

Generators of invariant linear system on tropical curves for finite isometry group

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Abstract. For a tropical curve Γ and a finite subgroup K of the isometry group of Γ , we prove, extending the work by Haase, Musiker and Yu ([6]), that the K -invariant part of the complete linear system associated to a K -invariant effective divisor on Γ is finitely generated.

Key words: tropical curves, invariant linear subsystem.

1. Introduction

Let $R(D)$ denote the set consisting of the constant function of $-\infty$ and rational functions corresponding to the complete linear system $|D|$ for an effective divisor D on a tropical curve Γ , where a tropical curve means a metric graph possibly with unbounded edges. $R(D)$ becomes a tropical semimodule over the tropical semifield $\mathbf{T} := (\mathbf{R} \cup \{-\infty\}, \max, +)$ with two operations:

$$(f \oplus g)(x) := \max\{f(x), g(x)\} \quad \text{and} \quad (a \odot f)(x) := a + f(x),$$

where $f, g \in R(D)$, $a \in \mathbf{R} \cup \{-\infty\}$ and $x \in \Gamma$. The projective space $R(D)/\mathbf{R}$ is naturally identified with the complete linear system $|D|$. Haase, Musiker and Yu showed that $R(D)$ is finitely generated ([6, Theorem 6]). A tropical subsemimodule R' of $R(D)$ corresponds to a linear subspace Λ of $|D|$. This linear subspace Λ is called a linear system associated to R' .

In this paper, we recall some basic facts of tropical curves in Section 2. Then in Section 3, we observe the K -invariant set $R(D)^K$ of $R(D)$ and prove that $R(D)^K$ is actually finitely generated, where K is a finite subgroup of the isometry group $\text{Isom}(\Gamma) := \{\sigma : \Gamma \rightarrow \Gamma \text{ is an isometry.}\}$. Here, a rational function f on Γ is K -invariant if for any $\sigma \in K$ and $x \in \Gamma$, $(f \circ \sigma)(x) = f(x)$ holds. Our proof is basically analogous to that of [6], but

it is not exactly the same, *i.e.* the K -invariant set S^K of the generator set S of $R(D)$ defined in [6, Lemma 6] is not a generator set of $R(D)^K$. We find such a set corresponding to S , which we call $S(D)_K$. The condition defining $S(D)_K$ is tangibly given from geometric information. Also, the construction of a harmonic morphism with degree $|K|$ from Γ to the quotient tropical curve Γ' of Γ by K precedes. We follow Chan's natural construction ([4]) with a little bit of adaptation. Finally, using the harmonic morphism, we prove that $R(D)^K$ is finitely generated as a tropical semimodule in our main theorem (Theorem 3.15). When D is K -invariant, we can identify $R(D)^K/\mathbf{R}$ with the K -invariant linear subsystem $|D|^K$ and then $|D|^K$ is finitely generated by $S(D)_K/\mathbf{R}$.

2. Preliminaries

In this section, we briefly recall the theories of tropical curves ([10]), divisors on tropical curves ([10]), harmonic morphisms of tropical curves ([4], [6], [9]), and chip-firing moves on tropical curves ([1], [3], [5], [6], [8], [11]), which we need later.

2.1. Tropical curves

In this paper, a *graph* means an unweighted, undirected finite connected nonempty multigraph. Note that we allow the existence of loops. For a graph G , the sets of vertices and edges are denoted by $V(G)$ and $E(G)$, respectively. The *valence* $\text{val}(v)$ of a vertex v of G is the number of edges emanating from v , where we count each loop with vertex v as two. A vertex v of G is a *leaf end* if v has valence one. A *leaf edge* is an edge of G having a leaf end.

An *edge-weighted graph* (G, l) is the pair of a graph G and a function $l : E(G) \rightarrow \mathbf{R}_{>0} \cup \{\infty\}$ called a *length function*, where l can take the value ∞ only on leaf edges. A *tropical curve* is the underlying topological space of an edge-weighted graph (G, l) together with an identification of each edge e of G with the interval $[0, l(e)]$. When $l(e) = \infty$, the interval $[0, \infty]$ is the one point compactification of the interval $[0, \infty)$ and the leaf end of e must be identified with ∞ . We regard this not just as a topological space but as almost a metric space. The distance from any point to ∞ is infinite. If $E(G) = \{e\}$ and $l(e) = \infty$, then we can identify either leaf ends of e with ∞ . For a point x on a tropical curve Γ obtained from (G, l) , if x is identified with ∞ , then x is called a *point at infinity*, else, x is said

to be a *finite point*. For the above tropical curve Γ , (G, l) is said to be its *model*. There are many possible models for Γ . We construct a model (G_\circ, l_\circ) called the *canonical model* of Γ as follows. Generally, we determine $V(G_\circ) := \{x \in \Gamma \mid \text{val}(x) \neq 2\}$, where for a point x in Γ , if x is not a point at infinity, then the *valence* $\text{val}(x)$ is the number of connected components of $U \setminus \{x\}$ with any sufficiently small connected neighborhood U of x , if x is a point at infinity, then $\text{val}(x) := 1$ except following two cases. Remark that this “valence” is defined for a point on a tropical curve and the “valence” in the first paragraph of this subsection is defined for a vertex of a graph and these are compatible to each other. When Γ is a circle, we determine $V(G_\circ)$ as the set consisting of one arbitrary point on Γ . When Γ is the tropical curve obtained from the graph consisting only of two edges with length of ∞ and three vertices adjacent to these edges (see Figure 2), $V(G_\circ)$ consists of the two endpoints of Γ (those are points at infinity) and any point on Γ other than two endpoints as the origin. Because connected components of $\Gamma \setminus V(G_\circ)$ consist of open intervals, then these lengths determine the length function l_\circ . If a model (G, l) of Γ has no loops, then (G, l) is said to be a *loopless model* of Γ . For a model (G, l) of Γ , the loopless model for (G, l) is obtained by regarding all midpoints of loops of G as vertices and by adding them to the set of vertices of G . The loopless model for the canonical model of a tropical curve is called the *canonical loopless model*.

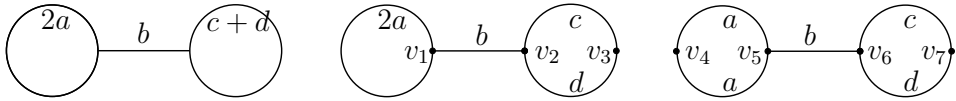


Figure 1. Examples of a tropical curve (the left figure) and its models (the center and the right figures). On each figure, each alphabet a, \dots, d means the length of the corresponding edge and v_1, \dots, v_7 mean vertices of the corresponding graph, respectively.

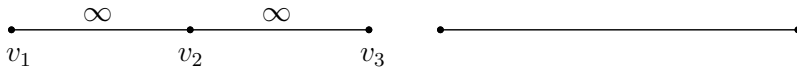


Figure 2. The left figure is a graph consisting only of two edges with length of ∞ and three vertices adjacent to these edges and the right figure is the tropical curve obtained from it. $\{v_1, v_2, v_3\}$ is the set of vertices of the left graph and the letters ∞ on the edges v_1v_2 and v_2v_3 mean that the lengths of them are ∞ . On the right tropical curve, the two points (two black dots on the figure) corresponding to v_1 and v_3 are points at infinity.

