Classical solutions of mean-field games

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A model problem for mean-field games [LL06a, LL06b, LL07, HMC06, HCM07] is the system

$$-V_t + H(D_x V, x) = \Delta V + g(\theta), \quad \theta_t - \operatorname{div}(\theta D_x V) = \Delta V. \tag{1}$$

together with initial-terminal conditions $V(x,T) = \psi(x)$, $\theta(x,0) = \theta_0$, and periodic boundary conditions in x. In [LL06b, LL07] the authors give conditions for existence of weak solutions to (1). In this talk we prove the following results (joint work with H. S. Morgado and G. Pires [GPSM13]):

Theorem 1. *Let $g(m) = m^\alpha$ and $H(p,x) = \frac{|p|^2}{2} + V(x)$, ψ and θ smooth, $\theta > 0$ If $d = 2$ and $\alpha > 0$, or if $d = 3$ and $\alpha < \frac{1}{2}$, then V is Lipschitz.*

Once this regularity is obtained then further regularity results can also be obtained by bootstrapping and using standard methods:

Theorem 2. *Under the conditions of Theorem 1 If $d = 2$ and $\alpha > 0$, or if $d = 3$ and $\alpha < \frac{1}{2}$, then $\ln m$ is Lipschitz.*

From the regularity obtained in the previous theorem then it is a routine matter to prove existence of smooth solutions. A number of further extensions are possible by considering more general Hamiltonians. A number of similar results for the time independent case can also be addressed similarly.

References

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