A Γ-structure on the Lagrangian Grassmannian

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This is joint work with Peter Albers and Jake Solomon. Given standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ the Lagrangian Grassmannian $\mathscr{L} = \mathscr{L}(n)$ is defined to be the manifold consisting of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. The Lagrangian Grassmannian can also be identified with the homogenous space $U(n)/O(n)$. Alternatively, we think of the Lagrangian Grassmannian as the space consisting of all linear, orthogonal antisymplectic involutions. Indeed, given an antisymplectic involution, the fixed point set is a Lagrangian subspace and this map identifies the two descriptions. Now thinking of points of the Lagrangian Grassmanian as linear, orthogonal, antisymplectic involutions we can define a product on the Lagrangian Grassmannian

$$\Theta : \mathscr{L} \times \mathscr{L} \to \mathscr{L}, \quad (R, S) \mapsto RSR.$$

Our main result is that for n odd, this product endows the Lagrangian Grassmannian with the structure of a Γ-manifold.

The notion of a Γ-manifold goes back to Hopf. Assume that M is a closed, orientable, connected manifold and $\Theta \colon M \times M \to M$ a smooth map which we think of as a product. If $x \in M$ plugging x into the left entry of Θ we get a map $\Theta_x := \Theta(x, \cdot) \colon M \to M$ which has a degree $\deg \Theta_x \in \mathbb{Z}$. Since M is connected, by invariance of the mapping degree under homotopies we see that $\deg \Theta_x$ is independent of the choice of x and we set $\ell(\Theta) := \deg(\Theta_x)$. Similarly, we can plug in x to the right entry of Θ and we get a degree $r(\Theta) \in \mathbb{Z}$ again independent of the choice of the point x. The left and right degree however do not need to agree. For example if Θ is a projection then one degree is zero while the other one is one. Now the tuple (M, Θ) is called a Γ-manifold if both degrees $\ell(\Theta$ and $r(\Theta)$ are different from zero. Note that this condition rules out trivial products like constant maps or projections to a factor. The interest in the existence of a Γ-structure comes from the discovery of Hopf that a Γ-structure endows the cohomology ring of the manifold with a Hopf algebra structure and therefore gives interesting topological information on the manifold. In particular, our main result allows us to recover some results by Fuks on the cohomology ring of the Lagrangian Grassmannian in the context of Hopf algebras.

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