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Degree formulas for the Euler characteristic of semialgebraic sets

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Abstract. We are interested in computing alternate sums of Euler characteristics of some particular semialgebraic sets, intersections of an algebraic one, smooth or with finitely many singularities, with sets given by just one polynomial inequality. We state theorems relating these alternate sums of characteristics to some topological degrees at infinity of polynomial mappings.

Key words: Morse theory, manifold with corners, Euler characteristic, semialgebraic sets.

1. Introduction

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ and $G = (G_1, \ldots, G_l) : \mathbb{R}^n \to \mathbb{R}^l$ be polynomial maps with $k, l \geq 1$ and let W be $F^{-1}(0)$. We are interested in the semialgebraic sets

$$W_G(\varepsilon) := W \cap \{(-1)^{\varepsilon_1} G_1 \geq 0\} \cap \cdots \cap \{(-1)^{\varepsilon_l} G_l \geq 0\},$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_l) \in \{0,1\}^l$, and we want to express the alternate sum

$$\sum_{\varepsilon \in \{0,1\}^l} (-1)^{|\varepsilon|} \chi(W_G(\varepsilon))$$

in terms of the topological degree at infinity of polynomial mappings involving F and G, with $|\varepsilon| = \sum_{i=1}^{l} \varepsilon_i$.

These semialgebraic sets are natural generalizations of those studied by Szafraniec in [6] and [7]. They are simply defined by equalities and inequalities. If we want to use Morse theory for manifold with corners, they are the natural objects to study. Basic semi-algebraic sets are semi-algebraic sets defined by only polynomial inequalities. They form one of the first natural class of semi-algebraic sets to study (see for example [1]). When $W = \mathbb{R}^n$, the sets that we consider in this paper are closed basic

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semialgebraic sets, and therefore very natural to study.

To give an idea, when $l = 1$, this alternate sum is equal to

$$\begin{aligned}&(-1)^{|0|}\chi\big(W \cap \{(-1)^0 G_1 \geq 0\}\big) + (-1)^{|1|}\chi\big(W \cap \{(-1)^1 G_1 \geq 0\}\big)\\ &\quad = \chi\big(W \cap \{G_1 \geq 0\}\big) - \chi\big(W \cap \{G_1 \leq 0\}\big);\end{aligned}$$

and when $l = 2$, it is equal to

$$\begin{aligned}&(-1)^{|(0,0)|}\chi\big(W \cap \{(-1)^0 G_1 \geq 0\} \cap \{(-1)^0 G_2 \geq 0\}\big)\\ &\qquad + (-1)^{|(1,0)|}\chi(W \cap \{(-1)^1 G_1 \geq 0\} \cap \{(-1)^0 G_2 \geq 0\})\\ &\qquad + (-1)^{|(1,1)|}\chi(W \cap \{(-1)^1 G_1 \geq 0\} \cap \{(-1)^1 G_2 \geq 0\})\\ &\qquad + (-1)^{|(0,1)|}\chi\big(W \cap \{(-1)^0 G_1 \geq 0\} \cap \{(-1)^1 G_2 \geq 0\}\big)\\ &\quad = \chi\big(W \cap \{G_1 \geq 0\} \cap \{G_2 \geq 0\}\big) - \chi(W \cap \{G_1 \leq 0\} \cap \{G_2 \geq 0\})\\ &\qquad + \chi(W \cap \{G_1 \leq 0\} \cap \{G_2 \leq 0\}) - \chi\big(W \cap \{G_1 \geq 0\} \cap \{G_2 \leq 0\}\big).\end{aligned}$$

There are already several results relating the Euler characteristic of an algebraic set to the topological degree of a polynomial mapping. First, when W is compact, Bruce [2] and Szafraniec [6] proved that there exists a polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ with an isolated critical point at the origin such that

$$\chi(W) = \frac{1}{2}\big((-1)^n - \deg_0 \nabla P\big),$$

where $\deg_0 \nabla P$ is the topological degree at the origin of the gradient of P.

