Finite generation questions for motivic cohomology

Thomas Geisser
名古屋大学
多元数理科学研究科

These are notes of a survey talk I gave on finite generation questions of motivic cohomology, translated into English and supplemented by references.

1 Classical results

The first finite generation results on motivic cohomology were proved in the 19th century.

Theorem 1 (Dirichlet's unit theorem) *Let $\mathcal{O}_K$ be the ring of integers in a number field K. Then $\mathcal{O}_K^\times$ is finitely generated of rank $r_1 + r_2 - 1$.*

Here $r_1 =$ is the number of real embeddings and $r_2 =$ is the number of pairs of complex embeddings of the numbe field K. For example, one has $\mathbb{Z}^\times = \{\pm 1\}$, $\mathcal{O}_{\mathbb{Q}(i)}^\times \cong \{\pm 1, \pm i\}$ and $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}^\times = \{\pm(1+\sqrt{2})^n, n \in \mathbb{Z}\}$.

If K is a number field, then the class group $\mathrm{Pic}(\mathcal{O}_K)$ is defined as the group of fractional ideals modulo the group of principal ideals. By unique factorization of ideals in Dedekind rings, we have an exact sequence

$$0 \to \mathcal{O}_K^\times \to K^\times \stackrel{div}{\to} \bigoplus_{\mathfrak{p}} \mathbb{Z} \to \mathrm{Pic}(\mathcal{O}_K) \to 0.$$

By definition, $\mathrm{Pic}(\mathcal{O}_K) = 1$ if and only if $\mathcal{O}_K$ is a principal ideal domain.

Theorem 2 *The class group $\mathrm{Pic}(\mathcal{O}_K)$ of a number field is finite.*

We give two generalization of this is from the middle of the 20th century. Define Milnor K-groups to be

$$K_*^M(K) = T^*K^\times / \langle a \otimes (1-a) \rangle$$

with $T^n A := A^{\otimes n}$ the tensor algebra for an abelian group A, and product given by concatenation. For example, $K_0^M(K) = \mathbb{Z}$, and $K_1^M(K) = K^\times$. The finite generation of the group of units is the case $n = 1$ of the following

Theorem 3 (Bass-Tate [2]) *Let K be a number field, $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$ and $k(\mathfrak{p}) = \mathcal{O}_k/\mathfrak{p}$. Then for $n \geq 2$, the kernel of*

$$K_n^M(K) \to \bigoplus_{\mathfrak{p}} K_{n-1}^M(K_\mathfrak{p})$$

is finite.

From now on, we always assume that X is a regular scheme, of finite type over $\mathbb{Z}$. i.e. X has a finite cover by $\operatorname{Spec} R$ for R a regular integral domain of the form $\mathbb{Z}[X_1, \cdots, X_n]/(f_1, \cdots, f_m)$. If $K(X)$ is the function field of X, then the group of invertible function on X and the Picard group of X fit into the exact sequence

$$0 \to \mathcal{O}(X)^\times \to K(X)^\times \stackrel{div}{\to} \bigoplus_{\mathfrak{p}} \mathbb{Z} \to \operatorname{Pic} X \to 0,$$

where the sum is over height one prime ideals, i.e. divisors. We have the following generalization of the finite generation of the group of units and Picard group of a number field:

Theorem 4 (Mordell-Weil, Roquette [26]) *The groups $\mathcal{O}(X)^\times$ and $\operatorname{Pic} X$ are finitely generated abelian groups.*

2 Bass conjecture

Quillen [24] defined in 1973 higher algebraic K-groups $K_i(X)$ for a scheme X using the category of finitely generated locally free modules, i.e. vector bundles. It is a generalization of the Grothendieck group $K_0(X)$ of X. This generalizes many invariants studied before. For example, $\operatorname{Pic} X$ is a direct factor of $K_0(X)$, and $\mathcal{O}(X)^\times$ direct factor of $K_1(X)$. For a local ring R, one has $K_0(R) = \mathbb{Z}$ and $K_1(R) = R^\times$.

