Equation system describing the radiation intensity and the air motion with the water phase transition

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Abstract. In this paper we consider the equation system describing the motion of the air and the variation of the radiation intensity and the quantity of water droplets in the air, including also the process of water phase transition. Under a suitable condition we prove the existence and uniqueness of the local solution. By eliminating the approximation by regularization of vapor density and by including the equation of radiation, this result improves previous ones.

Key words: Model of the atmosphere, phase transitions, radiation intensity, integral semilinear terms.

1. Introduction

Many attempts of modeling the atmosphere have been made (most classic ones are for example [9], [7]), but it seems that the complete description of atmospheric phenomena remains to be done. In [12] and [13] well-posedness and stability of the equation system for incompressible fluids with phase transition and without radiation intensity have been studied, whereas in [14] the well-posedness of an equation system describing rather completely the air motion and the water phase transition in the air is shown. But to obtain this result, the authors have slightly modified the equations, by replacing the vapor density by its approximation in the equations for the densities of water vapor, of water droplets and of ice crystals.

In this paper we consider the equation system made up of the equations considered in [14] and the equation of the radiation intensity. The introduction of the radiation is motivated by the fact that the absorbtion and diffusion of the radiation by the air can influence the state of atmosphere as the global warming effect. We prove the existence and the uniqueness of the local solution to this equation system, without modifying the function representing the vapor density, but we do not consider the equation for the density of ice crystals, excluding the phase transition to solid state; in fact,

the behavior of this equation would be similar to that of the equation for the density of water droplets.

From the technical point of view, for the part of the equations concerning the velocity and the temperature we follow the scheme of [14], which is largely based on the classical techniques of Solonnikov ([15] and others). For the part of the equations concerning the densities of the air, of water vapor and of water droplets, we use the techniques developed in [14] and [1] and, as is mentioned above, improve the proof. For the equation of radiation intensity, we use partially the techniques developed in [2], but we introduce also new techniques for the estimation of the solution in the norms of L^∞ and of L^2.

2.   Equation system

We will consider our equation system in a bounded domain $\Omega \subset \mathbb{R}^3$ with a sufficiently regular boundary $\partial\Omega$. We denote by $\varrho(t,x)$ ($t \geq 0$, $x \in \Omega$) the dry air density, by $\pi(t,x)$ the water vapor density, by $\sigma(t,m,x)$ ($m > 0$ is the mass of a droplet) the density of liquid water, by $T(t,x)$ the temperature of the air, by $v(t,x) = (v_1, v_2, v_3)$ the velocity of the air, by $u(t,m,x) = (u_1, u_2, u_3)$ the velocity of the droplets, and by $I_\lambda(x, q_1)$ the radiation intensity of wavelength λ. Moreover, we denote by η and ζ the viscosity coefficients, by κ the thermic conductivity, by c_v the specific heat at constant volume, and by L_{gl} the latent heat relative to the gas-liquid transition.

We assume that the pressure is given by

$$p = R_0 \left(\frac{\varrho}{\mu_a} + \frac{\pi}{\mu_h} \right) T,$$

where R_0, μ_a and μ_h are respectively the universal constant of gases, the average molar mass of the dry air and the molar mass of H_2O. We assume also that the external force is given by the gradient of a potential Φ.

In order to describe the motion of the air, the phase transition of water vapor, the variation of the density of water vapor, the motion of water droplets, the variation of the radiation intensity and their interactions, based on the fundamental equations given in [6] and [8] and their application to the mentioned phenomena (see [10], [14], [2]), we consider the following system of equations:

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho v) = 0, \tag{2.1}$$

$$\frac{\partial \pi}{\partial t} + \nabla \cdot (\pi v) = -H_{gl}(T, \pi, \sigma(m)), \tag{2.2}$$

$$\frac{\partial \sigma}{\partial t} + \nabla_{(m,x)} \cdot (\sigma \widetilde{U}_{4l}(u, T, \pi))$$
$$= [h_{gl}(T, \pi; m) + B_1(\sigma; m) - g_1(m)[\pi - \overline{\pi}_{vs}(T)]^-]\sigma$$
$$+ g_0(m)[N^* - \widetilde{N}(\sigma)]^+[\pi - \overline{\pi}_{vs}(T)]^+ + B_2(\sigma; m), \tag{2.3}$$

$$(\varrho + \pi)\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right)$$
$$= \eta \Delta v + \left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla \cdot v) - R_0 \nabla\left(\left(\frac{\varrho}{\mu_a} + \frac{\pi}{\mu_h}\right)T\right)$$
$$- \left[\int_0^\infty \sigma(m)dm + \varrho + \pi\right]\nabla\Phi, \tag{2.4}$$

$$(\varrho + \pi)c_v\left(\frac{\partial T}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial T}{\partial x_j}\right)$$
$$= \kappa \Delta T - R_0\left(\frac{\varrho}{\mu_a} + \frac{\pi}{\mu_h}\right)T\nabla \cdot v + \eta \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla \cdot v\right)\frac{\partial v_i}{\partial x_j}$$
$$+ \zeta(\nabla \cdot v)^2 - \nabla \cdot \mathcal{E} + L_{gl}H_{gl}, \tag{2.5}$$

$$- (q_1 \cdot \nabla)I_\lambda(t, x, q_1) = b_\lambda(t, x)I_\lambda(t, x, q_1) - J_\lambda(t, x, q_1, I_\lambda, T), \tag{2.6}$$

where

$$\nabla_{(m,x)} = (\partial_m, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}), \qquad \widetilde{U}_{4l}(u, T, \pi) = (mh_{gl}, u_1, u_2, u_3), \tag{2.7}$$

$$\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3), \qquad \mathcal{E}_j(t, x) = \int_0^\infty \int_{S^2} I_\lambda(t, x, q_1)q_{1j}dq_1 d\lambda,$$
$$j = 1, 2, 3, \tag{2.8}$$

and H_{gl}, h_{gl}, $B_1(\sigma; m)$, $B_2(\sigma; m)$, $J_\lambda(t, x, q_1, I_\lambda, T)$, $b_\lambda(t, x)$, $g_1(m)$, $g_0(m)$, N^*, $\widetilde{N}(\sigma)$ are the functions (or numbers) which we are going to precise below. In (2.4) $\int_0^\infty \sigma(m)\,dm$ is the total weight of water droplets per unit

volume, which is added to the weight of the air per unit volume in the equation of motion, while in (2.5), to the classical elements of heat balance, we add, as heat source, the latent heat L_{gl} multiplied by the quantity of phase transition H_{gl} as well as the divergence of the total intensity of directional radiation $\overrightarrow{\mathcal{E}}$. On the other hand, the right-hand side of the equation (2.3) represents the variation of σ due to the phase transition and the coagulation process; the meaning of each term will be given below.

We consider this equations system for $t \geq 0$, $x \in \Omega \subset \mathbb{R}^3$, $m > 0$ and $q_1 \in S^2 = \{q \in \mathbb{R}^3 : |q| = 1\}$. The function $I_\lambda(t, x, q_1)$ appearing in (2.6) and (2.8) depends on t, but the rôle of t is only that of parameter. So in the sequel we write simply $I_\lambda(x, q_1)$.

Concerning H_{gl} and h_{gl}, which represent the quantity of condensation (or evaporation) on all droplets and that on droplets of mass m, they can have some general form. But to fix the idea, we consider H_{gl} and h_{gl} having the form proposed in the modeling [14], that is

$$H_{gl}(T, \pi, \sigma(\cdot)) = K_1 \int_0^\infty \frac{S_l(m)}{m} \sigma(m) dm(\pi - \overline{\pi}_{vs}(T)), \qquad (2.9)$$

$$h_{gl} = h_{gl}(T, \pi, m) = K_1 \frac{S_l(m)}{m}(\pi - \overline{\pi}_{vs}(T)), \qquad (2.10)$$

where K_1 is a positive constant, $\overline{\pi}_{vs}(T)$ is the density of saturated vapor with respect to the liquid state and $S_l(m)$ represents the surface area of the droplet of mass m; for $S_l(m)$ we suppose that

$$S_l(\cdot) \in C^2(\mathbf{R}_+), \qquad S_l(\cdot) \geq 0, \qquad (2.11)$$

$$S_l(m) = 0 \quad \text{for} \ \ 0 \leq m \leq \frac{\overline{m}_a}{2} \quad (0 < \overline{m}_a), \qquad (2.12)$$

$$S_l(m) = c_l m^{2/3}, \quad \text{for} \ \ m \geq \overline{m}_A \quad (\overline{m}_a < \overline{m}_A < \infty); \qquad (2.13)$$

$\overline{m}_a$ and $\overline{m}_A$ should represent the lower and the upper bounds of the aerosols mass.