Throughout this paper, we will denote by $\deg_\infty \phi$ the topological degree at infinity of $\phi/\|\phi\| : S^N_R \to S^N_1$ for any mapping $\phi : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ such that $\phi^{-1}(0) \subsetneq B^{N+1}_R$.

When $k < n$ and W is a smooth $(n-k)$-dimensional manifold, Szafraniec constructs in [7] a polynomial map $H : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ such that $H^{-1}(0)$ is compact and such that $\chi(W) = (-1)^k \deg_\infty H$.

In [3], Dutertre studies semialgebraic sets with isolated singularities. From $f : \mathbb{R}^n \to \mathbb{R}$ a polynomial function with isolated critical points such that $f(0) > 0$, he constructs four polynomial mappings H, L, L_1 and L_2

in terms of f and ∇f such that $H^{-1}(0)$, $L^{-1}(0)$, $L_1^{-1}(0)$ and $L_2^{-1}(0)$ are compact. Then he gets the equalities:

- If n is even, then

$$\chi(f^{-1}(0)) = \deg_\infty H + \deg_\infty \nabla f - \deg_\infty L_2$$
$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - \deg_\infty L + \deg_\infty L_1.$$

- If n is odd, then

$$\chi(f^{-1}(0)) = \deg_\infty L - \deg_\infty L_1$$
$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - \deg_\infty H - \deg_\infty \nabla f + \deg_\infty L_2.$$

The aim of this paper is to obtain a similar result as Szafraniec's one in [7] for our sum $\sum_{i=1}^{l}(-1)^{|\varepsilon|}\chi(W_G(\varepsilon))$. Then we generalize the results of Dutertre in [3] to the sets $W_G(\varepsilon) \cap \{f \geq 0\}$, $W_G(\varepsilon) \cap \{f \leq 0\}$ and $W_G(\varepsilon) \cap \{f = 0\}$, when $W = \mathbb{R}^n$, namely we compute the alternate sums

$$\sum_{\varepsilon \in \{0,1\}^l} (-1)^{|\varepsilon|}\chi(W_G(\varepsilon) \cap \{f = 0\})$$

and

$$\sum_{\varepsilon \in \{0,1\}^l} (-1)^{|\varepsilon|}\big[\chi(W_G(\varepsilon) \cap \{f \geq 0\}) - \chi(W_G(\varepsilon) \cap \{f \leq 0\})\big].$$

We suppose $W = F^{-1}(0)$ smooth with $F = (F_1, \ldots, F_k)$, $G = (G_1, \ldots, G_l)$ and $k + l \leq n$, and we assume that all the intersections between the set $W = F^{-1}(0)$ and the sets $G_i^{-1}(0)$, $i = 1, \ldots, l$ are transverse (see Condition $(\star)$ on Page 3).

We will see that we can write $\sum_\varepsilon(-1)^{|\varepsilon|}\chi(W_G(\varepsilon))$ in terms of the topological degree at infinity of the following mapping L defined on $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l$ by:

$$L(x, \lambda, \mu)$$
$$= \Big(\nabla\omega(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \sum_{j=1}^{l} \mu_j \nabla G_j(x), F(x), \mu_1 G_1(x), \ldots, \mu_l G_l(x)\Big),$$

where $\omega(x) = (1/2)(x_1^2 + \cdots + x_n^2)$.

More precisely, we find the result (c.f. Theorem 2.7):

$$\sum_{\varepsilon \in \{0,1\}^l} (-1)^{|\varepsilon|} \chi(W_G(\varepsilon)) = (-1)^k \deg_\infty L.$$

To prove this theorem, we will use Morse theory for manifolds with corner, which is an application of stratified Morse theory ([4], [5]). For more details on this theory, we refer to [3].

In my thesis, I also obtained similar results for semialgebraic sets with isolated singularities.

The paper is organized as follows: In Section 2, we prove some technical lemmas that relate a Morse index to a topological degree, in order to give a similar result as the one of Szafraniec in [7].

Some computations are given in the example at the end of Section 2. They have been done with a program written by Andrzej Lecki. The author is very grateful to him and Zbigniew Szafraniec for giving her this program.