Conjecture 5 (Bass [1]) *The groups $K_i(X)$ are finitely generated.*

The conjecture is known in the following cases:

Theorem 6 (Quillen [25, 23, 14]) *1. For a finite field $\mathbb{F}_q$, we have*

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/q^n - 1 & i = 2n-1, \\ 0 & i > 0 \text{ even.} \end{cases}$$

2. *For a number field K, $K_i(\mathcal{O}_K)$ is finitely generated.*
3. *Let X be a (normal) curve over a finite field. Then $K_i(X)$ is finitely generated.*

This completely settles the one-dimensional case. In the second case, the ranks also have been calculated and are related to the order of zero of the Dedekind-zeta function $\zeta_K(s)$ at $s = 1 - i$:

Theorem 7 (Borel [5]) *For a number field K,*

$$\operatorname{rank} K_i(\mathcal{O}_K) = \begin{cases} r_1 + r_2 - 1 & i = 1, \\ r_1 + r_2 & i = 4n + 1 > 1, \\ r_2 & i = 4n - 1, \\ 0 & i > 0 \text{ even.} \end{cases}$$

In contrast, we have the following result in characteristic p

Theorem 8 (Harder [15], Soule [27]) *For a smooth proper curve over a finite field, $K_i(X)$ is torsion for $i > 0$.*

Thus the curve case in characteristic 0 and p are very different. The conjecture of Bass fails for non-regular X. For example, let $R = \mathbb{F}_p[t, \epsilon]/(\epsilon^2)$. Then $K_1(R) \supseteq R^\times$ contains an infinite $\mathbb{F}_p$-vector space with basis

$$e_i = (1 + \epsilon t^i)$$

because $(1 + \epsilon t^i)(1 - \epsilon t^i) = 1$.

3 Motivic cohomology

Motivic cohomology groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ have been conjectured to exist by Lichtenbaum [20] and Beilinson [3]. Later, definitions have been given by Bloch [4] and Voevodsky. Motivic cohomology is a finer invariant of X than K-theory. Indeed, for X smooth over a perfect field, there is a 4th quadrant spectral sequence

$$E_2^{s,t} = H^s_{\mathcal{M}}(X, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}(X)$$

which allows (in theory) to calculate K-groups from motivic cohomology groups. We have the following refinement of Bass conjecture:

Conjecture 9 (refined Bass conjecture) *The groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ are finitely generated for X is regular and of finite type over $\mathbb{Z}$.*

For example, the following groups are finitely generated (the latter by the Bass-Tate theorem)

- $H^1_{\mathcal{M}}(X, \mathbb{Z}(1)) \cong \mathcal{O}(X)^\times$
- $H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) \cong \mathrm{Pic}(X)$
- $H^2_{\mathcal{M}}(\mathcal{O}_K, \mathbb{Z}(2))$

Assume X is smooth over a perfect field. Then

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := CH^n(X, 2n - i)$$

is Bloch's higher Chow group [4]. This is a theorem [30], but we use it as the definition of motivic cohomology here.

To define $CH^n(X, j)$ consider the complex $z^n(X, -)$ defined as follows. Let $\Delta^j = \mathrm{Spec}\,\mathbb{Z}[T_0, \cdots t_j]/(\sum t_i = 1)$ be the algebraic j-simplex. Then $F \subseteq \Delta^j$ given by $t_{i_1} = \cdots = t_{i_m} = 0$ is called a face. Clearly $F \cong \Delta^{j-m}$.

Definition 10 *$z^n(X, j)$ is the free abelian group, generated by closed irreducible subvarieties $Z \subseteq X \times \Delta^j$ such that $codim_F(Z \cap F) = n$ for all faces (including $F = \Delta^j$).*

Differentials $z^n(X,j) \to z^n(X,j-1)$ are alternating sums of intersection with faces $\Delta^{j-1} \subseteq \Delta^j$ (there are $j+1$ ways of viewing Δ^{j-1} as a face of Δ^j). We obtain a complex of free abelian groups, and let $CH^n(X,j)$ be its jth homology. For example

$$H^{2n}_{\mathcal{M}}(X,\mathbb{Z}(n)) = \operatorname{coker} z^1(X,1) \to z^n(X,0)$$

is the usual Chow group $CH^n(X)$, and $H^i_{\mathcal{M}}(X,\mathbb{Z}(n)) = 0$ for $i > 2n$ or $i > n + \dim X$. These groups satisfy the following important properties:

- they are covariant for proper maps (with a change of n)
- they are contravariant for maps between smooth schemes
- they are homotopy invariant: $CH^n(X,i) \cong CH^n(X \times A^1, i)$