The terms $B_1(\sigma; m)$ and $B_2(\sigma; m)$ are defined by

$$B_1(\sigma; m) = -m\sigma(m) \int_0^\infty \beta(m, m')\sigma(m')dm',$$

$$B_2(\sigma; m) = \frac{m}{2} \int_0^m \beta(m - m', m')\sigma(m')\sigma(m - m')dm',$$

$$\beta(m_1, m_2) = \beta(m_2, m_1) \geq 0 \quad \forall m_1, m_2 \in \mathbb{R}_+$$

where $\beta(m, m')$ denotes the rate of occurrence of coagulation of two droplets with mass m and with mass m'. Here we suppose that for some $M > \overline{m}_a$,

$$\beta(m', m'') = 0 \quad \text{for} \quad m' + m'' \geq M. \tag{2.14}$$

The appearance of droplets of mass m is represented by $g_0(m)[N^* - \widetilde{N}(\sigma)]^+[\pi - \pi_{vs(l)}(T)]^+$, where N^* and $\tilde{N}(\sigma)$ should represent respectively the total number of aerosols susceptible to the formation of droplets in the unit volume and the number in the unit volume of aerosols already present in droplets. The disappearance of droplet of mass m is represented by $g_1(m)[\pi - \pi_{vs(l)}(T)]^-\sigma$. For the coefficient functions $g_0(m)$ and $g_1(m)$, we suppose that they are sufficiently regular and

$$\text{supp } g_0(\cdot) \subset [\overline{m}_a, \overline{m}_A], \qquad \text{supp } g_1(\cdot) \subset [0, \overline{m}_A]. \tag{2.15}$$

The functions $b_\lambda(t, x)$ and $J_\lambda(t, x, q_1, I_\lambda, T)$ are defined by the relations

$$b_\lambda(t, x) = (a_\lambda^{(1)} + r_\lambda^{(1)})\varrho(t, x) + (a_\lambda^{(2)} + r_\lambda^{(2)})\pi(t, x)$$

$$+ \int_0^\infty (a_\lambda^{(3)}(m) + r_\lambda^{(3)}(m))\sigma(t, m, x)dm, \tag{2.16}$$

$$J_\lambda(t, x, q_1, I_\lambda, T) = \frac{1}{4\pi}r_\lambda^{(1)}\varrho(t, x) \int_{S^2} I_\lambda(x, q_1')P_\lambda^{(1)}(q_1' \cdot q_1)dq_1'$$

$$+ \frac{1}{4\pi}r_\lambda^{(2)}\pi(t, x) \int_{S^2} I_\lambda(x, q_1')P_\lambda^{(2)}(q_1' \cdot q_1)dq_1'$$

$$+ \frac{1}{4\pi}\int_0^\infty r_\lambda^{(3)}(m)\sigma(t, m, x)dm \int_{S^2} I_\lambda(x, q_1')P_\lambda^{(3)}(q_1' \cdot q_1)dq_1'$$

$$+ \left(a_\lambda^{(1)}\varrho(t, x) + a_\lambda^{(2)}\pi(t, x)\right.$$

$$\left. + \int_0^\infty a_\lambda^{(3)}(m)\sigma(t, m, x)dm\right)B[\lambda, T(t, x)], \tag{2.17}$$

where $a_\lambda^{(1)}$ is the absorbtion coefficient, $r_\lambda^{(1)}$ is the diffusion coefficient of radiation and $P_\lambda^{(1)}(q_1' \cdot q_1)$ is the diffusion coefficient of radiation by the dry air of direction q_1' in the direction q_1, while $a_\lambda^{(2)}$, $r_\lambda^{(2)}$ and $P_\lambda^{(2)}(q_1' \cdot q_1)$ are respectively the absorption coefficient and the diffusion coefficient of radiation and the diffusion coefficient of radiation from the direction q_1' to the direction q_1 by the water vapor, and $a_\lambda^{(3)}(m)$, $r_\lambda^{(3)}(m)$ and $P_\lambda^{(3)}(q_1' \cdot q_1)$ are respectively the absorption coefficient and the diffusion coefficient of radiation and the diffusion coefficient of radiation from the direction q_1' to the direction q_1 by liquid water (or solid water); $\big(a_\lambda^{(1)}\varrho(t, x) + a_\lambda^{(2)}\pi(t, x) + \int_0^\infty a_\lambda^{(3)}(m)\sigma(m, t, x)dm\big)B[\lambda, T(x)]$ is the emission of radiation, where the function $B[\lambda, T]$, called *Planck's function*, is given by

$$B[\lambda, T] = \frac{2\pi c^2 h}{\lambda^5}(e^{ch/k\lambda T} - 1)^{-1} \tag{2.18}$$

(here c is the speed of light, h is the Planck constant and k is the Boltzmann constant).

For the velocity of water droplets, $u(m, t, x)$, we assume the relation

$$u(t, m, x) = v(t, x) - \frac{1}{\alpha_l(m)}\nabla\Phi, \tag{2.19}$$

where $\alpha_l(m)$ is the friction coefficient between the droplets with mass m and the air.

3. Position of the problem

Consider the equation system (2.1)–(2.6) in a bounded domain $\Omega \subset \mathbb{R}^3$ with the initial and boundary conditions

$$\varrho(0, \cdot) = \varrho_0(\cdot) \in W_p^1(\Omega), \quad \inf_{x \in \Omega} \varrho_0(x) > 0, \tag{3.1}$$

$$\pi(0, \cdot) = \pi_0(\cdot) \in W_p^1(\Omega), \quad \inf_{x \in \Omega} \pi_0(x) > 0, \tag{3.2}$$

$$\sigma(0, \cdot, \cdot) = \sigma_0(\cdot, \cdot) \in W_p^1(\mathbb{R}_+ \times \Omega), \quad \sigma_0(\cdot, \cdot) \geq 0, \tag{3.3}$$

$$v|_{\partial\Omega} = 0, \quad v(0, \cdot) = v_0(\cdot) \in W_p^{2-2/p}, \quad v_0|_{\partial\Omega} = 0, \tag{3.4}$$

$$\nabla T \cdot n\big|_{\partial\Omega} = 0, \quad T(0, \cdot) = T_0(\cdot) \in W_q^{2-2/q}. \tag{3.5}$$

For σ_0 we suppose

$$\exists \overline{M}' \geq \overline{m}_A > 0 \text{ such that } \sigma_0(m, \cdot) = 0 \quad \text{if } m \in]0, \overline{m}_a] \cup [\overline{M}', \infty[, \quad (3.6)$$

$$\partial_m \sigma_0 \in L^\infty(\mathbb{R}_+ \times \Omega), \quad (3.7)$$

we define

$$\Omega_M =]0, M[\times\Omega, \qquad \forall M > 0.$$

To specify the boundary conditions for $\{I_\lambda\}_{\lambda>0}$, it is convenient to transform the equation (2.6) into an integral equation, so that we can rewrite the equation (2.6) in the form

$$\frac{d}{d\alpha} I_\lambda(x+\alpha q_1, q_1) = -b_\lambda(t, x) I_\lambda(x+\alpha q_1, q_1) + J_\lambda(t, x+\alpha q_1, q_1, I_\lambda, T). \quad (3.8)$$

For $(x, q_1) \in \Omega \times S^2$ we define

$$\alpha^0_{(x,q_1)} = \inf\{\, \alpha \in \mathbb{R} \mid x + \alpha' q_1 \in \Omega, \ \forall \alpha' \in]\alpha, 0[\,\}. \quad (3.9)$$

The equation (3.8) with the condition

$$I_\lambda(x + \alpha^0_{(x,q_1)} q_1, q_1) = I^0_\lambda(x + \alpha^0_{(x,q_1)} q_1, q_1) \quad (3.10)$$

can be transformed into the integral equation

$$I_\lambda(x, q_1) = I^0_\lambda(x + \alpha^0_{(x,q_1)} q_1, q_1) e^{-I_b(x, \alpha^0_{(x,q_1)}, q_1)}$$

$$+ \frac{r^{(1)}_\lambda}{4\pi} \int_{\alpha^0_{(x,q_1)}}^0 \varrho(t, x + \alpha' q_1)$$

$$\times \int_{S^2} P^{(1)}_\lambda(q'_1 \cdot q_1) I_\lambda(x + \alpha' q_1, q'_1) e^{-I_b(x, \alpha', q_1)} dq'_1 d\alpha'$$

$$+ \frac{r^{(2)}_\lambda}{4\pi} \int_{\alpha^0_{(x,q_1)}}^0 \pi(t, x + \alpha' q_1)$$

$$\times \int_{S^2} P^{(2)}_\lambda(q'_1 \cdot q_1) I_\lambda(x + \alpha' q_1, q'_1) e^{-I_b(x, \alpha', q_1)} dq'_1 d\alpha'$$

$$+ \frac{1}{4\pi} \int_0^\infty r_\lambda^{(3)}(m) \int_{\alpha_{(x,q_1)}^0}^0 \sigma(t, m, x + \alpha' q_1)$$

$$\times \int_{S^2} I_\lambda(x + \alpha' q_1, q_1') P_\lambda^{(3)}(q_1' \cdot q_1) e^{-I_b(x, \alpha', q_1)} dq_1' d\alpha' dm$$

$$+ \int_{\alpha_{(x,q_1)}^0}^0 \left(a_\lambda^{(1)} \varrho(t, x + \alpha' q_1) + a_\lambda^{(2)} \pi(t, x + \alpha' q_1) \right.$$

$$\left. + \int_0^\infty a_\lambda^{(3)}(m) \sigma(t, m, x + \alpha' q_1) dm \right)$$

$$\times B[\lambda, T(t, x + \alpha' q_1)] e^{-I_b(x, \alpha', q_1)} d\alpha', \tag{3.11}$$

where

$$I_b(x, \alpha, q_1) = \int_\alpha^0 b_\lambda(t, x + \alpha' q_1) d\alpha'. \tag{3.12}$$

We remark that in (3.11) $(x + \alpha_{(x,q_1)}^0 q_1, q_1)$ should belong to the set

$$\Xi = \bigcup_{x^0 \in \partial\Omega} \left(\{x^0\} \times S_-^2(x^0) \right), \tag{3.13}$$

where

$$S_-^2(x^0) = \{q_1 \in S^2 \mid \exists \varepsilon > 0, \, x^0 + \alpha q_1 \in \Omega, \, \forall \alpha \in]0, \varepsilon[\} \tag{3.14}$$

$(x^0 \in \partial\Omega)$.