2. Smooth case

In this section, we compute the alternate sums:

$$\sum_{\varepsilon \in \{0,1\}^l} (-1)^{|\varepsilon|} \chi(W_G(\varepsilon)),$$

and as Szafraniec in [7], we want to write it in terms of the topological degree of a polynomial mapping.

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ and $G = (G_1, \ldots, G_l) : \mathbb{R}^n \to \mathbb{R}^l$ be polynomial maps, with $k, l \geq 1$ and $k + l \leq n$.

We define $W = \{x \in \mathbb{R}^n \mid F(x) = 0\}$ and assume it is a smooth manifold of dimension $n - k$ in $\mathbb{R}^n$ (or empty). We also make the following transversality assumption:

$$(\star) \quad \forall \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, l\},\ \forall x \in W \cap G_{i_1}^{-1}(0) \cap \cdots \cap G_{i_s}^{-1}(0),$$
$$\operatorname{rank}(DF(x), DG_{i_1}(x), \ldots, DG_{i_s}(x)) = k + s.$$

We are interested in the topology of the set

$$W_G(\varepsilon) := W \cap \{x \in \mathbb{R}^n | (-1)^{\varepsilon_1} G_1(x) \geq 0\} \cap \cdots \cap \{x \in \mathbb{R}^n | (-1)^{\varepsilon_l} G_l(x) \geq 0\}$$

for $\varepsilon \in \{0,1\}^l$ and in computing the value of the alternate sum

$$\sum_{\varepsilon \in \{0,1\}^l} (-1)^{|\varepsilon|} \chi(W_G(\varepsilon)),$$

where $|\varepsilon| = \sum_{i=1}^{l} \varepsilon_i$.

Let us remark that for $\varepsilon \in \{0,1\}^l$, $W_G(\varepsilon)$ is a manifold with corners, so it is also a Whitney stratified set, whose 2^l strata are given by:

$$S^{\varepsilon}_{I_j} = \left\{x \in W \mid (-1)^{\varepsilon_i} G_i(x) > 0 \text{ if } i \in I_j \text{ and } G_i(x) = 0 \text{ if } i \in \overline{I_j}\right\}$$

where for all $j \in \{1, \dots, l\}$, I_j is a subset $\{i_1, \dots, i_j\}$ of $\{1, \dots, l\}$ such that $i_1 < \cdots < i_j$, and where we write $\overline{I_j}$ its complement in $\{1, \dots, l\}$. Moreover, we denote by $I_0 = \emptyset$, and then we have $\overline{I_0} = \{1, \dots, l\}$.

For example, when $l = 2$, $G = (G_1, G_2) : \mathbb{R}^n \to \mathbb{R}^2$ and for a given $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0,1\}^2$, the four strata are:

$$\begin{aligned}
S^{\varepsilon}_{I_0} &= \{x \in W \mid G_1(x) = G_2(x) = 0\}, \\
S^{\varepsilon}_{I_1} &= \{x \in W \mid (-1)^{\varepsilon_1} G_1(x) > 0,\ G_2(x) = 0\}, \\
S^{\varepsilon}_{I'_1} &= \{x \in W \mid (-1)^{\varepsilon_2} G_2(x) > 0,\ G_1(x) = 0\}, \\
S^{\varepsilon}_{I_2} &= \{x \in W \mid (-1)^{\varepsilon_1} G_1(x) > 0,\ (-1)^{\varepsilon_2} G_2(x) > 0\}.
\end{aligned}$$

We will also consider the following submanifold of W:

$$W_{I_j} = \left\{x \in W \mid G_i(x) = 0 \text{ if } i \in \overline{I_j},\ G_i(x) \neq 0 \text{ if } i \in I_j\right\},$$

and so we have that for all $\varepsilon \in \{0,1\}^l$ and for all $j \in \{0, \dots, l\}$, $S^{\varepsilon}_{I_j} \subseteq W_{I_j}$. We denote by $\overline{W_{I_j}}$ the closure of W_{I_j} in W, that is

$$\overline{W_{I_j}} = \left\{x \in W \mid G_i(x) = 0\ \forall i \in \overline{I_j}\right\}.$$

Given a polynomial function ρ from $\mathbb{R}^n$ to $\mathbb{R}$, we are first going to study the critical points of ρ on the different strata. We need for this to recall the definition of a correct critical point on a manifold with corners.