For a field k, $H^i_{\mathcal{M}}(k,\mathbb{Z}(n)) = 0$ for $i > n$ (by dimension of cycles reasons) and

Theorem 11 (Nesterenko-Suslin [22], Totaro [29])

$$H^n_{\mathcal{M}}(k,\mathbb{Z}(n)) \cong K^M_n(k).$$

For low values of n, higher Chow groups agree with the invariants mentioned before:

Proposition 12 (Bloch [4]) *Let X be connected and smooth, then*

$$H^i_{\mathcal{M}}(X,\mathbb{Z}(0)) \cong \begin{cases} \mathbb{Z} & i = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

$$H^i_{\mathcal{M}}(X,\mathbb{Z}(1)) \cong \begin{cases} \mathcal{O}(X)^\times & i = 1, \\ \operatorname{Pic}(X) & i = 2, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, these groups are finitely generated for X over $\operatorname{Spec}\mathbb{Z}$. In general, $H^i_{\mathcal{M}}(X,\mathbb{Z}(2))$ and $H^i_{\mathcal{M}}(X,\mathbb{Z}(d))$ for $d = \dim X$ are understood a little, all other groups are a complete mystery. We don't even know that there is no negative motivic cohomology:

Conjecture 13 (Beilinson-Soule) *The groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ vanish for $i < 0$.*

Over a finite field, this conjecture has the following strengthening:

Conjecture 14 (Parshin) *If X is smooth and proper over a finite field, then $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ is torsion for $i \neq 2n$.*

We saw before that for a number field K

$$\operatorname{rank} H^1_{\mathcal{M}}(\mathcal{O}_K, \mathbb{Z}(n)) = \begin{cases} r_1 + r_2 & n \text{ odd} > 1, \\ r_2 & n \text{ even} > 0, \end{cases}$$

so again the behavior in characteristic p and 0 is very different.

4 Etale motivic cohomology

It is often easier to calculate the etale hypercohomology of a complex of sheaves. By contravariance, varying U we obtain a complex of presheaves defined by

$$U \mapsto \mathbb{Z}(n)(U) := z^n(U, -)[-2n].$$

This turns out that this is a complex of sheaves for the etale topology, in particular a complex of sheaves for the Zariski topology. We make the shift so that $H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) = CH^n(X, i - 2n)$ is the cohomology of $\mathbb{Z}(n)(X)$ in degree i. Bloch's proposition implies that

$$\mathbb{Z}(0) \cong \mathbb{Z}$$

$$\mathbb{Z}(1) = \mathbb{G}_m[-1], \text{ where } \mathbb{G}_m : U \mapsto \mathcal{O}(U)^\times.$$

With torsion coefficients, this complex of sheaves is well-understood:

Theorem 15 (Suslin-Voevodsky, G.-Levine [13, 12]) *Let X be smooth over a perfect field of characteristic p. Then there are quasi-isomorphisms of complexes of etale sheaves*

$$\mathbb{Z}/m(n) \cong \begin{cases} \mu_m^{\otimes n} & p \not| m \\ \nu_r^n[-n] & m = p^r \end{cases}$$

Here $\mu_n(U) = {}_m\mathcal{O}(U)^\times$ and $\nu_r^n \subseteq W_r\Omega^n_X$ is the logarithmic de Rham Witt sheaf.

So etale motivic cohomology with finite coefficients is etale cohomology considered since the 1960's!

Taking Zariski-hypercohomology, nothing changes

$$H^i(X_{\mathrm{Zar}}, \mathbb{Z}(n)) \cong H^i_{\mathcal{M}}(X, \mathbb{Z}(n)).$$

But taking etale hypercohomology, things become very interesting: For example,

$$H^3(X_{\mathrm{et}}, \mathbb{Z}(1)) \cong H^2(X_{\mathrm{et}}, \mathbb{G}_m) \cong \mathrm{Br}(X)$$

is the cohomological Brauer group of X classifying Azumaya algebras. It is a very deep conjecture that this is finite for X proper over $\mathbb{Z}$. More generally, one has

Conjecture 16 (Lichtenbaum [20]) *Let X be smooth and proper over a finite field. Then $H^i(X_{\mathrm{et}}, \mathbb{Z}(n))$ is*