For terminology, in a tropical curve Γ , an edge of Γ means an edge of the underlying graph G_\circ of the canonical model (G_\circ, l_\circ) . Let e be an edge of Γ which is not a loop. We regard e as a closed subset of Γ , *i.e.*, including the endpoints v_1, v_2 of e . The *relative interior* of e is $e^\circ = e \setminus \{v_1, v_2\}$. For a point x on Γ , a *half-edge* of x is a connected component of $U \setminus \{x\}$ with any connected neighborhood U of x which consists only of two valent points and x .

For a model (G, l) of a tropical curve Γ , we frequently identify a vertex v (resp. an edge e) of G with the point corresponding to v on Γ (resp. the closed subset corresponding to e of Γ).

2.2. Divisors on tropical curves

Let Γ be a tropical curve. An element of the free abelian group $\text{Div}(\Gamma)$ generated by points on Γ is called a *divisor* on Γ . For a divisor D on Γ , its *degree* $\deg(D)$ is defined by the sum of the coefficients over all points on Γ . We write the coefficient at x as $D(x)$. A divisor D on Γ is said to be *effective* if $D(x) \geq 0$ for any x in Γ . If D is effective, we write simply $D \geq 0$. The set of points on Γ where the coefficient(s) of D is not zero is called the *support* of D and written as $\text{supp}(D)$.

A continuous map $f : \Gamma \rightarrow \mathbf{R} \cup \{\pm\infty\}$ is a *rational function* on Γ if f is a constant function of $-\infty$ or a piecewise linear function with integer slopes, with a finite number of pieces and that can take the value $\pm\infty$ only at points at infinity. $\text{Rat}(\Gamma)$ denotes the set of rational functions on Γ . For a point x on Γ and f in $\text{Rat}(\Gamma)$ which is not constant $-\infty$, the sum of the outgoing slopes of f at x is denoted by $\text{ord}_x(f)$. Note that a constant ∞ function is not a rational function on Γ . If x is a point at infinity and f is not finite at there, we define $\text{ord}_x(f)$ as the outgoing slope from the interval $[z, x] \subset e$, where e is the edge of Γ adjacent to x and z is a point on e such that f is linear on $[y, x] \subset e$, y is not x and $z \in (y, x)$. Note when Γ is a singleton, for any f in $\text{Rat}(\Gamma)$, we define $\text{ord}_x(f) := 0$. This sum is 0 for all but finite number of points on Γ , and thus

$$\text{div}(f) := \sum_{x \in \Gamma} \text{ord}_x(f) \cdot x$$

is a divisor on Γ , which is called a *principal divisor*. Two divisors D and E on Γ are said to be *linearly equivalent* if $D - E$ is a principal divisor. We handle the values ∞ and $-\infty$ as follows: let f, g in $\text{Rat}(\Gamma)$ take the value ∞

and $-\infty$ at a point x at infinity on Γ respectively. When $\text{ord}_x(f) + \text{ord}_x(g)$ is negative, then $(f \odot g)(x) := \infty$ (see Introduction about the definition of \odot). When $\text{ord}_x(f) + \text{ord}_x(g)$ is positive, then $(f \odot g)(x) := -\infty$. Remark that the constant function of $-\infty$ on Γ does not determine a principal divisor. For a divisor D on Γ , the *complete linear system* $|D|$ is defined by the set of effective divisors on Γ being linearly equivalent to D .

Let us recall that the set $\mathbf{T} := \mathbf{R} \cup \{-\infty\}$ with two tropical operations:

$$a \oplus b := \max\{a, b\} \quad \text{and} \quad a \odot b := a + b,$$

where both a and b are in \mathbf{T} , becomes a semifield. Here, for any $a \in \mathbf{T}$, we handle $-\infty$ as follows:

$$\begin{aligned} a \oplus (-\infty) &= a, & (-\infty) \oplus a &= a, \\ a \odot (-\infty) &= -\infty, & (-\infty) \odot a &= -\infty. \end{aligned}$$

\mathbf{T} is called the *tropical semifield*. For a divisor D on a tropical curve, let $R(D)$ be the set of the constant function of $-\infty$ and rational functions $\neq -\infty$ such that $D + \text{div}(f)$ is effective. When $\deg(D)$ is negative, $|D|$ is empty, so is $R(D)$. Otherwise, by Theorem 10 in [7], $|D|$ is not empty and consequently so is $R(D)$. Hereafter, we treat only divisors of nonnegative degree.

Lemma 2.2.1 (cf. [6, Lemma 4]) *$R(D)$ becomes a tropical semimodule over \mathbf{T} by extending above tropical operations onto functions, giving point-wise sum and product.*

By the definition of $\text{ord}_x(f)$ for a point x at infinity and f in $\text{Rat}(\Gamma)$, we can prove Lemma 2.2.1 in the same way of [6, Lemma 4].

For a tropical subsemimodule M of $(\mathbf{R} \cup \{\pm\infty\})^\Gamma$, f in M is called an *extremal of M* when it implies $f = g_1$ or $f = g_2$ for any g_1 and g_2 in M satisfying $f = g_1 \oplus g_2$. Similarly, for a tropical subsemimodule M of \mathbf{R}^Γ , f in M is called an *extremal of M* when it implies $f = g_1$ or $f = g_2$ for any g_1 and g_2 in M satisfying $f = g_1 \oplus g_2$.

Remark 2.2.2 ([6, Proposition 8]) *Any finitely generated tropical subsemimodule M of \mathbf{R}^Γ is generated by the extremals of M .*

With the adaptation for $\pm\infty$, we can prove the following lemma almost in the same way as the above remark.

Lemma 2.2.3 *Any finitely generated tropical subsemimodule M of $R(D) \subset (\mathbf{R} \cup \{\pm\infty\})^{\Gamma}$ is generated by the extremals of M .*

Proof. Let $\{f_1, f_2, \dots, f_n\}$ be a generator set of M . Suppose that f_n is not an extremal. Then $f_n = g \oplus h$ for some $g, h \in M$ which are other than f_n . Since f_1, \dots, f_n generate M , we have

$$\begin{aligned} g &= (a_1 \odot f_1) \oplus \cdots \oplus (a_{n-1} \odot f_{n-1}) \oplus (a_n \odot f_n) \quad \text{and} \\ h &= (b_1 \odot f_1) \oplus \cdots \oplus (b_{n-1} \odot f_{n-1}) \oplus (b_n \odot f_n) \end{aligned}$$

for some $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{T}$. Since $g \leq f_n$ and $h \leq f_n$ pointwise, and $g \neq f_n$ and $h \neq f_n$, we must have $a_n < 0$ and $b_n < 0$. If there is a point x at infinity where $f_n(x) = \infty$ and where all other functions f_i have finite value or $-\infty$, then we can choose a point y on the edge e adjacent to x such that on the interval $[y, x] \subset e$, all functions f_1, \dots, f_n are linear. Then, since f_1, \dots, f_{n-1} must be constant on $[y, x]$ or decrement from y to x , $f_n|_{[y,x]} = (g \oplus h)|_{[y,x]} = \{(a_n \odot f_n) \oplus (b_n \odot f_n)\}|_{[y,x]}$, where $|_{[y,x]}$ means the restrictions of these functions on $[y, x]$. However, it contradicts to $a_n < 0$ and $b_n < 0$. Then

$$f_n = g \oplus h = (a_1 \odot f_1) \oplus \cdots \oplus (a_{n-1} \odot f_{n-1}) \oplus (b_1 \odot f_1) \oplus \cdots \oplus (b_{n-1} \odot f_{n-1}),$$

therefore f_n is in the tropical semimodule generated by f_1, \dots, f_{n-1} . We can remove non-extremals from any finite generator set by this way, so M is generated by the extremals. \square

2.3. Harmonic morphisms

Let Γ, Γ' be tropical curves, respectively, and $\varphi : \Gamma \rightarrow \Gamma'$ be a continuous map. The map φ is a *morphism* if there exist a model (G, l) of Γ and a model (G', l') of Γ' satisfying the following three conditions:

- (1) the image of the set of vertices of G by φ is a subset of the set of vertices of G' ,
- (2) the inverse image of the relative interior of any edge of G' by φ is the union of the relative interiors of a finite number of edges of G , and
- (3) for any edge e of G and any finite points x, y on e , the distance between $\varphi(x)$ and $\varphi(y)$ is a multiple of that of x and y by some nonnegative integer $\deg_e(\varphi)$ depending only on e .