Definition 2.1 Let $H = M \cap \{h_1 \geq 0, \ldots, h_k \geq 0\}$ be a manifold with corners where M is a smooth manifold. Let $\delta : M \to \mathbb{R}$ be a smooth function.

We say that $p \in H$ is a critical point of $\delta|_H$ if there exist $i_1, \ldots, i_s \in \{1, \ldots, k\}$ such that p is a critical point of $\delta|_{M\cap\{h_{i_1}=\cdots=h_{i_s}=0\}}$. In this case, we say that p is a correct critical point of $\delta|_H$ if p is not a critical point of δ restricted to $M \cap (\cap_{i\in I}\{h_i = 0\})$ for any proper subset I of $\{i_1, \ldots, i_s\}$. Moreover, we say that p is **pointing inward** H if $\nabla\delta(p) = \sum_{j=1}^{s} \alpha_{i_j} \nabla h_{i_j}(p)$ where $\alpha_{i_1} > 0, \ldots, \alpha_{i_s} > 0$. Otherwise we say that p is **pointing outward** H.

We remark that in our case, $p \in S^{\varepsilon}_{I_j}$ is a correct critical point of $\rho_{|W_G(\varepsilon)}$ if and only if p is a critical point of $\rho|_{\overline{W_{I_j}}}$ and is not a critical point of $\rho|_{\overline{W_I}}$ for any set $I \subseteq \{1, \ldots, l\}$ such that $I_j \subsetneq I$.

For $(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l$, we consider the mapping:

$$L(x,\lambda,\mu) = \Bigg(\nabla\rho(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \sum_{j=1}^{l} \mu_j \nabla G_j(x), \\ F(x), \mu_1 G_1(x), \ldots, \mu_l G_l(x)\Bigg),$$

and for $(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{l-j}$, $j \in \{0, \ldots, l\}$, and $\overline{I_j} = (i_1, \ldots, i_{l-j})$, we introduce the following 2^l mappings:

$$L_{\overline{I_j}}(x,\lambda,\mu) = \Bigg(\nabla\rho(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \sum_{s=1}^{l-j} \mu_{i_s} \nabla G_{i_s}(x), \\ F(x), \mu_{i_1} G_{i_1}(x), \ldots, \mu_{i_{l-j}} G_{i_{l-j}}(x)\Bigg).$$

Lemma 2.2 *Let $p \in S^{\varepsilon}_{I_j}$. The point p is a critical point of $\rho|_{W_G(\varepsilon)}$ if, and only if, there exists a unique $(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^l$ such that $L(p, \lambda, \mu) = 0$.*

Moreover,

$$p \text{ is correct} \iff \mu_i \neq 0 \quad \forall i \in \overline{I_j} \text{ and } \mu_i = 0 \text{ otherwise.}$$

Proof. From [7, Lemma 1.2], we know that

$$p \text{ is a critical point of } \rho|_{S^{\varepsilon}_{I_j}} \iff \exists!(\lambda, \bar{\mu}) \in \mathbb{R}^k \times \mathbb{R}^{l-j}, \quad L_{\overline{I_j}}(p, \lambda, \bar{\mu}) = 0.$$

Hence

$$L(p, \lambda, \mu) = 0 \text{ where } \mu = (\mu_1, \ldots, \mu_l) \text{ with } \mu_i = \begin{cases} 0 & \text{if } i \in I_j \\ \bar{\mu_i} & \text{if } i \in \overline{I_j} \end{cases}.$$

The converse is obvious. It is straightforward to prove the unicity of (λ, μ).

Now let us suppose that p is correct. As $p \in S^{\varepsilon}_{I_j}$ and $L(p, \lambda, \mu) = 0$, we necessarily have $\mu_i = 0$ for all $i \in I_j$ and as p is correct, $\mu_i \neq 0$ for all $i \in \overline{I_j}$.