For the diffusion rate $P_\lambda^{(i)}(q_1' \cdot q_1)$ and for the diffusion coefficients $r_\lambda^{(i)}$ $(i = 1, 2, 3)$ we suppose

$$P_\lambda^{(i)}(q_1' \cdot q_1) \geq 0 \quad \forall q_1', q_1 \in S^2, \qquad \frac{1}{4\pi} \int_{S^2} P_\lambda^{(i)}(q_1' \cdot q_1) dq_1 = 1 \quad \forall q_1' \in S^2. \tag{3.15}$$

$$\sup_{x \in \Omega} \int_0^\infty r_\lambda^{(3)}(m) \sigma_0(x, m) dm \leq 4, \qquad \int_0^\infty (a_\lambda^{(3)}(m) + r_\lambda^{(3)}(m)) dm) < \infty, \tag{3.16}$$

moreover, we assume that there is a strictly positive constant ε_1 such that

$$\sup_{x \in \Omega, -1 \le c \le 1} \left(r_\lambda^{(1)} P_\lambda^{(1)}(c) \varrho_0(x) + r_\lambda^{(2)} P_\lambda^{(2)}(c) \pi_0(x) \right) \le \frac{\varepsilon_1}{2}, \tag{3.17}$$

$$\sup_{\lambda \in \mathbb{R}_+} \left(K_b \sup_{-1 \le c \le 1} P_\lambda^{(3)}(c) \right)^{1/2} + \frac{\varepsilon_1}{2} < 1, \tag{3.18}$$

where

$$K_b = \sup_{x \in \Omega, q_1 \in S^2} \left(1 - e^{-2 I_{b^0}(x, \alpha_{(x, q_1)}^0, q_1)} \right), \tag{3.19}$$

with $I_{b^0}(x, \alpha_{(x, q_1)}^0, q_1)$ defined in an analogous way to (3.12) but with

$$b_\lambda^0(x) = 2(a_\lambda^{(1)} + r_\lambda^{(1)}) \varrho_0(x) + 2(a_\lambda^{(2)} + r_\lambda^{(2)}) \pi_0(x)$$

$$+ 2 \int_0^\infty (a_\lambda^{(3)} + r_\lambda^{(3)}) \sigma_0(m, x) dm + \varepsilon_2$$

instead of b_λ, where ε_2 is a strictly positive constant and sufficiently small.

It is not restrictive to suppose that the diameter of the domain Ω is equal to 1 because we can transform a generic bounded domain into a domain with diameter equal to 1 by a simple change of variables.

For the function Φ we suppose

$$\Phi \in C^3(\Omega), \qquad \nabla \Phi \cdot n = 0 \quad \text{on} \ \partial\Omega \tag{3.20}$$

(n is the unit outward normal vector to $\partial\Omega$).

The main result of this paper is the following theorem.

Theorem 3.1 *Let us assume $p > 4$, $2q > p > q > 3$ and the conditions (3.1)–(3.5), (3.10), (3.15)–(3.18). Then there exists a number $\bar{t} > 0$ such that the problem (2.1)–(2.6) admits a unique solution $(\varrho, \pi, \sigma, v, T, I_\lambda)$ with the following properties*:

$$\varrho \in C^0([0, \bar{t}]; W_p^1(\Omega)), \qquad \inf_{(t,x) \in Q_{\bar{t}}} \varrho(t, x) > 0, \tag{3.21}$$

$$\pi \in C^0([0, \bar{t}]; W_p^1(\Omega)), \quad \pi \ge 0, \tag{3.22}$$

$$\sigma \in C^0([0,\overline{t}]; W_p^1(\mathbf{R}_+ \times \Omega)), \quad \sigma \geq 0,$$

$$\partial_m \sigma \in C^0([0,\overline{t}]; L^\infty(\mathbf{R}_+ \times \Omega)), \tag{3.23}$$

$$v \in W_p^{2,1}([0,\overline{t}] \times \Omega), \qquad T \in W_q^{2,1}([0,\overline{t}] \times \Omega), \quad T > 0, \tag{3.24}$$

$$I_\lambda \in L^\infty(\Omega \times S^2) \tag{3.25}$$

with

$$\|\varphi\|_{W_p^{2,1}([0,t]\times\Omega)}^p = \|\varphi\|_{L^p(0,t;W_p^2(\Omega))}^p + \|\partial_t \varphi\|_{L^p([0,t]\times\Omega)}^p$$

4.   Equation of radiation intensity

In this section, supposing that ϱ, π, σ and T are given, we prove the existence and uniqueness of the solution to the equation (3.11) with fixed λ and t, and give an estimate of the difference of two solutions of (3.11) with the same λ and t but different ϱ, π, σ and T.

Lemma 4.1   *Let be $I_\lambda^0(x^0, q_1)$ a non-negative measurable function defined on Ξ. We suppose that*

$$\sup_{(x^0,q_1)\in\Xi} I_\lambda^0(x^0, q_1) < \infty. \tag{4.1}$$

If the functions $\varrho(t,\cdot)$, $\pi(t,\cdot)$ and $T(t,\cdot)$ are given in $L^\infty(\Omega)$ and $\sigma(t,\cdot,\cdot)$ is given in $L^\infty(\mathbb{R}_+ \times \Omega)$, then the equation (3.11) admits a unique solution I_λ in $L^\infty(\Omega \times S^2)$ and we have

$$\sup_{(x,q_1)\in\Omega\times S^2} I_\lambda(x, q_1) \leq \frac{1}{\varepsilon_b} \left[\sup_{(x^0,q_1)\in\Xi} I_\lambda^0(x^0, q_1) + \sup_{\overline{T}_0^{(-)}/2 \leq T \leq 3\overline{T}_0^{(+)}/2} B[\lambda, T] \right], \tag{4.2}$$

where

$$\varepsilon_b = \varepsilon_b(t) = \inf_{(x,q_1)\in\Omega\times S^2} e^{-I_b(x,\alpha_{x,q_1}^0,q_1)},$$

$$\overline{T}_0^{(-)} = \inf_{x\in\Omega} T_0(x), \qquad \overline{T}_0^{(+)} = \sup_{x\in\Omega} T_0(x).$$

Proof. First, supposing that $I_\lambda(x, q_1)$ is a solution to (3.11), we prove the inequality (4.2), setting

$$\overline{A} = \sup_{(x,q_1)\in\Omega\times S^2} I_\lambda(x, q_1), \qquad \overline{B} = \sup_{\overline{T}_0^{(-)}/2 \leq T \leq 3\overline{T}_0^{(+)}/2} B[\lambda, T],$$

$$\overline{I} = \sup_{(x^0,q_1)\in\Xi} I_\lambda^0(x^0, q_1).$$

From the equation (3.11), we deduce

$$I_\lambda(x, q_1) \leq \overline{I} + \overline{A} \int_{\alpha_{(x,q_1)}^0}^0 b_\lambda(t, x + \alpha' q_1) e^{-I_b(x,\alpha',q_1)} d\alpha'$$

$$+ \overline{B} \int_{\alpha_{(x,q_1)}^0}^0 b_\lambda(t, x + \alpha' q_1) e^{-I_b(x,\alpha',q_1)} d\alpha'.$$

As

$$\int_{\alpha_{(x,q_1)}^0}^0 b_\lambda(t, x + \alpha' q_1) e^{-I_b(x,\alpha',q_1)} d\alpha' < 1 - \varepsilon_b$$

(see (2.16)), we obtain

$$\overline{A} \leq \overline{I} + (1 - \varepsilon_b)(\overline{A} + \overline{B}).$$

From this inequality we deduce (4.2).

To prove the existence and the uniqueness of the solution I_λ, we denote by $G(I_\lambda)$ the second member of (3.11). So we have

$$|G(I_\lambda^{[1]})(x, q_1) - G(I_\lambda^{[2]})(x, q_1)|$$

$$\leq \sup_{(x,q_1)\in\Omega\times S^2} |I_\lambda^{[1]} - I_\lambda^{[2]}| \frac{1}{4\pi} \int_{\alpha_{(x,q_1)}^0} \int_{S^2} \left(r_\lambda^{(1)} \varrho(t, x + \alpha' q_1) P_\lambda^{(1)}(q_1' \cdot q_1) \right.$$

$$+ r_\lambda^{(2)} \pi(t, x + \alpha' q_1) P_\lambda^{(2)}(q_1' \cdot q_1)$$

$$\left. + \int_0^\infty r_\lambda^{(3)}(m) \sigma(t, m, x + \alpha' q_1) P_\lambda^{(3)}(q_1' \cdot q_1) dm \right) e^{-I_b(x,\alpha',q_1)} dq_1' d\alpha'.$$

$$(4.3)$$

Hence, using the properties on $P_\lambda^{(i)}$ (see (3.15)), we obtain

$$
\left| G(I_\lambda^{[1]})(x, q_1) - G(I_\lambda^{[2]})(x, q_1) \right|
$$

$$
\leq \int_{\alpha_{(x,q_1)}^0}^0 b_\lambda(t, x + \alpha' q_1) e^{-I_b(x,\alpha',q_1)} d\alpha' \sup_{(x,q_1)\in\Omega\times S^2} \left| I_\lambda^{[1]} - I_\lambda^{[2]} \right|
$$

$$
\leq \left(1 - e^{-I_b(x,\alpha_{x,q_1}^0,q_1)}\right) \sup_{(x,q_1)\in\Omega\times S^2} \left| I_\lambda^{[1]} - I_\lambda^{[2]} \right|, \tag{4.4}
$$

where $I_b(x, \alpha', q_1)$ is defined by (3.12). That is, the operator $G(\cdot)$ is a contraction in $L^\infty(\Omega \times S^2)$, so that the equation (3.11) has a unique solution in $L^\infty(\Omega \times S^2)$. □

Now, we are going to prove an estimate for the divergence of $\mathcal{E}$.

