and there are many examples of infinite classes of distinct simply connected smooth (symplectic, complex) 4-manifolds which are mutually homeomorphic. Hence it is a fundamental and important question in the study of 4-manifolds which unimodular symmetric bilinear integral forms are realized as the intersection form of a simply connected smooth (symplectic, complex) 4-manifold - called an *existence problem*, and how many distinct smooth (symplectic, complex) structures exist on it - called a *uniqueness problem*. Geometers and topologists call these *'geography problems of 4-manifolds'*.

The geography problem asks which lattice points in the $(\frac{b^+ +1}{2}, 3\sigma + 2e)$-plane are 'populated' by simply connected minimal smooth (symplectic, complex) 4-manifolds. These coordinates are chosen because of their relation to complex surfaces, where $\chi = \frac{b^+ +1}{2}$ and $c_1^2 = 3\sigma + 2e$. The geography problem for complex surfaces of general type has been studied extensively by algebraic surface theorists. For example, it is well-known that minimal complex surfaces of general type satisfy Noether inequality ($2\chi - 6 \leq c_1^2$) and Bogomolov-Miyaoka-Yau inequality ($c_1^2 \leq 9\chi$). Many remarkable results on the geography problems of smooth and symplectic 4-manifolds have also been obtained using Donaldson theory and Seiberg-Witten theory. For example, all known simply connected minimal symplectic 4-manifolds satisfy $0 \leq c_1^2 \leq 9\chi$ and most lattice points satisfying $0 \leq c_1^2 \leq 9\chi$ are populated by simply connected minimal

smooth and symplectic 4-manifolds. Furthermore, it has also been proved that most known simply connected smooth 4-manifolds with $b_2^+ > 1$ and odd admit infinitely many distinct smooth structures.

One the other hand, in the case of $b_2^+ = 1$ (equivalently, $p_g = 0$ in complex category) and $c_1^2 > 0$, until 2003 the only previously known simply connected, minimal, symplectic 4-manifolds (or complex surfaces) are rational surfaces and Barlow surface ([BHPV]). Barlow surface has $c_1^2 = 1$ ([B]). So the natural question arises if there are other simply connected symplectic 4-manifolds with $b_2^+ = 1$ or complex surfaces of general type with $p_g = 0$ except Barlow surface.

Since I discovered a new simply connected symplectic 4-manifold with $b_2^+ = 1$ and $c_1^2 = 2$ ([P]) in 2004 by using a rational blow-down surgery, and Y. Lee and myself constructed a new family of simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $c_1^2 = 2$ ([LP]) by modifying the symplectic 4-manifold in 2006, many new simply connected symplectic 4-manifolds with $b_2^+ = 1$ and complex surfaces of general type with $p_g = 0$ have been constructed ([FS, PPS, PSS, SS, SSW]) and now it is one of the most active research areas in 4-manifolds to find a new family of smooth (symplectic, complex) 4-manifolds with $b_2^+ = 1$ (equivalently, $p_g = 0$ in complex category).

The aim of this talk is to review briefly recent developments in this area. In particular, I'd like to survey the existence and the uniqueness problems of simply connected 4-manifolds with $b_2^+ = 1$ in three levels - smooth category, symplectic category and complex category.

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