Conversely, if $\mu_i = 0$ for all $i \in I_j$ and $\mu_i \neq 0$ for all $i \in \overline{I_j}$, then

$$\nabla \rho(p) + \sum \lambda_i \nabla F(p) + \sum_{i \in \overline{I_j}} \mu_i \nabla G_i(p) = 0.$$

We see that p is a correct critical point of $\rho|_{W_G(\varepsilon)}$. □

Lemma 2.3 *Let $p \in S^{\varepsilon}_{I_j}$ be a correct critical point of $\rho|_{W_G(\varepsilon)}$ and $(\lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^l$ such that $L(p, \lambda, \mu) = 0$. Then p is a correct critical point of $\rho|_{W_G(\varepsilon)}$ pointing inward if, and only if, $(-1)^{\varepsilon_i} \mu_i < 0$ for all $i \in \overline{I_j}$.*

Proof. From the previous lemma, p is a correct critical point of $\rho|_{W_G(\varepsilon)}$ if and only if $\exists!(\lambda, \mu)$ such that $L(p, \lambda, \mu) = 0$ with $\mu_i = 0$ for all $i \in I_j$ and $\mu_i \neq 0$ for all $i \in \overline{I_j}$. Denoting by Π the orthogonal projection on T_pW, we have

$$\Pi(\nabla \rho(p)) = -\sum_{i=1}^{l} \mu_i \Pi(\nabla G_i(p)) = -\sum_{i \in \overline{I_j}} \mu_i \Pi(\nabla G_i(p)).$$

We know that for all $i \in \overline{I_j}$, $\Pi((-1)^{\varepsilon_i} \nabla G_i(p))$ points inward $W_G(\varepsilon)$, so

$$\Pi(\nabla \rho(p)) = \sum_{i \in \overline{I_j}} (-(-1)^{\varepsilon_i} \mu_i) \Pi((-1)^{\varepsilon_i} \nabla G_i(p))$$

points inward if and only if $-(-1)^{\varepsilon_i} \mu_i > 0$, i.e. $(-1)^{\varepsilon_i} \mu_i < 0$ for all $i \in \overline{I_j}$. □

In the next lemma, we want to relate the sign of the differential of L at one zero (p, λ, μ) of L to the Morse index at p of ρ restricted to the stratum that contains p.

Lemma 2.4 *Let $p \in S^{\varepsilon}_{I_j}$ be a correct critical point of $\rho|_{W_G(\varepsilon)}$ and let $(\lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^l$ be such that $L(p, \lambda, \mu) = 0$. Then p is a Morse critical point if, and only if,* $\det(DL(p, \lambda, \mu)) \neq 0$. *Moreover, we get that*

$$\operatorname{sign}\det(DL(p,\lambda,\mu)) = \left(\prod_{i\in I_j} \operatorname{sign}(G_i(p)) \prod_{i\in\overline{I_j}} \operatorname{sign}(\mu_i)\right)(-1)^{\tau(p)+k+l-j},$$

where $\tau(p)$ is the Morse index of $\rho|_{S^{\varepsilon}_{I_j}}$ at p.

Proof. Let (p, λ, μ) be a zero of L:

$$DL(p,\lambda,\mu) = \left[\begin{array}{c|c|ccc} (\star) & B & \nabla G_1(p) & \dots & \nabla G_l(p) \\ \hline {}^tB & (0) & & (0) & \\ \hline \mu_1^t \nabla G_1(p) & & G_1(p) & & \\ \vdots & (0) & & \ddots & \\ \mu_l^t \nabla G_l(p) & & & & G_l(p) \end{array}\right],$$

with $B = (\nabla F_1(p), \dots, \nabla F_k(p)) \in M_{n,k}(\mathbb{R})$ and $(\star) \in M_n(\mathbb{R})$.