Lemma 4.2 *Let us assume $\varrho(t, \cdot), \pi(t, \cdot) \in L^p(\Omega)$, $\sigma(t, \cdot, \cdot) \in L^p(\Omega; L^\infty(\mathbb{R}_+))$, $I_\lambda \in L^\infty(\Omega \times S^2)$, $B[\lambda, T(\cdot)] \in L^\infty(\Omega)$. Then there exists a positive constant c such that*

$$
\|\nabla \cdot \mathcal{E}\|_{L^q(\Omega)} \leq c\left(\left\| \int_0^\infty I_\lambda(x, q) d\lambda \right\|_{L^\infty(\Omega\times S^2)} + \left\| \int_0^\infty B[\lambda, T(x)] d\lambda \right\|_{L^\infty(\Omega)} \right)
$$

$$
\times \left(\|\varrho\|_{L^p(\Omega)} + \|\pi\|_{L^p(\Omega)} + \|\sigma\|_{L^p(\Omega; L^\infty(\mathbb{R}_+))} \right). \tag{4.5}
$$

Proof. From the definitions (2.6) and (2.8), we have

$$
\|\nabla \cdot \mathcal{E}\|_{L^q(\Omega)} \leq \left\| \int_0^\infty \int_{S^2} b_\lambda(x, q_1) I_\lambda(x, q_1) dq_1 d\lambda \right\|_{L^q(\Omega)}
$$

$$
+ \left\| \int_0^\infty \int_{S^2} J_\lambda(x, q_1, I_\lambda, T) dq_1 d\lambda \right\|_{L^q(\Omega)}.
$$

As $p > q$, by elementary calculus (see also the conditions (3.15), (3.16)), we obtain the inequality (4.5) from the definitions (2.16) and (2.17). □

We prove also some estimates for the difference between two possible functions representing the radiation intensity I_λ.

Let $I_\lambda^{[i]}(x, q_1)$, $i = 1, 2$, two functions in $L^\infty(\Omega \times S^2)$ verifying the equation (3.11) with $\varrho = \varrho_i$, $\pi = \pi_i$, $\sigma = \sigma_i$, $T = T_i$, $i = 1, 2$. Then we have

$$I_\lambda^{[1]}(x, q_1) - I_\lambda^{[2]}(x, q_1) = \Delta_0 I_\lambda + \Delta_1 I_\lambda + \Delta_2 I_\lambda, \tag{4.6}$$

where

$$\Delta_0 I_\lambda = I_\lambda^0(x + \alpha_{(x,q_1)}^0 q_1, q_1)(e^{-I_{b_1}(x,\alpha^0,q_1)} - e^{-I_{b_2}(x,\alpha^0,q_1)})$$

$$+ \frac{1}{4\pi} \int_{\alpha_{(x,q_1)}^0}^0 \int_{S^2} \left(r_\lambda^{(1)} P_\lambda^{(1)}(q_1' \cdot q_1)(\varrho_1 - \varrho_2)(x + \alpha' q_1) \right.$$

$$+ r_\lambda^{(2)} P_\lambda^{(2)}(q_1' \cdot q_1)(\pi_1 - \pi_2)(x + \alpha' q_1)$$

$$\left. + \int_0^\infty r_\lambda^{(3)}(m) P_\lambda^{(3)}(q_1' \cdot q_1)(\sigma_1 - \sigma_2)(m, x + \alpha' q_1) dm \right)$$

$$\times e^{-I_{b_2}(x,\alpha',q_1)} I_\lambda^{[2]}(x + \alpha' q_1, q_1') dq_1' d\alpha'$$

$$+ \frac{1}{4\pi} \int_{\alpha_{(x,q_1)}^0}^0 \int_{S^2} \left(r_\lambda^{(1)} P_\lambda^{(1)}(q_1' \cdot q_1)\varrho_1(x + \alpha' q_1) \right.$$

$$+ r_\lambda^{(2)} P_\lambda^{(2)}(q_1' \cdot q_1)\pi_1(x + \alpha' q_1)$$

$$\left. + \int_0^\infty r_\lambda^{(3)}(m) P_\lambda^{(3)}(q_1' \cdot q_1)\sigma_1(m, x + \alpha' q_1) dm \right)$$

$$\times (e^{-I_{b_1}(x,\alpha',q_1)} - e^{-I_{b_2}(x,\alpha',q_1)}) I_\lambda^{[2]}(x + \alpha' q_1, q_1') dq_1' d\alpha'$$

$$+ \int_{\alpha_{(x,q_1)}^0}^0 \left(a_\lambda^{(1)} \varrho_1(x + \alpha' q_1) + a_\lambda^{(2)} \pi_1(x + \alpha' q_1) \right.$$

$$\left. + \int_0^\infty a_\lambda^{(3)}(m)\sigma_1(m, x + \alpha' q_1) \right)$$

$$\times \left[(B[\lambda, T_1] - B[\lambda, T_2]) e^{-I_{b_1}(x,\alpha',q_1)} \right.$$

$$\left. + (e^{-I_{b_1}(x,\alpha',q_1)} - e^{-I_{b_2}(x,\alpha',q_1)}) B[\lambda, T_2] \right] d\alpha'$$

$$+ \int_{\alpha_{(x,q_1)}^0}^0 \left(a_\lambda^{(1)}(\varrho_1 - \varrho_2)(x + \alpha' q_1) + a_\lambda^{(2)}(\pi_1 - \pi_2)(x + \alpha' q_1) \right.$$

$$\left. + \int_0^\infty a_\lambda^{(3)}(m)(\sigma_1 - \sigma_2)(m, x + \alpha' q_1) dm \right)$$

$$\times e^{-I_{b_2}(x,\alpha',q_1)} B[\lambda, T_2] d\alpha', \tag{4.7}$$

$$\Delta_1 I_\lambda = \frac{1}{4\pi} \int_{\alpha^0_{(x,q_1)}}^0 \int_{S^2} \left(r_\lambda^{(1)} P_\lambda^{(1)}(q_1' \cdot q_1) \varrho_1(x + \alpha' q_1) \right.$$

$$\left. + r_\lambda^{(2)} P_\lambda^{(2)}(q_1' \cdot q_1) \pi_1(x + \alpha' q_1) \right)$$

$$\times e^{-I_{b_1}(x,\alpha',q_1)} (I_\lambda^{[1]} - I_\lambda^{[2]})(x + \alpha' q_1, q_1') dq_1' d\alpha', \tag{4.8}$$

$$\Delta_2 I_\lambda = \frac{1}{4\pi} \int_{\alpha^0_{(x,q_1)}}^0 \int_{S^2} \int_0^\infty r_\lambda^{(3)}(m) P_\lambda^{(3)}(q_1' \cdot q_1) \sigma_1(m, x + \alpha' q_1) dm$$

$$\times e^{-I_{b_1}(x,\alpha',q_1)} (I_\lambda^{[1]} - I_\lambda^{[2]})(x + \alpha' q_1, q_1') dq_1' d\alpha'; \tag{4.9}$$

here $I_{b_i}(x, \alpha', q_1)$ denotes the function $I_b(x, \alpha, q_1) = \int_\alpha^0 b_\lambda(t, x + \alpha' q_1) d\alpha'$ with $\varrho = \varrho_i$, $\pi = \pi_i$, $\sigma = \sigma_i$, $i = 1, 2$.

Lemma 4.3 *Let us assume* (3.16) *and* (3.17). *Then we have*

$$|\Delta_1 I_\lambda| \le \frac{\varepsilon_1}{4\pi} \int_{\alpha^0_{(x,q_1)}}^0 \int_{S^2} |I_\lambda^{[1]} - I_\lambda^{[2]}|(x + \alpha' q_1, q_1') dq_1' d\alpha', \tag{4.10}$$

$$|\Delta_2 I_\lambda| \le K_b^{1/2} \left(\frac{1}{4\pi} \int_{S^2} P_\lambda^{(3)}(q_1' \cdot q_1) \right.$$

$$\left. \times \int_{\alpha^0_{(x,q_1)}}^0 (I_\lambda^{[1]} - I_\lambda^{[2]})^2 (x + \alpha' q_1, q_1') d\alpha' dq_1' \right)^{1/2}, \tag{4.11}$$

where

$$K_b = \sup_{x \in \Omega, q_1 \in S^2} \left(1 - e^{-2 I_{b_1}(x, \alpha^0_{(x,q_1)}, q_1)} \right). \tag{4.12}$$

Proof. The inequality (4.10) results immediately from (3.17).
On the other hand, using the Cauchy-Schwartz inequality, we have

$$|\Delta_2 I_\lambda| \le \frac{1}{4\pi} \int_{S^2} P_\lambda^{(3)}(q_1' \cdot q_1) \left(\int_{\alpha^0_{(x,q_1)}}^0 \left(\int_0^\infty r_\lambda^{(3)}(m) \sigma_1(m, x + \alpha' q_1) dm \right)^2 \right.$$

$$\left. \times e^{-2 I_{b_1}(x,\alpha',q_1)} d\alpha' \right)^{1/2}$$

$$\times \left(\int_{\alpha^0_{(x,q_1)}}^{0} (I_\lambda^{[1]} - I_\lambda^{[2]})^2 (x + \alpha' q_1, q'_1) d\alpha' \right)^{1/2} dq'_1.$$