If an edge e is mapped to a vertex of G' by φ , then $\deg_e(\varphi) = 0$. The morphism φ is said to be *finite* if $\deg_e(\varphi) > 0$ for any edge e of G . For any half-edge h of any point on Γ , we define $\deg_h(\varphi)$ as $\deg_e(\varphi)$, where e is the edge of G containing h .

Let Γ' be not a singleton and x a point on Γ . The morphism φ is *harmonic at x* if the number

$$\deg_x(\varphi) := \sum_{x \in h \mapsto h'} \deg_h(\varphi)$$

is independent of the choice of half-edge h' emanating from $\varphi(x)$, where h is a connected component of the inverse image of h' by φ emanating from x . The morphism φ is *harmonic* if it is harmonic at all points on Γ . If φ is a finite harmonic morphism, then the number

$$\deg(\varphi) := \sum_{x \mapsto x'} \deg_x(\varphi)$$

is independent of the choice of a point x' on Γ' , and is said the *degree* of φ , where x is an element of the inverse image of x' by φ . It is sufficient to check that for any vertex of G' , which is the graph of a loopless model of Γ' , the sum is same. Let x'_1 and x'_2 be vertices of G' both adjacent to an edge e' of G' . Let h'_1 be the half-edge of x'_1 contained in e' . Then

$$\begin{aligned} \sum_{x_1 \mapsto x'_1} \deg_{x_1}(\varphi) &= \sum_{x_1 \mapsto x'_1} \left(\sum_{x_1 \in h_1 \mapsto h'_1} \deg_{h_1}(\varphi) \right) \\ &= \sum_{x_1 \mapsto x'_1} \left(\sum_{x_1 \in e_1 \mapsto e'} \deg_{e_1}(\varphi) \right) \\ &= \sum_{e \mapsto e'} \deg_e(\varphi). \end{aligned}$$

Similarly,

$$\sum_{x_2 \mapsto x'_2} \deg_{x_2}(\varphi) = \sum_{e \mapsto e'} \deg_e(\varphi).$$

If Γ' is a singleton and Γ is not a singleton, for any point x on Γ , we define

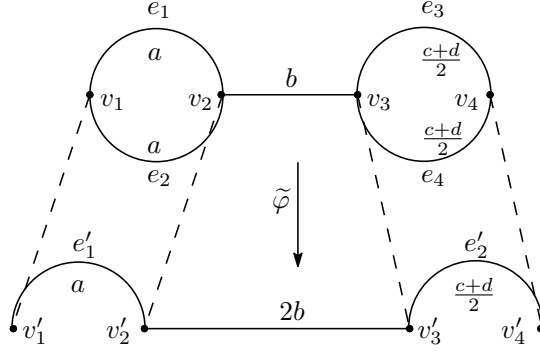


Figure 3. $\tilde{\varphi}$ is a finite harmonic morphism of tropical curves with degree two. v_i, v'_i (resp. e_i, e'_i) are the vertices (resp. edges) of the metric graphs, respectively. Each alphabet $a, b, 2b, (c+d)/2$ means the length of the corresponding edge, respectively. Each v_i is mapped to v'_i , and two edges e_1 and e_2 are mapped to e'_1 , and two edges e_3 and e_4 are mapped to e'_2 by φ .

$\deg_x(\varphi)$ as zero so that we regard φ as a harmonic morphism of degree zero. If both Γ and Γ' are singletons, we regard φ as a harmonic morphism which can have any number of degree.

Let $\varphi : \Gamma \rightarrow \Gamma'$ be a finite harmonic morphism between tropical curves. For f in $\text{Rat}(\Gamma)$, the *push-forward* of f is the function $\varphi_* f : \Gamma' \rightarrow \mathbf{R} \cup \{\pm\infty\}$ defined by

$$\varphi_* f(x') := \sum_{\substack{x \in \Gamma \\ \varphi(x) = x'}} \deg_x(\varphi) \cdot f(x).$$

Here, when there exist two points $x_1, x_2 \in \varphi^{-1}(x')$ such that $f(x_1) = \infty$ and $f(x_2) = -\infty$, we define as follows. We choose a point y' on the edge e' adjacent to x' such that f is linear on all connected components of $\varphi^{-1}([y', x'])$. Since each $y \in \varphi^{-1}(y')$ is not a point at infinity, the value $\varphi_* f(y')$ is finite. As $y' \rightarrow x'$, if $\varphi_* f(y') \rightarrow \infty$, then we set $\varphi_* f(x') := \infty$. If $\varphi_* f(y') \rightarrow -\infty$, then $\varphi_* f(x') := -\infty$. If $\varphi_* f(y') \rightarrow a \in \mathbf{R}$, then $\varphi_* f(x') := a$. This definition is clearly independent of the choice of y' . The *pull-back* of f' in $\text{Rat}(\Gamma')$ is the function $\varphi^* f' : \Gamma \rightarrow \mathbf{R} \cup \{\pm\infty\}$ defined by $\varphi^* f' := f' \circ \varphi$. We define the *push-forward* $\varphi_* : \text{Div}(\Gamma) \rightarrow \text{Div}(\Gamma')$ on divisors by

$$\varphi_*(D) := \sum_{x \in \Gamma} D(x) \cdot \varphi(x).$$

$\deg(\varphi_*(D)) = \deg(D)$ and $\varphi_*(\operatorname{div}(f)) = \operatorname{div}(\varphi_*f)$ hold for any divisor D on Γ and any f in $\operatorname{Rat}(\Gamma) \setminus \{-\infty\}$ (cf. [2, Proposition 4.2]). In fact, since $(\varphi_*(D))(x') = \sum_{x \in \varphi^{-1}(x')} D(x)$ for any $x' \in \Gamma'$,

$$\begin{aligned} \deg(\varphi_*(D)) &= \sum_{x' \in \Gamma'} (\varphi_*(D))(x') = \sum_{x' \in \Gamma'} \left(\sum_{x \in \varphi^{-1}(x')} D(x) \right) \\ &= \sum_{x \in \Gamma} D(x) = \deg(D). \end{aligned}$$