We know that $p \in S^{\varepsilon}_{I_j}$ so $\mu_i = 0 \;\; \forall i \in I_j$ and $G_i(p) = 0 \;\; \forall i \in \overline{I_j}$, which we denote by $\overline{I_j} = (i_1, \dots, i_{l-j})$. Then we get:

$$\begin{aligned} &DL(p,\lambda,\mu) \\ &= \left(\prod_{i\in I_j} G_i(p) \prod_{i\in\overline{I_j}} \mu_i\right) \left[\begin{array}{c|c|c} (\star) & B & \nabla G_{i_1}(p) \dots \nabla G_{i_{l-j}}(p) \\ \hline {}^tB & (0) & (0) \\ \hline {}^t\nabla G_{i_1}(p) & & \\ \vdots & (0) & (0) \\ {}^t\nabla G_{i_{l-j}}(p) & & \end{array}\right], \\ &= \left(\prod_{i\in I_j} G_i(p) \prod_{i\in\overline{I_j}} \mu_i\right) DL_{\overline{I_j}}(p,\lambda,\bar{\mu}), \end{aligned}$$

where $\bar{\mu} \in \mathbb{R}^{l-j}$ is such that $\bar{\mu}_i = \mu_i \; \forall i \in \overline{I_j}$.

From [7, Lemma 1.4], we know that

$$p \text{ is Morse } \iff \det(DL_{\overline{I_j}}(p,\lambda,\bar{\mu})) \neq 0,$$

and we have

$$\det(DL(p,\lambda,\mu)) = \left(\prod_{i\in I_j} G_i(p) \prod_{i\in\overline{I_j}} \mu_i \right) \det\left(DL_{\overline{I_j}}(p,\lambda,\bar{\mu})\right).$$

So

$$\det(DL(p,\lambda,\mu)) \neq 0.$$

Next, again from [7, Lemma 1.4], we know that

$$\operatorname{sign}\det\left(DL_{\overline{I_j}}(p,\lambda,\bar{\mu})\right) = (-1)^{\tau(p)}(-1)^{k+l-j},$$

hence we obtain

$$\operatorname{sign}\det(DL(p,\lambda,\mu)) = \left(\prod_{i\in I_j} \operatorname{sign}(G_i(p)) \prod_{i\in\overline{I_j}} \operatorname{sign}(\mu_i) \right) (-1)^{\tau(p)+k+l-j}. \quad \square$$

Remark 2.5 If $k+l=n$ and p is a correct critical point of $\rho|_{S^{\varepsilon}_{I_0}}$, its Morse index $\tau(p)$ is 0.

Now we choose $\rho(x)=\omega(x)=(1/2)(x_1^2+\cdots+x_n^2)$ so we have $\nabla\omega(x)=x$. For all positive constant R, we set:

$$B_R^{n+k+l} = \{(x,\lambda,\mu)\in\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^l \mid \|x\|^2+\|\lambda\|^2+\|\mu\|^2 \leq R^2\},$$
$$S_R^{n+k+l-1} = \{(x,\lambda,\mu)\in\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^l \mid \|x\|^2+\|\lambda\|^2+\|\mu\|^2 = R^2\}.$$

We would like to compute $\deg_\infty L$, the topological degree at infinity of $L/\|L\| : S_R^{n+k+l-1} \to S^{n+k+l-1}$ for some $R>0$ but we need to prove first that it is well defined.

Lemma 2.6 *The set $L^{-1}(0)$ is compact.*

Proof. Let us study the vanishing set of L.

We easily find that

$$L^{-1}(0) = \bigcup_{j,I_j} \big\{(p,\lambda,\mu) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l \mid (p,\lambda,\bar{\mu}) \in L^{-1}_{\overline{I_j}}(0) \text{ with } \bar{\mu}_i = \mu_i \quad \forall i \in \overline{I_j} \text{ and } \mu_i = 0 \text{ otherwise}\big\},$$

which we can also write

$$L^{-1}(0) = \bigcup_{j,I_j} L^{-1}_{\overline{I_j}}(0) \times \{0\}^j \quad \text{(up to changes of indices in the coordinate system)}.$$