We remark that the condition (3.16) and the definitions of $I_{b_1}(x, \alpha', q_1)$ and $b_\lambda(t, x)$ imply

$$\int_{\alpha^0_{(x,q_1)}}^{0} \left(\int_0^\infty r_\lambda^{(3)}(m) \sigma_1(m, x + \alpha' q_1) dm \right)^2 e^{-2 I_{b_1}(x, \alpha', q_1)} d\alpha'$$

$$\leq \int_{\alpha^0_{(x,q_1)}}^{0} \left(2 \int_0^\infty r_\lambda^{(3)}(m) \sigma_1(m, x + \alpha' q_1) dm \right) e^{-2 I_{b_1}(x, \alpha', q_1)} d\alpha'$$

$$\leq 1 - e^{-2 I_{b_1}(x, \alpha^0_{(x,q_1)}, q_1)}.$$

So we deduce

$$|\Delta_2 I_\lambda| \leq \frac{K_b^{1/2}}{4\pi} \int_{S^2} P_\lambda^{(3)}(q'_1 \cdot q_1) \left(\int_{\alpha^0_{(x,q_1)}}^{0} (I_\lambda^{[1]} - I_\lambda^{[2]})^2 (x + \alpha' q_1, q'_1) d\alpha' \right)^{1/2} dq'_1.$$

As $(1/4\pi) \int_{S^2} P_\lambda^{(3)}(q'_1 \cdot q_1) dq'_1 = 1$, by applying again the Cauchy-Schwartz inequality to the right-hand side of the last inequality, we obtain (4.11). $\square$

Lemma 4.4 *Let $\varphi(x) \geq 0$ for all $x \in \Omega$. Then we have*

$$\frac{1}{4\pi} \int_\Omega \int_{S^2} \int_{\alpha^0_{(x,q_1)}}^{0} \varphi(x + \alpha' q_1) d\alpha' dq_1 dx \leq \int_\Omega \varphi(x) dx. \qquad (4.13)$$

Proof. From

$$\int_{S^2} \int_{\alpha^0_{(x,q_1)}}^{0} \varphi(x + \alpha' q_1) d\alpha' dq_1 \leq \int_\Omega \varphi(x') \frac{1}{|x - x'|^2} dx'$$

we obtain

$$\frac{1}{4\pi} \int_\Omega \int_{S^2} \int_{\alpha^0_{(x,q_1)}}^{0} \varphi(x + \alpha' q_1) d\alpha' dq_1 dx \leq \frac{1}{4\pi} \int_\Omega \left(\int_\Omega \varphi(x') \frac{1}{|x - x'|^2} dx' \right) dx$$

$$= \int_\Omega \varphi(x')dx' \left(\frac{1}{4\pi} \int_\Omega \frac{1}{|x-x'|^2}dx \right)$$

$$= \int_\Omega \varphi(x')dx' \left(\frac{1}{4\pi} \int_0^{\mathrm{dist}(x',\partial\Omega)} \frac{1}{r^2}\mu_2(\Sigma_r^{x'})dr \right),$$

where $\mu_2(\cdot)$ is the Hausdorf measure of dimension 2 and

$$\Sigma_r^{x'} = \{x \in \Omega \mid |x-x'| = r\}.$$

As

$$\mu_2(\Sigma_r^{x'}) \le 4\pi r^2, \qquad \mathrm{dist}(x',\partial\Omega) \le 1,$$

we deduce (4.13). $\qquad\qquad\square$

Lemma 4.5 *We assume the condition* (3.18). *Then there exists a constant C such that*

$$\int_0^\infty \|I_\lambda^{[1]} - I_\lambda^{[2]}\|_{L^2(\Omega \times S^2)}^2 d\lambda$$

$$\le C\left[\|\varrho_2 - \varrho_1\|_{L^2(\Omega)}^2 + \|\pi_2 - \pi_1\|_{L^2(\Omega)}^2 \right.$$

$$\left. + \|\sigma_2 - \sigma_1\|_{L^2(\Omega_{\overline{M}_1})}^2 + \|T_2 - T_1\|_{L^2(\Omega)}^2\right]. \qquad (4.14)$$

Proof. Using (4.10) and the Cauchy-Schwartz inequality, we have

$$\int_\Omega \int_{S^2} \Delta_1 I_\lambda (I_\lambda^{[1]} - I_\lambda^{[2]})dq_1 dx$$

$$\le \frac{\varepsilon_1}{4\pi} \left(\int_\Omega \int_{S^2} |I_\lambda^{[1]} - I_\lambda^{[2]}|^2 dq_1 dx \right)^{1/2}$$

$$\times \left(\int_\Omega dx \int_{S^2} dq_1 \left(\int_{\alpha_{(x,q_1)}^0}^0 \int_{S^2} |I_\lambda^{[1]} - I_\lambda^{[2]}|(x+\alpha'q_1, q_1')dq_1' d\alpha' \right)^2 \right)^{1/2}.$$

Now, taking into account the condition $|\alpha_{(x,q_1)}^0| \le 1$, we have

$$\frac{1}{\sqrt{4\pi}} \int_{\alpha^0_{(x,q_1)}}^{0} \int_{S^2} |I_\lambda^{[1]} - I_\lambda^{[2]}|(x + \alpha' q_1, q_1') dq_1' d\alpha'$$

$$\leq \left(\int_{\alpha^0_{(x,q_1)}}^{0} \int_{S^2} |I_\lambda^{[1]} - I_\lambda^{[2]}|^2 (x + \alpha' q_1, q_1') dq_1' d\alpha' \right)^{1/2}.$$

Changing the order of the integration with respect to q_1 and with respect to q_1' and applying Lemma 4.4 to $\varphi(x + \alpha' q_1) = |I_\lambda^{[1]} - I_\lambda^{[2]}|(x + \alpha' q_1, q_1')$ for each fixed q_1', we obtain

$$\frac{1}{(4\pi)^2} \int_\Omega dx \int_{S^2} dq_1 \left(\int_{\alpha^0_{(x,q_1)}}^{0} \int_{S^2} |I_\lambda^{[1]} - I_\lambda^{[2]}|(x + \alpha' q_1, q_1') dq_1' d\alpha' \right)^2$$

$$\leq \frac{1}{4\pi} \int_{S^2} dq_1' \int_\Omega dx \int_{S^2} \int_{\alpha^0_{(x,q_1)}}^{0} |I_\lambda^{[1]} - I_\lambda^{[2]}|^2 (x + \alpha' q_1, q_1') d\alpha' dq_1$$

$$\leq \int_{S^2} dq_1' \int_\Omega dx |I_\lambda^{[1]} - I_\lambda^{[2]}|^2 (x, q_1') = \|I_\lambda^{[1]} - I_\lambda^{[2]}\|^2_{L^2(\Omega \times S^2)}.$$

We deduce

$$\int_\Omega \int_{S^2} \Delta_1 I_\lambda (I_\lambda^{[1]} - I_\lambda^{[2]}) dq_1 dx \leq \varepsilon_1 \|I_\lambda^{[1]} - I_\lambda^{[2]}\|^2_{L^2(\Omega \times S^2)}. \tag{4.15}$$

On the other hand, according to (4.11) we have

$$\int_\Omega \int_{S^2} \Delta_2 I_\lambda (I_\lambda^{[1]} - I_\lambda^{[2]}) dq_1 dx$$

$$\leq K_b^{1/2} \|I_\lambda^{[1]} - I_\lambda^{[2]}\|_{L^2(\Omega \times S^2)}$$

$$\times \left(\frac{1}{4\pi} \int_\Omega dx \int_{S^2} dq_1 \int_{S^2} P_\lambda^{(3)}(q_1' \cdot q_1) \right.$$

$$\times \left. \int_{\alpha^0_{(x,q_1)}}^{0} (I_\lambda^{[1]} - I_\lambda^{[2]})^2 (x + \alpha' q_1, q_1') d\alpha' dq_1' \right)^{1/2}.$$

From Lemma 4.4, we obtain

$$\frac{1}{4\pi} \int_\Omega dx \int_{S^2} dq_1 \int_{S^2} P_\lambda^{(3)}(q_1' \cdot q_1) \int_{\alpha^0_{(x,q_1)}}^0 (I_\lambda^{[1]} - I_\lambda^{[2]})^2(x + \alpha' q_1, q_1') d\alpha' dq_1'$$

$$\leq \sup_{-1 \leq c \leq 1} P_\lambda^{(3)}(c) \int_\Omega \int_{S^2} (I_\lambda^{[1]} - I_\lambda^{[2]})^2(x, q_1') dq_1' dx$$

$$= \sup_{-1 \leq c \leq 1} P_\lambda^{(3)}(c) \|I_\lambda^{[1]} - I_\lambda^{[2]}\|_{L^2(\Omega \times S^2)}^2.$$

Consequently, we have

$$\int_\Omega \int_{S^2} \Delta_2 I_\lambda (I_\lambda^{[1]} - I_\lambda^{[2]}) dq_1 dx$$

$$\leq \left(K_b \sup_{-1 \leq c \leq 1} P_\lambda^{(3)}(c) \right)^{1/2} \|I_\lambda^{[1]} - I_\lambda^{[2]}\|_{L^2(\Omega \times S^2)}^2. \qquad (4.16)$$