For any $f \in \operatorname{Rat}(\Gamma) \setminus \{-\infty\}$, we may break Γ and Γ' into sets S and S' of segments along which f and φ_*f , respectively, are linear as follows: each segment $s \in S$ is mapped linearly to some $s' \in \Gamma'$. Then at any point x' on Γ' other than a point at infinity, we have

$$\varphi_*(\operatorname{div}(f))(x') = \sum_{x \in \varphi^{-1}(x')} \operatorname{div}(f)(x) = \sum_{x \in \varphi^{-1}(x')} \left(\sum_{s=xy \in S} \frac{f(y) - f(x)}{l(s)} \right)$$

and

$$\begin{aligned} &\operatorname{div}(\varphi_*f)(x') \\ &= \sum_{s'=x'y' \in S'} \frac{(\varphi_*f)(y') - (\varphi_*f)(x')}{l'(s')} \\ &= \sum_{s'=x'y' \in S'} \frac{1}{l'(s')} \\ &\quad \times \left\{ \sum_{y \in \varphi^{-1}(y')} \left(\sum_{y \in s \rightarrow s'} \frac{l'(s')}{l(s)} \right) f(y) - \sum_{x \in \varphi^{-1}(x')} \left(\sum_{x \in s \rightarrow s'} \frac{l'(s')}{l(s)} \right) f(x) \right\} \\ &= \sum_{s'=x'y' \in S'} \left\{ \sum_{y \in \varphi^{-1}(y')} \left(\sum_{y \in s \rightarrow s'} \frac{1}{l(s)} \right) f(y) - \sum_{x \in \varphi^{-1}(x')} \left(\sum_{x \in s \rightarrow s'} \frac{1}{l(s)} \right) f(x) \right\} \\ &= \sum_{s'=x'y' \in S'} \left\{ \sum_{\substack{s=xy \in S \\ \varphi(s)=s'}} \left(\frac{f(y)}{l(s)} - \frac{f(x)}{l(s)} \right) \right\} \end{aligned}$$

$$= \sum_{x \in \varphi^{-1}(x')} \left(\sum_{s=xy \in S} \frac{f(y) - f(x)}{l(s)} \right).$$

Here, $l(s)$ and $l'(s')$ mean the lengths of s and s' , respectively. For a point z' at infinity on Γ' , there exists a unique segment $s' \in S'$ adjacent to z' . As f and $\varphi_* f$ are linear on any segment $s \in \varphi_*^{-1}(s')$ and s' , respectively, we may choose any point $w' \in (s')^\circ$ and

$$\begin{aligned} \varphi_*(\operatorname{div}(f))(z') &= \sum_{z \in \varphi^{-1}(z')} \operatorname{div}(f)(z) = \sum_{z \in \varphi^{-1}(z')} \operatorname{ord}_z(f) \\ &= \sum_{w \in \varphi^{-1}(w')} \operatorname{ord}_w(g) = \sum_{w \in \varphi^{-1}(w')} \operatorname{div}(g)(w) = \varphi_*(\operatorname{div}(g))(w'), \end{aligned}$$

where g is the rational function on Γ defined by $g(x) := f(x)$ if $x \in \Gamma$ is not on $\varphi^{-1}([w', z'])$, and $g(x) := f(w)$ if x is on the segment $[w, z] \subset \varphi^{-1}([w', z'])$. Since $\deg_z(\varphi) = \deg_w(\varphi)$, by the definition of g , $\operatorname{div}(\varphi_* f)(z') = \operatorname{div}(\varphi_* g)(w')$. Therefore, we have

$$\varphi_*(\operatorname{div}(f))(z') = \varphi_*(\operatorname{div}(g))(w') = \operatorname{div}(\varphi_* g)(w') = \operatorname{div}(\varphi_* f)(z').$$

In conclusion, $\varphi_*(\operatorname{div}(f)) = \operatorname{div}(\varphi_* f)$ holds.

2.4. Chip firing moves

In [6], Haase, Musiker and Yu used the term *subgraph* of a tropical curve as a compact subset of the tropical curve with a finite number of connected components and defined the *chip firing move* $\operatorname{CF}(\Gamma_1, l)$ by a subgraph Γ_1 of a tropical curve Γ and a positive real number l as the rational function $\operatorname{CF}(\Gamma_1, l)(x) := -\min(l, \operatorname{dist}(x, \Gamma_1))$, where $\operatorname{dist}(x, \Gamma_1)$ is the infimum of the lengths of the shortest path to arbitrary points on Γ_1 from x . They proved that every rational function on a tropical curve is an (ordinary) sum of chip firing moves (plus a constant) ([6, Lemma 2]) with the concept of a *weighted chip firing move*. This is a rational function on a tropical curve having two disjoint proper subgraphs Γ_1 and Γ_2 such that the complement of the union of Γ_1 and Γ_2 in Γ consists only of open line segments and such that the rational function is constant on Γ_1 and Γ_2 and linear (smooth) with integer slopes on the complement. A weighted chip firing move is an (ordinary) sum of chip firing moves (plus a constant) ([6, Lemma 1]).

With unbounded edges, their definition of chip firing moves needs a little correction. Let Γ_1 be a subgraph of a tropical curve Γ which does not have any connected components consisting only of points at infinity and l a positive real number or infinity. The *chip firing move* by Γ_1 and l is defined as the rational function $\text{CF}(\Gamma_1, l)(x) := -\min(l, \text{dist}(x, \Gamma_1))$. Then we can define a *weighted chip firing move* on a tropical curve possibly with unbounded edges as the same way in the first paragraph of this subsection.

Lemma 2.4.1 *A weighted chip firing move on a tropical curve is a linear combination of chip firing moves having integer coefficients (plus a constant).*

Proof. We use the same notations as in the proof of Lemma 1 in [6]. Let f be a weighted chip firing move, Γ_1 and Γ_2 the subgraphs corresponding to f and L_i 's open line segments between Γ_1 and Γ_2 . Suppose $f(\Gamma_2) - f(\Gamma_1) =: l > 0$. All we have to do is to show the construction for the case with $l = \infty$ since when l is finite, the statement is clear by the proof of [6, Lemma 1]. As $l = \infty$, at least one of Γ_1 and Γ_2 consists only of points at infinity. When the one is Γ_2 , by thinking $-f$ instead of f , it suffices to check the case that Γ_1 consists only of points at infinity.

Supposing that Γ_1 has only one point gives only two situations. Firstly, Γ_2 contains a finite point. Then f can be written as $s \cdot \text{CF}(\Gamma_2, \infty)$ plus a constant, where s is the slope of f in the direction from Γ_1 to Γ_2 on the complement. Secondly, Γ_2 consists only of one point at infinity. In this case, Γ is of the type of Figure 2. Taking a finite point x , then f can be written as $s \cdot (\text{CF}(f^{-1}([f(x), \infty]), \infty) - \text{CF}(f^{-1}([-\infty, f(x)]), \infty))$ plus a constant with same s as the first situation.

Suppose that Γ_1 has plural points. Γ_2 must contain at least one finite point. Let x_i be the intersection of Γ_1 and the closure of L_i . Note that $\Gamma_1 = \{x_1, \dots, x_k\}$, where k is no less than two. With the slope s_i of f on $e_i := L_i \sqcup \{x_i\}$ in the direction from Γ_1 to Γ_2 , f is $\sum_{i=1}^k (s_i \cdot \text{CF}(\Gamma \setminus e_i, \infty))$ plus a constant. \square

The next lemma is proven in the same way of [6, Lemma 2] and shows the appropriateness of this definition.