From [7, Lemma 2.1], the set $L^{-1}_{\overline{I_j}}(0)$ is compact for all $j \in \{0,\dots,l\}$ and so is the product $L^{-1}_{\overline{I_j}}(0) \times \{0\}$. As the union on j and I_j is finite, we can conclude that the set $L^{-1}(0)$ is compact. □

Theorem 2.7 *Let $R > 0$ be such that $L^{-1}(0) \subsetneq B_R^{n+k+l}$. If the sets W and $G_i^{-1}(0)$, $i = 1,\dots,l$ satisfy the transversality condition $(\star)$, then*

$$\sum_{\varepsilon} (-1)^{|\varepsilon|} \chi(W_G(\varepsilon)) = (-1)^k \deg_\infty L.$$

Proof. As in the proof of Theorem 2.2 in [7], we can choose a function $\tilde{\omega} : \mathbb{R}^n \to \mathbb{R}$ equal to ω on $\mathbb{R}^n \setminus B_R$, such that $\tilde{\omega}|_W$ and $\tilde{\omega}|_{W_G(\varepsilon)}$, with $\varepsilon \in \{0,1\}^l$, have only Morse correct critical points, and such that $\tilde{\omega}$ uniformly approaches ω for the Whitney $\mathcal{C}^2$-topology.

For $(x,\lambda,\mu) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l$, we consider

$$\tilde{L}(x,\lambda,\mu) = \Big(\nabla\tilde{\omega}(x) + \sum_i \lambda_i \nabla F_i(x) + \sum_i \mu_i \nabla G_i(x), F(x), \mu_1 G_1(x), \dots, \mu_l G_l(x)\Big).$$

As $L^{-1}(0) \subseteq B_R^{n+k+l}$, for $\tilde{\omega}$ close enough to ω, we also have that $\tilde{L}^{-1}(0) \subseteq B_R^{n+k+l}$. Let $(x,\lambda,\mu) \in \tilde{L}^{-1}(0)$. Then there exists $j \in \{0,\dots,l\}$ such that $x \in W_{I_j}$ and from Lemma 2.2, x is a critical point of $\tilde{\omega}|_{\overline{W_{I_j}}}$. But we have supposed that $\tilde{\omega}|_{W_{I_j}}$ only had Morse correct critical points, so from Lemma 2.4, we know that $\det D\tilde{L}(x,\lambda,\mu) \neq 0$. Thus, $0 \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l$ is a regular value of $\tilde{L}$. Moreover, the set $\tilde{L}^{-1}(0)$ is finite.

We now want to compute the degree of $\tilde{L}$, which is equal to the degree of L since if ω and $\tilde{\omega}$ are close enough, $L/\|L\|$ and $\tilde{L}/\|\tilde{L}\|$ are homotopic on a sufficiently big sphere.

Let us define $C_{\tilde{L}} = \{(p,\lambda,\mu) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l \mid \tilde{L}(p,\lambda,\mu) = 0\}$ the zero set of $\tilde{L}$.

Then we consider:

$$F_{I_j} = \{\mu \in \mathbb{R}^l \mid \mu_i = 0 \ \forall i \in I_j,\ \mu_i \neq 0 \ \forall i \in \overline{I_j}\}$$

and the following subset of $C_{\tilde{L}}$:

$$C^{\varepsilon}_{I_j} = \big\{(p,\lambda,\mu) \in C_{\tilde{L}} \mid \mu \in F_{I_j},\ (-1)^{\varepsilon_i} G_i(p) > 0 \\ \text{if } i \in I_j,\ (-1)^{\varepsilon_i+1}\mu_i > 0 \text{ if } i \in \overline{I_j}\big\}.$$

From Morse theory for manifolds with corners ([3]), for a given ε, we have

$$\chi(W_G(\varepsilon)) = \sum_{j=0}^{l} \sum_{I_j} \sum_{p \in \Pi_n(C^{\varepsilon}_{I_j})} (-1)^{\tau(p)},$$

where $\tau(p)$ is the Morse index of $\tilde{\omega}|_{W_{I_j}}$ at p and Π_n the projection of $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l$ on $\mathbb{R}^n$. Hence

$$\sum_{\varepsilon} (-1)^{|\varepsilon|} \chi(W_G(\varepsilon)) = \sum_{\varepsilon} (-1)^{|\varepsilon|} \sum_{j=0}^{l} \sum_{I_j} \sum_{p \in \Pi_n(C^{\varepsilon}_{I_j})} (-1)^{\tau(p)}.$$