For the term $\Delta_0 I_\lambda$, using the intermediate value theorem, we have

$$|B[\lambda, T_1] - B[\lambda, T_2]| \leq \sup_{\overline{T}_0^{(-)}/2 \leq T \leq 3\overline{T}_0^{(+)}/2} \frac{\partial}{\partial T} B[\lambda, T] |T_2 - T_1|,$$

while the other terms of $\Delta_0 I_\lambda$ can be estimated, without difficulties, by a usual way. Thus applying Lemma 4.4, we find the inequality

$$\int_\Omega \int_{S^2} \Delta_0 I_\lambda (I_\lambda^{[1]} - I_\lambda^{[2]}) dq_1 dx$$

$$\leq C_\lambda \big(\|\varrho_2 - \varrho_1\|_{L^2(\Omega)} + \|\pi_2 - \pi_1\|_{L^2(\Omega)}$$

$$+ \|\sigma_2 - \sigma_1\|_{L^2(\Omega_{\overline{M}_1})} + \|T_2 - T_1\|_{L^2(\Omega)} \big) \|I_\lambda^{[1]} - I_\lambda^{[2]}\|_{L^2(\Omega \times S^2)},$$

where C_λ is a constant depending on I_λ^0, $B[\lambda, T_2]$ and $(\partial/\partial T)B[\lambda, T]$. $\qquad \square$

5.   Linear equations for densities

In this section we study the equations (2.1)–(2.3) for ϱ, π, σ with a given $(v, T) = (\overline{v}, \overline{T})$. In the following proofs, we will write simply c to denote a constant if its specific value will not be used in the sequel; of course, constants c in different inequalities are different in general.

We introduce the functions spaces

$$\Theta_{t_1}^{(v)} = \{v \in W_p^{2,1}(Q_{t_1})|\, v \text{ satisfies (3.4)}\}, \tag{5.1}$$

$$\Theta_{t_1}^{(T)} = \{T \in W_q^{2,1}(Q_{t_1})|\, T \text{ satisfies (3.5)}\}, \tag{5.2}$$

where $Q_{t_1} =]0, t_1[\times\Omega$.

Let be $(\overline{v}, \overline{T}) \in \Theta_{t_1}^{(v)} \times \Theta_{t_1}^{(T)}$. We consider the equations (2.1)–(2.3) for ϱ, π, σ with $(v, T) = (\overline{v}, \overline{T})$

$$\partial_t \varrho + \nabla \cdot (\varrho \overline{v}) = 0, \tag{5.3}$$

$$\partial_t \pi + \nabla \cdot (\pi \overline{v}) = -H_{gl}(\overline{T}, \pi, \sigma), \tag{5.4}$$

$$\frac{\partial \sigma}{\partial t} + \nabla_{(m,x)} \cdot (\sigma \widetilde{U}_{4l}(\overline{u}, \overline{T}, \pi))$$
$$= \big[h_{gl}(\overline{T}, \pi; m) + B_1(\sigma; m) - g_1(m)[\pi - \overline{\pi}_{vs(l)}(\overline{T})]^- \big]\sigma$$
$$+ g_0(m)[N^* - \widetilde{N}(\sigma)]^+ [\pi - \overline{\pi}_{vs(l)}(\overline{T})]^+ + B_2(\sigma; m), \tag{5.5}$$

where

$$\overline{u} = \overline{v} - \frac{1}{\alpha_l(m)} \nabla \Phi. \tag{5.6}$$

Before the complete analysis of the equation system (5.3)–(5.5), in this section, we study the equation (5.3), which is linear, and the linearized equations for (5.4) and (5.5).

For the equation (5.3) we know the following result (see for example [3], [4]).

Lemma 5.1   *Let be $p > 4$ and $\overline{v} \in \Theta_{t_1}^{(v)}$.   The equation (5.3) with the condition (3.1) admits a unique solution*

$$\varrho \in C^0([0, t_1]; W_p^1(\Omega))$$

and we have

$$\|\varrho(t, \cdot)\|_{W_p^1(\Omega)}^p \leq q_\varrho(t), \quad 0 < \alpha_{\varrho(t)} \leq \varrho(t, x) \leq \beta_{\varrho(t)} < \infty \ \ in \ \ Q_{t_1}, \tag{5.7}$$

where

$$\alpha_\varrho(t) = \inf_{x \in \Omega} \varrho_0(x) \exp(-cR_{(\overline{v},t)}t^{(p-1)/p}),$$

$$\beta_\varrho(t) = \sup_{x \in \Omega} \varrho_0(x) \exp(cR_{(\overline{v},t)}t^{(p-1)/p}), \tag{5.8}$$

$$q_{\varrho(t)} = \|\varrho_0\|^p_{W^1_p(\Omega)} \exp(cR_{(\overline{v},t)}t^{(p-1)/p}), \quad R_{(\overline{v},t)} = \|\overline{v}\|_{W^{2,1}_p(Q_t)}.$$

For the linearized equation of (5.4), we have the following lemma.

Lemma 5.2 *Let be $p > 4$, $\overline{v} \in \Theta^{(v)}_{t_1}$, $\overline{T} \in \Theta^{(T)}_{t_1}$, $\overline{\pi} \in C^0([0, t_1]; W^1_p(\Omega))$, $\overline{\sigma} \in C^0([0, t_1]; W^1_p(\mathbf{R}_+ \times \Omega))$. We suppose that there exists a positive number $\overline{M}_1$ such that $\mathrm{supp}(\overline{\sigma}(t, \cdot, \cdot)) \subset \Omega_{\overline{M}_1}$ for every $t \in [0, t_1]$. Then the equation*

$$\partial_t \pi + \nabla \cdot (\pi \overline{v}) = -H_{gl}(\overline{T}, \overline{\pi}, \overline{\sigma}) \tag{5.9}$$

with the initial condition (3.2) admits a unique solution $\pi \in C^0([0, t_1]; W^1_p(\Omega))$ and we have

$$\|\pi(t, \cdot)\|^p_{W^1_p(\Omega)} \le q_\pi(t), \tag{5.10}$$

where

$$q_{\pi(t)} = \left[\|\pi_0\|^p_{W^1_p(\Omega)} + cR_{(\overline{\sigma},t)}(R_{(\overline{T},t)}t^{(q-1)/q} + R_{(\overline{\pi},t)}t)\right]$$

$$\times \exp\left(c(R_{(\overline{v},t)}t^{(p-1)/p} + R_{(\overline{\sigma},t)}(R_{(\overline{T},t)}t^{(q-1)/q} + R_{(\overline{\pi},t)}t))\right),$$

$$R_{(\overline{T},t)} = \|\overline{T}\|_{W^{2,1}_q(Q_t)}, \qquad R_{(\overline{\pi},t)} = \|\overline{\pi}\|_{C^0([0,t];W^1_p(\Omega))}, \tag{5.11}$$

$$R_{(\overline{\sigma},t)} = \|\overline{\sigma}\|_{C^0([0,t],W^1_p(\mathbb{R}_+ \times \Omega))}.$$

Proof. See [14]. $\square$

Lemma 5.3 *Let be $\overline{v} \in \Theta^{(v)}_{t_1}$, $\overline{T} \in \Theta^{(T)}_{t_1}$, $\overline{\pi} \in C^0([0, t_1]; W^1_p(\Omega))$, $\overline{\sigma} \in C^0([0, t_1]; W^1_p(\mathbf{R}_+ \times \Omega))$. Moreover we assume that $\partial_m \overline{\sigma} \in C^0([0, t_1]; L^\infty(\mathbf{R}_+ \times \Omega))$. Then there exists a positive constant $\overline{M}_1$ such that, provided that $\overline{\sigma}(m, \cdot, \cdot) = 0$ for $m \notin]\overline{m}_a/2, \overline{M}_1[$, the equation*

$$\frac{\partial \sigma}{\partial t} + \nabla_{(m,x)} \cdot (\sigma \widetilde{U}_{4l}(\overline{u}, \overline{T}, \overline{\pi}))$$

$$= \left[h_{gl}(\overline{T}, \overline{\pi}; m) + B_1(\overline{\sigma}; m) - g_1(m)[\overline{\pi} - \overline{\pi}_{vs(l)}(\overline{T})]^-\right]\sigma$$

$$+ g_0(m)[N^* - \widetilde{N}(\overline{\sigma})]^+[\overline{\pi} - \overline{\pi}_{vs(l)}(\overline{T})]^+ + B_2(\overline{\sigma}; m), \qquad (5.12)$$

with the initial conditions (3.3)–(3.6) *admit a unique solution* $\sigma \in C^0([0, t_1]; W_p^1(\mathbf{R}_+ \times \Omega))$ *satisfying the following relations*

$$\partial_m \sigma \in C^0([0, t_1]; L^\infty(\mathbf{R}_+ \times \Omega)), \qquad (5.13)$$

$$\sigma(\cdot, m, \cdot) = 0 \quad for \;\; m \notin \left]\frac{\overline{m_a}}{2}, \overline{M}_1\right[, \qquad (5.14)$$

$$\|\sigma(t, \cdot, \cdot)\|_{W_p^1(\mathbf{R}_+ \times \Omega)}^p = \|\sigma(t, \cdot, \cdot)\|_{W_p^1(\Omega_{\overline{M}_1})}^p \le q_\sigma(t), \qquad (5.15)$$