Lemma 2.4.2 *Every rational function on a tropical curve is a linear combination of chip firing moves having integer coefficients (plus a constant).*

A point on Γ with valence two is said to be a *smooth point*. We sometimes refer to an effective divisor D on Γ as a *chip configuration*, i.e. for a point x on Γ , $D(x)$ means the number of chips at x . Here, we can think of chips as any small discrete object. We say that a subgraph Γ_1 of Γ can *fire on D* if for each boundary point of Γ_1 there are at least as many chips as the number of edges pointing out of Γ_1 . A set of points on a tropical curve Γ is said to be *cut set* of Γ if the complement of that set in Γ is disconnected. Note that a cut set need not to be a (sub)graph and the complement also need not to be a graph.

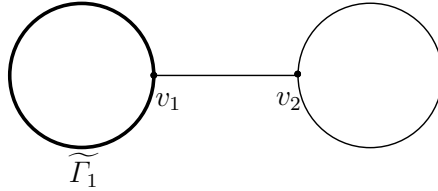


Figure 4. Let $\widetilde{\Gamma}$ be the tropical curve of the figure and $\widetilde{\Gamma}_1$ the left circle of $\widetilde{\Gamma}$. $\widetilde{\Gamma}_1$ can fire on v_1 but $\widetilde{\Gamma} \setminus \widetilde{\Gamma}_1$ cannot fire on it. Both $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma} \setminus \widetilde{\Gamma}_1$ can fire on $2v_1$. The set of vertices of the canonical model of $\widetilde{\Gamma}$ is $\{v_1, v_2\}$. Let l be the distance between v_1 and v_2 . Then $v_1 + \text{div}(\text{CF}(\widetilde{\Gamma}_1, l)) = v_2$.

3. Generators of $R(D)^K$

In this section, for an effective divisor D on a tropical curve Γ and a finite subgroup K of the isometry group $\text{Isom}(\Gamma) := \{\sigma : \Gamma \rightarrow \Gamma \mid \sigma \text{ is an isometry.}\}$ of Γ , we find a generator set of the K -invariant set $R(D)^K$ of $R(D)$ and then, show that $R(D)^K$ is finitely generated as a tropical semi-module. Here, a rational function f on Γ is K -invariant if for any $\sigma \in K$ and $x \in \Gamma$, $(f \circ \sigma)(x) = f(x)$ holds. When D is K -invariant, $R(D)/\mathbf{R}$ is identified with the K -invariant linear system $|D|^K$, so $|D|^K$ is finitely generated by the generators of $R(D)^K$ modulo tropical scaling (except by $-\infty$).

Lemma 3.1 ([6, Lemma 6]) *Let Γ be a tropical curve, D be a divisor on Γ and S be the set of rational functions f in $R(D)$ such that the support of $D + \text{div}(f)$ does not contain any cut set of Γ consisting only of smooth points. Then*

- (1) S contains all the extremals of $R(D)$,
- (2) S is finite modulo tropical scaling (except by $-\infty$), and

(3) S generates $R(D)$ as a tropical semimodule.

Theorem 3.2 ([6, Theorem 14]) *Let (G, l) be a model of Γ and let S_G be the set of functions $f \in R(D)$ such that the support of $D + \text{div}(f)$ does not contain an interior cut set (i.e. a cut set consisting of points in interior of edges in the graph G). Then*

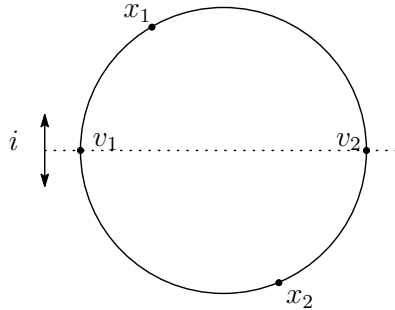
- (1) S_G contains the set S from Remark 3.1, and
- (2) S_G is finite modulo tropical scaling (except by $-\infty$).

Though in the above lemma and theorem they assume that $R(D)$ is a subset of \mathbf{R}^Γ , the proofs are applied even in the case that $R(D)$ is a subset of $(\mathbf{R} \cup \{\pm\infty\})^\Gamma$ with preparations in Section 2. Also, the above lemma (resp. theorem) throws the relation between S (resp. S_G) and D into relief, hence hereafter we write S (resp. S_G) for D as $S(D)$ (resp. $S_G(D)$). Note that a tropical subsemimodule of $R(D)$ is not always finitely generated. Consider the tropical subsemimodule R' of $R(0)$ corresponding to $|0| \setminus \{0\}$ on a tropical curve $[0, 1]$. By Theorem 3.1, $R(0)$ is finitely generated. On the other hand, for any point $x \in (0, 1]$, R' contains all rational functions which are constant on $[x, 1]$ and have slope minus one in the direction from 0 to x on $[0, x]$. Since such rational functions corresponding to x are not contained in the tropical subsemimodule generated by all such rational functions corresponding to any point $y \in (0, 1] \setminus \{x\}$, R' is not finitely generated.

Let Γ be a tropical curve, D an effective divisor on Γ and K a finite subgroup of the isometry group of Γ . One can expect the relation between $R(D)$ and $S(D)$ to be analogous to that of their K -invariant counterparts $R(D)^K$ and $S(D)^K$, but in vain. Indeed, the next example objects.

Example 3.3 Let $\tilde{\Gamma}$ be a circle and let a map $i : \tilde{G}_0 \rightarrow \tilde{G}_0$ which transfers two edges to each other, where \tilde{G}_0 is the underlying graph of the canonical loopless model of $\tilde{\Gamma}$. For a point x_1 on $\tilde{\Gamma}$, we choose another point x_2 on $\tilde{\Gamma}$ such that $i(x_1) \neq x_2$. For the group \tilde{K} generated by i and the effective divisor $\tilde{D} = x_1 + x_2$, although $S(\tilde{D})^{\tilde{K}}$ is empty, $R(\tilde{D})^{\tilde{K}}$ is not empty since $R(\tilde{D})^{\tilde{K}}$ contains all constant functions, and in fact, it consists only of constant functions. It means that $S(\tilde{D})^{\tilde{K}}$ is not a generator set of $R(\tilde{D})^{\tilde{K}}$. (See the below figure. Here, v_1 and v_2 are the vertices of \tilde{G}_0 .)

Now, let us find a generator set for $R(D)^K$ that corresponds to $S(D)$ for $R(D)$. In the above situation, K acts on Γ naturally. We define $V_1(K)$ as



the set of points x on Γ such that there exists a point y in any neighborhood of x whose stabilizer of K is not equal to that of x .

Lemma 3.4 $V_1(K)$ is a finite set.

Proof. We assume that Γ is not the tropical curve obtained from the graph consisting only of two edges with length of ∞ and three vertices adjacent to these edges, *i.e.* Γ is not of the type of Figure 2. Let $\sigma : \Gamma \rightarrow \Gamma$ be an isometry. Then, for any edge e of Γ , the image $\sigma(e)$ agrees completely with e or the intersection $e \cap \sigma(e)$ is contained in the set of the endpoints of e . In fact, if $|e \cap \sigma(e)|$ is infinite, then $\sigma(e)$ is contained in e because e is an edge of Γ . It means that $\sigma(e) = e$. If $|e \cap \sigma(e)|$ is finite and $e \cap \sigma(e)$ contains a point on Γ other than endpoints of e , then that point has the valence of greater than two. It contradicts to the fact that e is an edge of Γ .