- *finite for $i \neq 2n, 2n+2$*
- *finitely generated for $i = 2n$*
- *cofinitely generated for $i = 2n+2$ (i.e. $(finite) \oplus (\mathbb{Q}/\mathbb{Z})^r$)*

This conjecture is older than motivic cohomology itself, because Lichtenbaum was conjecturing the existence of a complex $\mathbb{Z}(n)$ with the above properties (among others). The easiest example that cofinitely generated groups appear is

$$\begin{aligned} H^2((\mathbb{F}_q)_{\mathrm{et}}, \mathbb{Z}(0)) \cong H^2(\mathrm{Gal}(\mathbb{F}_q), \mathbb{Z}) \cong H^1(\mathrm{Gal}(\mathbb{F}_q), \mathbb{Q}/\mathbb{Z}) \\ \cong \mathrm{Hom}(\mathrm{Gal}(\mathbb{F}_q), \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

It is easy to see that $H^i(X_{\mathrm{Zar}}, \mathbb{Q}(n)) \cong H^i(X_{\mathrm{et}}, \mathbb{Q}(n))$ but what about integral coefficients? To compare motivic cohomology to its etale version we have

Theorem 17 (Rost-Voevodsky, formerly Bloch-Kato conjecture) *For any field k, and $m \in k^\times$ the map $K_n^M(k)/m \to H^n(k_{\mathrm{et}}, \mathbb{Z}/m(n))$ is surjective.*

The analog for $m = p^r$ a power of the characteristic had been shown be Bloch-Kato and Gabber before. This implies the so called Beilinson-Lichtenbaum conjecture

Corollary 18 (Suslin-Voevodsky, G.-Levine [13, 12]) *The Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture. In particular, for any smooth X over a perfect field,*

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \cong H^i(X_{\mathrm{et}}, \mathbb{Z}(n))$$

for $i \leq n+1$.

Another way to related motivic cohomology to etale cohomology is the cycle map

$$c_{i,n} : H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \to H^i(X_{\mathrm{et}}, \mathbb{Z}(n)) \to \lim_r H^i(X_{\mathrm{et}}, \mathbb{Z}/l^r(n)).$$

Conjecture 19 (Tate [28]) *If X is smooth and proper over a finite field, then*

$$c_{2n,n} : CH^n \otimes \mathbb{Z}_l \cong H^{2n}_{\mathcal{M}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to \lim_r H^{2n}(X_{\mathrm{et}}, \mathbb{Z}/l^r(n)).$$

has finite cokernel.

One can show that Tate's conjecture for X and $n = 1$ is equivalent to the vanishing of the Tate module $T_l \operatorname{Br}(X) := \lim_r {}_{l^r} \operatorname{Br}(X)$ of the Brauer group, and this follows from the above mentioned finiteness conjecture of the Brauer group. More generally:

Proposition 20 *Tate's conjecture for X and n*
$\Leftrightarrow H^{2n+1}(X_{\mathrm{et}}, \mathbb{Z}(n))$ has no torsion divisible subgroup.

In particular, it is implied by the finite generation conjecture of Lichtenbaum.

5 Weil-etale version

It is a fact that "etale cohomology over k" is the same as "etale cohomology over $\bar{k}$ followed by Galois cohomology". More precisely, for $\bar{X} = X \times_k \bar{k}$ the base extension to the algebraic closure of k, we have a quasi-isomorphism

$$R\Gamma(X_{\mathrm{et}}, \mathcal{F}) \cong R\Gamma(\mathrm{Gal}(k), R\Gamma(\bar{X}_{\mathrm{et}}, \mathcal{F})).$$

Hence there is a spectral sequence

$$H^s(\mathrm{Gal}(k), H^t(\bar{X}_{\mathrm{et}}, \mathbb{Z}(n)) \Rightarrow H^{s+t}(X_{\mathrm{et}}, \mathbb{Z}(n)).$$

Over a finite field, we know that etale motivic cohomology is not finitely generated for $i = 2n+2$, for example $H^2(\mathrm{Gal}(\mathbb{F}_q), \mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. But this obstruction, coming from Galois cohomology, is (conjecturally) the only obstruction! If we replace $\mathrm{Gal}(\mathbb{F}_q) = \hat{\mathbb{Z}}$ by the Weil group $G := \langle \varphi \rangle \cong \mathbb{Z} \subseteq \mathrm{Gal}(\mathbb{F}_q)$, then $H^1(G, \mathbb{Z}) = \mathbb{Z}$ and $H^2(G, \mathbb{Z}) = 0$. In fact, $H^0(G, M) = M^G$ and $H^1(G, \mathbb{Z}) = M_G$, and the higher cohomology groups vanish.