$$\|\partial_m \sigma(t, \cdot)\|_{L^\infty(\mathbf{R}_+ \times \Omega)}$$

$$\le \Bigg[\|\partial_m \sigma_0\|_{L^\infty(\mathbf{R}_+ \times \Omega)} + c \int_0^t \Big(1 + \|\overline{\pi}(s, \cdot)\|_{L^\infty(\Omega)}^2 + \|\overline{\sigma}(s, \cdot)\|_{L^\infty(\mathbf{R}_+ \times \Omega)}^2$$

$$+ \|\sigma(s, \cdot)\|_{L^\infty(\mathbf{R}_+ \times \Omega)}^2 + \|\partial_m \overline{\sigma}(s, \cdot)\|_{L^\infty(\mathbf{R}_+ \times \Omega)}^2 \Big)ds\Bigg]$$

$$\times \exp\Bigg[c \int_0^t \Big(1 + \|\nabla_x \cdot \overline{v}(s, \cdot)\|_{L^\infty(\Omega)} + \|\overline{\pi}(s, \cdot)\|_{L^\infty(\Omega)}$$

$$+ \|\overline{\sigma}(s, \cdot)\|_{L^\infty(\mathbf{R}_+ \times \Omega)} \Big)ds\Bigg], \qquad (5.16)$$

where

$$q_\sigma(t) = \Big\{\|\sigma_0\|_{W_p^1(\Omega_{\overline{M}_1})}^p + c[(1 + R_{(\overline{\pi},t)}^2 + R_{(\overline{\sigma},t)}^2)t$$

$$+ R_{(\overline{v},t)}t^{(p-1)/p} + R_{(\overline{T},t)}^2 t^{(q-2)/q}]\Big\}$$

$$\times \exp\Big\{c[(1 + R_{(\overline{\pi},t)}^2 + R_{(\overline{\sigma},t)}^2)t + R_{(\overline{v},t)}t^{(p-1)/p} + R_{(\overline{T},t)}^2 t^{(q-2)/q}]\Big\}. \qquad (5.17)$$

Proof. For the proof of the existence and uniqueness of the solution in $C^0([0, t_1]; W_p^1(\Omega))$ and the relation (5.14), see [14]. The relation (5.13) will follow from (5.16).

For the inequality (5.15), by taking into account (3.20), (5.6) and (5.14), from (5.5) and integration by parts we deduce

$$\int_{\Omega_{\overline{M}_1}} \sigma^{p-1} \nabla_{(m,x)} \sigma \cdot \widetilde{U}_{4l}(\overline{u}, \overline{T}, \overline{\pi}) dm dx$$

$$= -\frac{1}{p} \int_{\Omega_{\overline{M}_1}} \sigma^p \nabla_{(m,x)} \cdot \widetilde{U}_{4l}(\overline{u}, \overline{T}, \overline{\pi}) dm dx, \tag{5.18}$$

$$\int_{\Omega_{\overline{M}_1}} \left|\nabla_{(m,x)} \sigma\right|^{p-2} \nabla_{(m,x)} \sigma \cdot \left(\widetilde{U}_{4l}(\overline{u}, \overline{T}, \overline{\pi}) \cdot \nabla_{(m,x)}\right) \nabla_{(m,x)} \sigma dm dx$$

$$= -\frac{1}{p} \int_{\Omega_{\overline{M}_1}} \left|\nabla_{(m,x)} \sigma\right|^{p} \nabla_{(m,x)} \cdot \widetilde{U}_{4l}(\overline{u}, \overline{T}, \overline{\pi}) dm dx. \tag{5.19}$$

Multiplying the equation (5.12) by σ^{p-1} and integrating it on $\Omega_{\overline{M}_1}$, thanks to (5.18), we obtain

$$\frac{d}{dt} \|\sigma\|_{L^p(\Omega_{\overline{M}_1})}^p = (1-p) \int_{\Omega_{\overline{M}_1}} \sigma^p \nabla_{(m,x)} \cdot \widetilde{U}_{4l}(\overline{u}, \overline{T}, \overline{\pi}) dm dx$$

$$+ p \int_{\Omega_{\overline{M}_1}} \sigma^p a_1^*(t, m, x) dm dx + p \int_{\Omega_{\overline{M}_1}} \sigma^{p-1} b_1(t, m, x) dm dx \tag{5.20}$$

where

$$a_1^*(t, m, x) = h_{gl}(\overline{T}, \overline{\pi}; m) + B_1(\overline{\sigma}; m) - g_1(m)[\overline{\pi} - \overline{\pi}_{vs}(\overline{T})]^-,$$

$$b_1(t, m, x) = g_0(m)[N^* - \widetilde{N}(\overline{\sigma})]^+ [\overline{\pi} - \overline{\pi}_{vs(l)}(\overline{T})]^+ + B_2(\overline{\sigma}; m).$$

On the other hand, applying the differential operator $|\nabla_{(m,x)} \sigma|^{p-2} \nabla_{(m,x)} \sigma \cdot \nabla_{(m,x)}$ to the equation (5.12) and integrating it on $\Omega_{\overline{M}_1}$, we have

$$\frac{d}{dt} \|\nabla_{(m,x)} \sigma\|_{L^p(\Omega_{\overline{M}_1})}^p$$

$$= -p \int_{\Omega_{\overline{M}_1}} |\nabla_{(m,x)} \sigma|^{p-2} \nabla_{(m,x)} \sigma \cdot \nabla_{(m,x)} \left[\nabla_{(m,x)} \cdot (\sigma \widetilde{U}_{4l}(\overline{u}, \overline{T}, \overline{\pi}))\right] dm dx$$

$$+ p \int_{\Omega_{\overline{M}_1}} |\nabla_{(m,x)}\sigma|^{p-2} \nabla_{(m,x)}\sigma \cdot \nabla_{(m,x)}[a_1^*(m,x,t)\sigma + b_1(t,m,x)]dmdx.$$

$$(5.21)$$

Remembering the definitions of $a_1^*(t,m,x)$ and $b_1(t,m,x)$ and using repeatedly the Sobolev and Hölder inequalities, we deduce

$$\left| \int_{\Omega_{\overline{M}_1}} \sigma^p \nabla_{(m,x)} \cdot \widetilde{U}_{4l}(\overline{u},\overline{T},\overline{\pi})dmdx \right| + \left| \int_{\Omega_{\overline{M}_1}} \sigma^p a_1^*(t,m,x)dmdx \right|$$

$$\leq c\big(1 + \|\overline{\pi}\|_{W_p^1(\Omega)} + \|\overline{T}\|_{W_q^2(\Omega)} + \|\overline{u}\|_{W_p^2(\Omega_{\overline{M}_1})} + \|\overline{\sigma}\|_{W_p^1(\Omega_{\overline{M}_1})}\big)$$

$$\times \|\sigma\|_{L^p(\Omega_{\overline{M}_1})}^p, \qquad (5.22)$$

$$\left| \int_{\Omega_{\overline{M}_1}} \sigma^{p-1} b_1(t,m,x)dmdx \right| \leq \|\sigma\|_{L^p(\Omega_{\overline{M}_1})}^{p-1} \|b_1\|_{L^p(\Omega_{\overline{M}_1})}$$

$$\leq c\big(1 + \|\overline{\pi}\|_{L^p(\Omega)}^2 + \|\overline{T}\|_{W_q^1(\Omega)}^2 + \|\overline{\sigma}\|_{W_p^1(\Omega_{\overline{M}_1})}^2\big)\big(\|\sigma\|_{L^p(\Omega_{\overline{M}_1})}^p + 1\big), \quad (5.23)$$

$$\left| \int_{\Omega_{\overline{M}_1}} |\nabla_{(m,x)}\sigma|^{p-2} \nabla_{(m,x)}\sigma \cdot \nabla_{(m,x)}\big[\nabla_{(m,x)} \cdot (\sigma\widetilde{U}_{4l}(\overline{u},\overline{T},\overline{\pi}))\big]dmdx \right|$$

$$\leq c\big(1 + \|\overline{\pi}\|_{W_p^1(\Omega)} + \|\overline{T}\|_{W_q^2(\Omega)} + \|\overline{u}\|_{W_p^2(\Omega_{\overline{M}_1})}\big)\|\sigma\|_{W_p^1(\Omega_{\overline{M}_1})}^p, \qquad (5.24)$$

$$\left| \int_{\Omega_{\overline{M}_1}} |\nabla_{(m,x)}\sigma|^{p-2} \nabla_{(m,x)}\sigma \cdot \nabla_{(m,x)}(a_1^*(t,m,x)\sigma)dmdx \right|$$

$$\leq c\big(1 + \|\overline{\pi}\|_{W_p^1(\Omega)} + \|\overline{T}\|_{W_q^2(\Omega)} + \|\overline{\sigma}\|_{W_p^1(\Omega_{\overline{M}_1})}\big)\|\sigma\|_{W_p^1(\Omega_{\overline{M}_1})}^p, \qquad (5.25)$$

$$\left| \int_{\Omega_{\overline{M}_1}} |\nabla_{(m,x)}\sigma|^{p-2} \nabla_{(m,x)}\sigma \cdot \nabla_{(m,x)}b_1(t,m,x)dmdx \right|$$

$$\leq c\big(1 + \|\overline{\pi}\|_{W_p^1(\Omega)}^2 + \|\overline{T}\|_{W_q^2(\Omega)}^2 + \|\overline{\sigma}\|_{W_p^1(\Omega_{\overline{M}_1})}^2\big)$$

$$\times \big(\|\sigma\|_{W_p^1(\Omega_{\overline{M}_1})}^p + 1\big). \qquad (5.26)$$

From (5.20)–(5.26), taking into account (5.6), we obtain

$$\frac{d}{dt}\|\sigma\|^p_{W^1_p(\Omega_{\overline{M}_1})} \le c\big(1 + \|\overline{v}\|_{W^2_p(\Omega)} + \|\overline{\pi}\|^2_{W^1_p(\Omega)} + \|\overline{T}\|^2_{W^2_q(\Omega)} + \|\overline{\sigma}\|^2_{W^1_p(\Omega_{\overline{M}_1})}\big)$$
$$\times \big(\|\sigma\|^p_{W^1_p(\Omega_{\overline{M}_1})} + 1\big),$$

from this inequality we deduce (5.15) with $q_\sigma(t)$ defined as in (5.17).