But for $(p,\lambda,\mu) \in C^{\varepsilon}_{I_j}$, we know that

$$\text{sign} D\tilde{L}(p,\lambda,\mu) = \text{sign}\Big(\prod_{i \in I_j} G_i(p)\Big) \text{sign}\Big(\prod_{i \in \overline{I_j}} \mu_i\Big) (-1)^{k+l-j+\tau(p)}$$

and we easily see that

$$\text{sign}\Big(\prod_{i \in I_j} G_i(p)\Big) \text{sign}\Big(\prod_{i \in \overline{I_j}} \mu_i\Big) = \prod_{i \in I_j} (-1)^{\varepsilon_i} \prod_{i \in \overline{I_j}} (-1)^{\varepsilon_i+1}$$

$$= \left(\prod_{i=1}^{l}(-1)^{\varepsilon_i}\right)(-1)^{\mathrm{card}(\overline{I_j})} = (-1)^{|\varepsilon|}(-1)^{l-j}.$$

As

$$\deg_\infty \tilde{L} = \sum_{\varepsilon}\sum_{j=0}^{l}\sum_{I_j}\sum_{p\in\Pi_n(C^{\varepsilon}_{I_j})} \times \operatorname{sign}\left(\prod_{i\in I_j} G_i(p)\right)\operatorname{sign}\left(\prod_{i\in\overline{I_j}}\mu_i\right)(-1)^{\tau(p)}(-1)^{k+l-j},$$

and

$$\deg_\infty \tilde{L} = \deg_\infty L,$$

we can conclude that

$$\sum_{\varepsilon}(-1)^{|\varepsilon|}\chi(W_G(\varepsilon)) = (-1)^k \deg_\infty L. \qquad \square$$

Example 2.8

$$f(x_1,x_2) = -(x_1+3)^3x_2 - 3x_1^2 + x_2^2 + x_1,$$
$$G(x_1,x_2) = x_1^2 + x_2^2 - 30.$$

Here, the transversality condition $(\star)$ is satisfied.
We have

$$L : \mathbb{R}^{2+1+1} \longrightarrow \mathbb{R}^{2+1+1}$$
$$(x,\lambda,\mu) \longmapsto (\nabla\omega(x) + \lambda\nabla f(x) + \mu\nabla G(x), f(x), \mu G(x)).$$

We get that $\deg_\infty L = -2$ and thus

$$\sum_{\varepsilon\in\{0,1\}}(-1)^{|\varepsilon|}\chi(W_G(\varepsilon)) = \chi(f^{-1}(0)\cap\{G\geq 0\}) - \chi(f^{-1}(0)\cap\{G\leq 0\})$$
$$= -\deg_\infty L = 2.$$

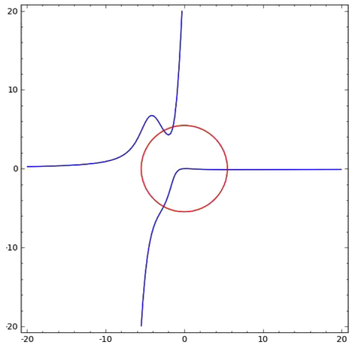

Figure 1. Curves $G^{-1}(0)$ and $f^{-1}(0)$.

The sets $f^{-1}(0) \cap \{G \geq 0\}$ and $f^{-1}(0) \cap \{G \leq 0\}$ are respectively four and two lines so are respectively homotopic to four and two points. Hence their respective Euler characteristic are 4 and 2. Thus,

$$\chi(f^{-1}(0) \cap \{G \geq 0\}) - \chi(f^{-1}(0) \cap \{G \leq 0\}) = 4 - 2 = 2.$$

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