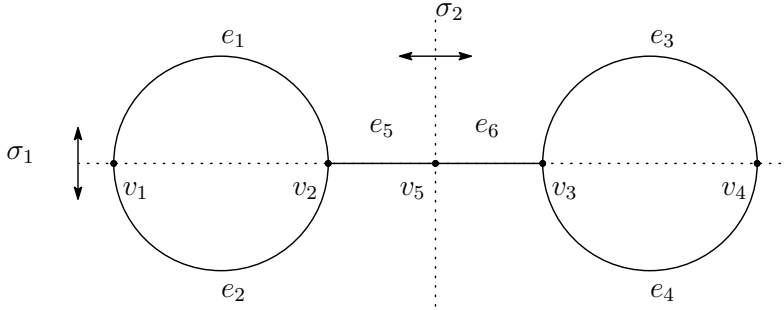
From the above argument, for any edge e of Γ , we can roughly classify the situations into four. First, σ is the identity map on e , *i.e.*, σ fixes all points on e . Second, σ gives a mirror image of e . In this case, if Γ is a circle consisting of e , the fixed points on e by σ are only antipodal points on the axis of symmetry of σ , otherwise, the midpoint of e is fixed by σ , moreover when e is a loop, then the vertex connected to e is also fixed by σ . Third, σ acts as a proper rotation on e . This is possible only when Γ is a circle, and σ gives no fixed points on e . Finally, σ maps e onto other edge of Γ , then only the endpoints of e may be fixed by σ .

Consequently, under the above assumption, since K is a finite set and Γ has finite vertices and edges, $V_1(K)$ is a finite set.

Let us suppose that Γ is the tropical curve obtained from the graph consisting only of two edges with length of ∞ and three vertices adjacent to these edges, *i.e.* Γ is of the type of Figure 2. Since K is a finite set,

any σ in K is not a proper translation of Γ . Each isometry of Γ other than translations fixes only one point on Γ . Thus, also in this case, $V_1(K)$ is a finite set. Note that there can exist only one inversion. If there were two distinct, these two can generate a translation, leading $|K|$ to infinity. \square

Example 3.5 Let $\tilde{\Gamma}$ be the tropical curve of the following figure whose loops have a same length. $\{v_1, \dots, v_4\}$ is the set of vertices of the canonical loopless model of $\tilde{\Gamma}$. v_5 is the midpoint of the edge v_2v_3 , *i.e.* the lengths of e_5 and e_6 are same. σ_1 is the isometry on $\tilde{\Gamma}$ which maps each v_i to v_i itself, e_1 to e_2 , e_3 to e_4 , e_5 to e_5 itself and e_6 to e_6 itself. σ_2 is the isometry on $\tilde{\Gamma}$ which maps v_1 to v_4 , v_2 to v_3 , e_1 to e_3 , e_2 to e_4 and e_5 to e_6 . Let \tilde{K} be the finite isometry group on $\tilde{\Gamma}$ generated by σ_1 and σ_2 . Then, $V_1(\tilde{K})$ consists of v_1, \dots, v_5 .



We set (G_0, l_0) as the canonical loopless model of Γ . By Lemma 3.4, we obtain the model (G_1, l_1) of Γ by setting the union of $V(G_0)$ and $V_1(K)$ as the set $V(G_1)$ of vertices. Naturally, we can regard that K acts on $V(G_1)$ and also on $E(G_1)$. Thus, the sets $V(G'_1)$ and $E(G'_1)$ are defined as the quotient sets of $V(G_1)$ and $E(G_1)$ by K , respectively. Let G'_1 be the graph obtained by setting $V(G'_1)$ as the set of vertices and $E(G'_1)$ as the set of edges. Since G_1 is connected, G'_1 is also connected. We obtain the loopless graph G' from G'_1 and the loopless model (G_2, l_2) of Γ from the inverse image of $V(G')$ by the map defined by K . Note that $V(G_2)$ contains $V(G_1)$. Since K is a finite subgroup of the isometry group of Γ , the length function $l' : E(G') \rightarrow \mathbf{R}_{>0} \cup \{\infty\}$, $[e] \mapsto |K_e| \cdot l_2(e)$ is well-defined, where $[e]$ and K_e mean the equivalence class of e and the stabilizer of e , respectively. Let Γ' be the tropical curve obtained from (G', l') . Then, Γ' is the quotient tropical curve of Γ by K .

For any edge e of G_2 , by the orbit-stabilizer theorem and Lagrange's theorem (cf. [13, Chapter 6]), $|K_e|$ is a positive integer. Here, the orbit-stabilizer theorem and Lagrange's theorem are as follows:

Theorem 3.6 (the orbit-stabilizer theorem) *Let G be a finite group and x an element of G . Then there is a bijection between G/G_x and G_x such that hG_x corresponds to hx .*

Theorem 3.7 (Lagrange's theorem) *Let G be a finite group and H a subgroup of G . Then $|G| = [G : H]|H|$.*

Thus, for (G_2, l_2) and (G', l') , there exists only one morphism $\varphi : \Gamma \rightarrow \Gamma'$ that satisfies $\deg_e(\varphi) = |K_e|$ for any edge e of G_2 .

We obtain the following lemma as an extension of [4, Lemma 2.2].

Lemma 3.8 *φ is a finite harmonic morphism of degree $|K|$.*

Proof. Clearly, φ is finite. Now we check that φ is harmonic and its degree is $|K|$. If both Γ and Γ' are singletons, then the statement holds. Assume that both Γ and Γ' are not singletons. Since K is a finite subgroup of the isometry group of Γ , for any point x' on Γ' and any half-edge h' of x' , each half-edge whose image by φ is h' has the same dilation factor $\deg_h(\varphi)$, where h is a half-edge of any point $x \in \varphi^{-1}(x')$ whose image is h' . Therefore, for the edge e of G_2 containing h and its image $e' := \varphi(e)$ by φ , the following hold:

$$\begin{aligned} \sum_{x \mapsto x'} \deg_x(\varphi) &= \sum_{x \mapsto x'} \left(\sum_{x \in \tilde{h} \mapsto h'} \deg_{\tilde{h}}(\varphi) \right) = \sum_{x \mapsto x'} \left(\sum_{x \in \tilde{h} \mapsto h'} \deg_h(\varphi) \right) \\ &= \sum_{x \mapsto x'} \left(\sum_{x \in \tilde{e} \mapsto e'} \deg_e(\varphi) \right) = \sum_{\tilde{e} \mapsto e'} \deg_e(\varphi) = |Ke| \cdot |K_e| = |K|, \end{aligned}$$

where \tilde{h} is a half-edge of x mapped to h' by φ and \tilde{e} is an edge adjacent to x mapped to e' by φ and \tilde{e} is an edge mapped to e' by φ and Ke denotes the orbit of e by K . Note that we use the orbit-stabilizer theorem at the last equality. Accordingly, we get the conclusion. \square

Note that whether Γ is a singleton or not agrees with whether Γ' is a singleton.

Is $R(D)^K$, the K -invariant set of $R(D)$, identical to $\varphi^*(R(\varphi_*(D)))$? Nor is it.

Example 3.9 Assume the situation of Example 3.3. Let $\tilde{\varphi}$ be the harmonic morphism from $\tilde{\Gamma}$ to the quotient metric graph $\tilde{\Gamma}/\tilde{K}$. For a rational function f which decreases from $\tilde{\varphi}(x_1)$ to $\tilde{\varphi}(x_2)$ with slope one and is constant on other graph, however f is an element of $R(\tilde{\varphi}_*(\tilde{D}))$, the pull-back of f by $\tilde{\varphi}$ is not in $R(\tilde{D})^{\tilde{K}}$.