Definition 21 (Lichtenbaum [21]) *Weil-etale cohomology is defined to be*

$$H^i_W(X, \mathbb{Z}(n)) := H^i R\Gamma(G, R\Gamma(\bar{X}_{\mathrm{et}}, \mathbb{Z}(n))).$$

The Leray spectral sequence for composition of functors degenerates into short exact sequences

$$0 \to H^{i-1}(\bar{X}_{\mathrm{et}}, \mathbb{Z}(n))_G \to H^i_W(X, \mathbb{Z}(n)) \to H^i(\bar{X}_{\mathrm{et}}, \mathbb{Z}(n))^G \to 0.$$

Theorem 22 (G.[8]) *There are long exact sequences*

$$\begin{aligned} \cdots \to H^i(X_{\mathrm{et}}, \mathbb{Z}(n)) &\to H^i_W(X, \mathbb{Z}(n)) \\ &\to H^{i-1}(X_{\mathrm{et}}, \mathbb{Q}(n)) \to H^{i+1}(X_{\mathrm{et}}, \mathbb{Z}(n)) \to \cdots . \end{aligned}$$

In particular,

$$H^i(X_{\mathrm{et}}, \mathbb{Z}/m(n)) \cong H^i_W(X, \mathbb{Z}/m(n))$$
$$H^i_W(X, \mathbb{Q}(n)) \cong H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \oplus H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}(n))$$

A typical example is the case $n = 0, i = 1$ and $X = \mathbb{F}_q$. In this case the exact sequence

$$H^1((\mathbb{F}_q)_{\mathrm{et}}, \mathbb{Z}) \to H^1_W(\mathbb{F}_q, \mathbb{Z}) \to H^0_{\mathcal{M}}(\mathbb{F}_q, \mathbb{Q}) \to H^2((\mathbb{F}_q)_{\mathrm{et}}, \mathbb{Z})$$

is the sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, and this explains how a $\mathbb{Q}/\mathbb{Z}$ in etale cohomology corresponds to a $\mathbb{Z}$ in Weil-etale cohomology.

Conjecture 23 (refined Lichtenbaum conjecture) *For X smooth and proper over a finite field, $H^i_W(X, \mathbb{Z}(n))$ is finitely generated for all i, n.*

Theorem 24 (G.[8, 9], Kahn [17]) *The refined Lichtenbaum conjecture is equivalent to the conjunction of Tate's conjecture and Beilinson's conjecture that over a finite field, rational and numerical equivalence of cycles agree up to torsion.*

The refined Lichtenbaum conjecture implies Parshin's conjecture, the Beilinson-Soule vanishing conjecture for schemes in characteristic p, and a formula for special values of zeta-functions for varieties over finite fields.

The author was also able to obtain results by applying the method "replace the Galois group by the Weil-group" to other theories, like higher Chow groups [10] and Suslin homology [11] of singular schemes. In each case, one obtains groups which are (conjecturally) finitely generated out of groups which were not finitely generated due to Galois cohomology.

Recently, there is work by Jannsen, Kerz and Shuji Saito dealing with the case $n = d = \dim X$. They can show:

Theorem 25 (Jannsen, Kerz, Saito [16, 19]) *Let $m \in \mathbb{F}_q^\times$ and X smooth and proper over $\mathbb{F}_q$. Then $H^i_{\mathcal{M}}(X, \mathbb{Z}/m(d))$ is finite for all i.*

The case m a power of the characteristic holds under resolution of singularities. This can be combined with the rational case:

Proposition 26 (G. [10]) *Let X be smooth and proper of dimension d over $\mathbb{F}_q$. Then Parshin's conjecture implies that $H^i_{\mathcal{M}}(X, \mathbb{Z}(d))$ is finitely generated for all i.*

The converse also holds in the sense that finite generation of $H^i_{\mathcal{M}}(X, \mathbb{Z}(d))$ implies that $H^i_{\mathcal{M}}(X, \mathbb{Q}(d))$ vanishes for $i \neq 2d$ under the hypothesis of the theorem.

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