Next, for $R(D)^K$, the following holds.

Lemma 3.10 $R(D)^K$ is a tropical semimodule.

Proof. Let c be in \mathbf{R} , f, g in $R(D)^K$ and σ in K . Since $R(D)$ is a tropical semimodule by Lemma 2.2.1, $c \odot f$ and $f \oplus g$ are in $R(D)$. It is obvious that \odot and \oplus are associative and that \oplus is distributive over \odot from right, both $(c \odot f) \oplus \sigma$ and $(f \oplus g) \oplus \sigma$ are in $R(D)^K$. \square

Note that $R(D + \text{div}(f))^K = R(D)^K \odot (-f)$ for any K -invariant rational function f .

The following lemma is an extension of [6, Lemma 5].

Lemma 3.11 Let f be in $\text{Rat}(\Gamma)$. Then, f is an extremal of $R(D)^K$ if and only if there are not two proper K -invariant subgraphs Γ_1 and Γ_2 covering Γ such that each can fire on $D + \text{div}(f)$.

Proof. First, let us show the “if” part. Suppose that there are two such subgraphs Γ_1 and Γ_2 . We can assume that each Γ_i does not have any connected component consisting only of points at infinity. Each Γ_i defines a chip firing move g_i for a small positive number so that g_i is zero on Γ_i and they are nonpositive. As Γ_1 and Γ_2 are K -invariant, so g_1 and g_2 are in $R(D + \text{div}(f))^K$. Since $g_1 \oplus g_2 = 0$ on Γ , we can write f as $(f + g_1) \oplus (f + g_2)$, i.e. f is not an extremal of $R(D)^K$.

Next, let us show the “only if” part. Suppose $f = g_1 \oplus g_2$ for some g_1 and g_2 in $R(D)^K \setminus \{f\}$. For $i = 1, 2$, there exists \tilde{g}_i in $R(D + \text{div}(f))^K$ such that $g_i = \tilde{g}_i \odot f$. Let Γ_i be the closure of the loci where $\tilde{g}_i = 0$. Then, the union of Γ_1 and Γ_2 is Γ and each Γ_i is proper. Since \tilde{g}_i is K -invariant, so is Γ_i . Then, each Γ_i can fire on $D + \text{div}(f)$. \square

Lemma 3.12 Let E be an effective divisor on Γ and A a K -invariant subset of $\text{supp}(E)$. If $\varphi(A)$ is a cut set of Γ' and $E(x) \geq \text{val}(x) - 1$ for any

x in A , then there exists a K -invariant subgraph Γ_1 of Γ which contains infinitely many points, can fire on E and whose boundary points are in A .

Proof. For such A , let $\Gamma'_1, \dots, \Gamma'_n$ be distinct connected components of $\Gamma' \setminus \varphi(A)$ respectively. Note that n is no less than two since $\varphi(A)$ is a cut set of Γ' . Clearly, for any i , the inverse image of the closure of Γ'_i by φ is a K -invariant subgraph of Γ we want. \square

We call a point on Γ not being a vertex of G_2 a K -ordinary point. Note that if a subgraph of Γ has a K -ordinary point, topologically saying, it should have infinitely many points.

Lemma 3.13 *Let Γ_1 be a K -invariant subgraph of Γ . If Γ_1 contains infinitely many points and if the set $\partial\Gamma_1$ of its boundary points contains at least one K -ordinary point, then $\varphi(\partial\Gamma_1)$ is a cut set of Γ' and contains a point on Γ' not being a vertex of G' .*

Proof. For such Γ_1 , obviously $\varphi(\partial\Gamma_1)$ contains a point on Γ' not being a vertex of G' . It is sufficient to check that $\varphi(\partial\Gamma_1)$ is a cut set of Γ' . Let Γ_2 be the closure of the complement set of Γ_1 in Γ . This Γ_2 is K -invariant and contains a K -ordinary point. Thus, Γ_2 is a subgraph containing infinitely many points. Consequently, $\Gamma' \setminus \varphi(\partial\Gamma_1) = \varphi(\Gamma_1 \cup \Gamma_2) \setminus \varphi(\partial\Gamma_1) = (\varphi(\Gamma_1) \setminus \varphi(\partial\Gamma_1)) \sqcup (\varphi(\Gamma_2) \setminus \varphi(\partial\Gamma_1))$. Hence, $\varphi(\partial\Gamma_1)$ is a cut set of Γ' . \square

The next corollary follows from Lemma 3.12 and Lemma 3.13.

Corollary 3.14 *Let E be an effective divisor on Γ . For a subset of the support of $\varphi_*(E)$, we consider the following condition $(*)$:*

- $(*)$ *it is a cut set of Γ' containing no vertices of G' and whose inverse image by φ is a subset of the support of E .*
- (1) *For a subset A of $\text{supp}(E)$ whose image by φ satisfies $(*)$, there exists a K -invariant subgraph Γ_1 of Γ which contains infinitely many points, can fire on E and whose boundary points are in A .*
- (2) *Let Γ_1 be a K -invariant subgraph of Γ . If Γ_1 contains infinitely many points and can fire on E and if the set of its boundary points consists only of K -ordinary points, then the image of the set of boundary points of Γ_1 by φ satisfies $(*)$.*

By Corollary 3.14, it is natural to define $S(D)_K$ as

$$S(D)_K := \left\{ f \in R(D)^K \left| \begin{array}{l} \exists A \subset \text{supp}(\varphi_*(D + \text{div}(f))) \text{ such that} \\ (1) A \text{ is a cut set of } \Gamma', \\ (2) A \cap V(G') = \emptyset, \text{ and} \\ (3) \varphi^{-1}(A) \subset \text{supp}(D + \text{div}(f)). \end{array} \right. \right\}.$$

In fact, this $S(D)_K$ is the set corresponding to $S(D)$, *i.e.* $S(D)_K$ is a generator set of $R(D)^K$.

Theorem 3.15 *In the above situation, the following hold:*

- (1) $S(D)_K$ contains all the extremals of $R(D)^K$,
- (2) $S(D)_K$ is finite modulo tropical scaling (except by $-\infty$), and
- (3) $S(D)_K$ generates $R(D)^K$ as a tropical semimodule.

Proof. (1) Suppose f is in the difference set of $R(D)^K$ from $S(D)_K$, then there exists a cut set A' of Γ' contained in $\text{supp}(\varphi_*(D + \text{div}(f)))$, containing no vertices of G' and such that $\varphi^{-1}(A') \subset \text{supp}(D + \text{div}(f))$. By (1) of Corollary 3.14, there exists a K -invariant subgraph Γ_1 of Γ which contains infinitely many points, can fire on $D + \text{div}(f)$ and whose boundary points are in $\varphi^{-1}(A')$. Then, the closure of $\Gamma \setminus \Gamma_1$ can also fire on $D + \text{div}(f)$. Therefore, by Lemma 3.11, f is not an extremal of $R(D)^K$.

(2) The push-forward of a rational function on Γ induces a natural map $S(D)_K/\mathbf{R} \rightarrow S_{G'}(\varphi_*(D))/\mathbf{R}$, $[f] \mapsto [\varphi_*(f)]$. In fact, for any f in $S(D)_K$, $\varphi_*(D + \text{div}(f)) = \varphi_*(D) + \varphi_*(\text{div}(f)) = \varphi_*(D) + \text{div}(\varphi_*(f))$, thus, $\varphi_*(f)$ is in $R(\varphi_*(D))$. From $f \in S(D)_K$, there exist no cut sets of Γ' contained in $\text{supp}(\varphi_*(D) + \varphi_*(\text{div}(f)))$, containing no vertices of G' and whose inverse image by φ is a subset of $\text{supp}(D + \text{div}(f))$. This means that $\varphi_*(f)$ is in $S_{G'}(\varphi_*(D))$. Also, for any pair of f_1 and f_2 in $[f]$, there exists c in \mathbf{R} satisfying $f_2 = f_1 + c$. Since $\varphi_*(f_2) = \varphi_*(f_1 + c) = \varphi_*(f_1) + \varphi_*(c) = \varphi_*(f_1) + c$, the map is well-defined. Now we show that the map is injective. Let $[f]$ and $[g]$ be distinct elements of $S(D)_K/\mathbf{R}$, thus $\text{div}(f)$ differs from $\text{div}(g)$. Since both f and g are K -invariant, so their images $\varphi_*(\text{div}(f))$ and $\varphi_*(\text{div}(g))$ are different, *i.e.* the map is injective. By Remark 3.2, we get the conclusion.

(3) Suppose $f \in R(D)^K$. Let $N(f)$ be the number of distinct K -orbits in the union of all K -invariant subsets, whose image by φ is a cut set of Γ' containing no vertices of G' , of $\text{supp}(D + \text{div}(f))$. We prove (3) by induction for $N(f)$. If $N(f) = 0$, then $f \in S(D)_K$ from the definition of

$S(D)_K$. Assume that $f \in \langle S(D)_K \rangle$ for all $N(f) \leq k$, where $\langle S(D)_K \rangle$ means the tropical semimodule generated by $S(D)_K$. We consider the case where $N(f) = k + 1$ and $f \notin S(D)_K$. Let A be a subset of $\text{supp}(D + \text{div}(f))$ whose image by φ is a cut set of Γ' containing no vertices of G' . By (1) of Corollary 3.14, there exists a K -invariant subgraph Γ_1 of Γ which can fire on $D + \text{div}(f)$ and whose boundary points are in A . Let Γ_2 be the closure of the complement of Γ_1 in Γ . For any $x \in \partial\Gamma_i$, we write the distance between x and its closest vertex of G_1 as l_x . Let $l_i := \min\{l_x \mid x \in \partial\Gamma_i\}$ and $g_i := \text{CF}(\Gamma_i, l_i)$. Then, for both $i = 1, 2$, $f \odot g_i$ is not equal to f and is in $R(D)^K$ since $f, g_i \in R(D)^K$ and $f = (f \odot g_1) \oplus (f \odot g_2)$. By the definition of g_i , $N(f) > N(f \odot g_i)$ and $f \odot g_i \in \langle S(D)_K \rangle$, then $f \in \langle S(D)_K \rangle$. \square

By Lemma 2.2.3 and the above theorem, we obtain the following corollary, which is an extension of [6, Corollary 9].

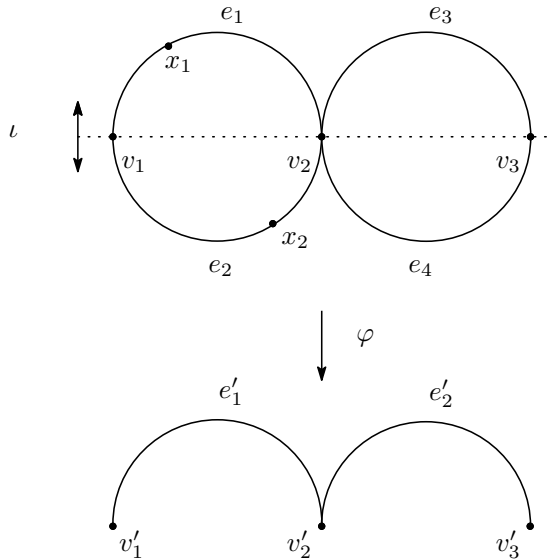
Corollary 3.16 *Let Γ be a tropical curve, D an effective divisor on Γ and K a finite subgroup of the isometry group of Γ . Then, the tropical semimodule $R(D)^K$ is generated by the extremals of $R(D)^K$. This generating set is minimal and unique up to tropical scalar multiplication.*

If D is K -invariant, $R(D)^K/\mathbf{R}$ is naturally identified with the subset $|D|^K$ of $|D|$ consisting of all K -invariant elements of $|D|$. In fact, let D be a K -invariant effective divisor on Γ . For any $D' \in |D|^K$, there exists $f \in R(D)$ such that $D' = D + \text{div}(f)$. Since both D and D' are K -invariant, $D' = D + \text{div}(f \circ \sigma)$ for any $\sigma \in K$. Thus $0 = \text{div}(f) - \text{div}(f \circ \sigma) = \text{div}(f - f \circ \sigma)$ and there exists $c \in \mathbf{R}$ such that $f - f \circ \sigma = c$, i.e. $f = c \odot f \circ \sigma$. Since the order k of σ is finite, $f = c \odot \cdots \odot c \odot f$ holds, where c is multiplied k times. As k is not zero, c must be zero. Therefore f is K -invariant and then $f \in R(D)^K$. Conversely, $[g] \in R(D)^K/\mathbf{R}$ corresponds to an element $D + \text{div}(g)$ in $|D|^K$. In conclusion, the following holds from Theorem 3.15.

Theorem 3.17 *Let Γ be a tropical curve, D an effective divisor on Γ and K a finite subgroup of the isometry group of Γ . If D is K -invariant, then the K -invariant linear subsystem $|D|^K$ of $|D|$ is finitely generated by $S(D)_K/\mathbf{R}$.*

In some cases, like the following example, the image of $S(D)_K/\mathbf{R}$ by the map in the proof of (2) of Theorem 3.15 is not contained in $S(\varphi_*(D))/\mathbf{R}$. This is the reason we think $S_{G'}(\varphi_*(D))$ instead of $S(\varphi_*(D))$.

Example 3.18 Let $\tilde{\Gamma}$ be a tropical curve which shape is as the following upper figure, where each for $i = 1, 2$, e_i and e_{i+1} have a same length. Let ι be the isometry on $\tilde{\Gamma}$ which maps each v_i to v_i itself, e_1 to e_2 , e_2 to e_1 , e_3 to e_4 and e_4 to e_3 . ι generates a finite group \tilde{K} consisting of only two elements, ι and the identity map id . \tilde{K} defines the natural map $\tilde{\varphi}$, which is a finite harmonic morphism of degree two, from $\tilde{\Gamma}$ to the quotient tropical curve $\tilde{\Gamma}/\tilde{K} =: \tilde{\Gamma}'$ (more precisely, see after of Example 3.5). Then, by $\tilde{\varphi}$, each v_i is mapped to v'_i , e_1 and e_2 are mapped to e'_1 , and e_3 and e_4 are mapped to e'_2 . Let $\tilde{D} := x_1 + x_2 + 2v_2 \in \text{Div}(\tilde{\Gamma})$. $S(\tilde{D})_{\tilde{K}}/\mathbf{R}$ consists of three elements $[f_1], [f_2]$ and $[f_3]$ such that $\tilde{D} + \text{div}(f_1) = \tilde{D}$ (i.e. f_1 is a constant function), $\tilde{D} + \text{div}(f_2) = x_1 + x_2 + 2v_1$ and $\tilde{D} + \text{div}(f_3) = x_1 + x_2 + 2v_3$.



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