To prove the inequality (5.16), we consider $r \ge p$. Applying the operator $|\partial_m\sigma|^{r-2}\partial_m\sigma\partial_m$ to the equation (5.12) and integrating it on $\Omega_{\overline{M}_1}$, we have

$$\frac{d}{dt}\|\partial_m\sigma\|_{L^r(\Omega_{\overline{M}_1})}$$
$$\le c\Big[\big(1 + \|\nabla_x \cdot \overline{u}(s,\cdot)\|_{L^\infty(\Omega_{\overline{M}_1})} + \|\overline{\pi}\|_{L^\infty(\Omega)} + \|\overline{\sigma}\|_{L^\infty(\Omega_{\overline{M}_1})}\big)\|\partial_m\sigma\|_{L^r(\Omega_{\overline{M}_1})}$$
$$+ \|\overline{\sigma}\|_{L^\infty(\Omega_{\overline{M}_1})}\big(\|\overline{\sigma}\|_{L^r(\Omega_{\overline{M}_1})} + \|\partial_m\overline{\sigma}\|_{L^r(\Omega_{\overline{M}_1})}\big)$$
$$+ \big(1 + \|\overline{\pi}\|_{L^\infty(\Omega)} + \|\overline{\sigma}\|_{L^\infty(\Omega_{\overline{M}_1})}\big)\|\sigma\|_{L^r(\Omega_{\overline{M}_1})}\Big]. \tag{5.27}$$

Applying Gronwall's lemma and taking the limit for $r \to \infty$ we obtain (5.16). $\qquad\square$

6. Equations for the water densities with given velocities and temperature

To prove the existence and uniqueness of the solution (π, σ) to the non-linear equation system (5.4)–(5.5) and obtain its estimates with given temperature $T = \overline{T}$ and velocity $v = \overline{v}$, we use the following lemma.

Lemma 6.1 *Let be* $\overline{v} \in \Theta^{(v)}_{t_1}$, $\overline{T} \in \Theta^{(T)}_{t_1}$ *and* $R_0 > 0$. *We assume that*

$$\|\overline{v}\|_{W^{2,1}_p(Q_{t_1})} \le R_0, \qquad \|\overline{T}\|_{W^{2,1}_q(Q_{t_1})} \le R_0. \tag{6.1}$$

Then there exists $t_2 = t_2(R_0)$, $0 < t_2 \le t_1$, *such that, if* $\overline{\pi} \in C^0([0,t_2]; W^1_p(\Omega))$, $\overline{\sigma} \in C^0([0,t_2]; W^1_p(\mathbf{R}_+ \times \Omega))$ *and* $\partial_m\overline{\sigma} \in C^0([0,t_1]; L^\infty(\mathbf{R}_+ \times \Omega))$ *with the conditions*

$$\|\overline{\pi}\|_{C^0([0,t_2]; W^1_p(\Omega))} \le \|\pi_0\|_{W^1_p(\Omega)} + 1,$$
$$\|\overline{\sigma}\|_{C^0([0,t_2]; W^1_p(\mathbf{R}_+\times\Omega))} \le \|\sigma_0\|_{W^1_p(\mathbf{R}_+\times\Omega)} + 1,$$

$$\|\partial_m \overline{\sigma}\|_{C^0([0,t_2];L^\infty(\mathbf{R}_+\times\Omega))} \le \|\partial_m \sigma_0\|_{L^\infty(\mathbf{R}_+\times\Omega)} + 1, \qquad (6.2)$$

$$\overline{\sigma}(\cdot, m, \cdot) = 0 \quad for \ m \notin \left]\frac{\overline{m}_a}{2}, \overline{M}_1\right[,$$

then the solution (π, σ) of the equations (5.9), (5.12) with the initial conditions (3.2)–(3.3) satisfies the conditions

$$\|\pi\|_{C^0([0,t_2];W_p^1(\Omega))} \le \|\pi_0\|_{W_p^1(\Omega)} + 1,$$

$$\|\sigma\|_{C^0([0,t_2];W_p^1(\mathbf{R}_+\times\Omega))} \le \|\sigma_0\|_{W_p^1(\mathbf{R}_+\times\Omega)} + 1,$$

$$\|\partial_m \sigma\|_{C^0([0,t_2];L^\infty(\mathbf{R}_+\times\Omega))} \le \|\partial_m \sigma_0\|_{L^\infty(\mathbf{R}_+\times\Omega)} + 1, \qquad (6.3)$$

$$\sigma(\cdot, m, \cdot) = 0 \quad for \ m \notin \left]\frac{\overline{m}_a}{2}, \overline{M}_1\right[.$$

Proof. The lemma follows from the relations (5.10), (5.14), (5.15), (5.16) (see also (5.11), (5.17)). □

Lemma 6.2 *Let be $\overline{v}$, $\overline{T}$, $\overline{M}_1$, R_0, $t_2 = t_2(R_0)$ as in Lemma 6.1. Then there exists $t_3 \in]0, t_2]$ such that the equation system (5.4)–(5.5) with the initial conditions (3.2)–(3.3) admit a unique solution $(\pi, \sigma) \in C^0([0, t_3]; W_p^1(\Omega)) \times C^0([0, t_3]; W_p^1(\mathbf{R}_+ \times \Omega)))$, satisfying the conditions*

$$\|\pi\|_{C^0([0,t_3];W_p^1(\Omega))} \le \|\pi_0\|_{W_p^1(\Omega)} + 1,$$

$$\|\sigma\|_{C^0([0,t_3];W_p^1(\mathbf{R}_+\times\Omega))} \le \|\sigma_0\|_{W_p^1(\mathbf{R}_+\times\Omega)} + 1,$$

$$\|\partial_m \sigma\|_{C^0([0,t_2];L^\infty(\mathbf{R}_+\times\Omega))} \le \|\partial_m \sigma_0\|_{L^\infty(\mathbf{R}_+\times\Omega)} + 1, \qquad (6.4)$$

$$\sigma(\cdot, m, \cdot) = 0 \quad for \ m \notin \left]\frac{\overline{m}_a}{2}, \overline{M}_1\right[.$$

Proof. For $0 < t \le t_2$, we define the set

$$A_{[t]} = \{(\pi, \sigma) \text{ satisfies the conditions } (C.1)\text{–}(C.4)\}, \qquad (6.5)$$

where

$$\pi \in C^0([0, t]; W_p^1(\Omega)), \quad \sigma \in C^0([0, t]; W_p^1(\mathbf{R}_+ \times \Omega)), \qquad (C.1)$$

$$\pi(0, \cdot) = \pi_0(\cdot), \quad \sigma(0, \cdot, \cdot) = \sigma_0(\cdot, \cdot), \qquad (C.2)$$

$$\|\pi\|_{C^0([0,t];W_p^1(\Omega))} \leq \|\pi_0\|_{W_p^1(\Omega)} + 1,$$

$$\|\sigma\|_{C^0([0,t];W_p^1(\mathbf{R}_+\times\Omega))} \leq \|\sigma_0\|_{W_p^1(\mathbf{R}_+\times\Omega)} + 1,$$

$$\|\partial_m\sigma\|_{C^0([0,t_2];L^\infty(\mathbf{R}_+\times\Omega))} \leq \|\partial_m\sigma_0\|_{L^\infty(\mathbf{R}_+\times\Omega)} + 1, \tag{C.3}$$

$$\sigma(t',m,x) = 0 \quad \text{for} \quad m \notin \left]\frac{\overline{m_a}}{2}, \overline{M}_1\right[, \ x \in \Omega, \ 0 \leq t' \leq t. \tag{C.4}$$

From Lemma 6.1, the map $G_{1,t} : A_{[t]} \to A_{[t]}$ that sends $(\overline{\pi}, \overline{\sigma}) \in A_{[t]}$ to the solution $(\pi, \sigma) = G_{1,t}(\overline{\pi}, \overline{\sigma})$ of the equations (5.9), (5.12) is well defined.

Now, we prove that there exists t_3, $0 < t_3 \leq t_2$, such that G_{1,t_3} is a contraction with respect to the metric of $C^0([0,t_3]; L^p(\Omega)) \times C^0([0,t_3]; L^p(\Omega_{\overline{M}_1}))$ on the closed set $A_{[t_3]}$. For this, we consider two elements $(\overline{\pi}_1, \overline{\sigma}_1)$, $(\overline{\pi}_2, \overline{\sigma}_2)$ of $A_{[t]}$ with their values by the map $G_{1,t}$ are $(\pi_i, \sigma_i) = G_{1,t}(\overline{\pi}_i, \overline{\sigma}_i)$, $i = 1, 2$; we denote by

$$\overline{\Pi} = \overline{\pi}_2 - \overline{\pi}_1, \quad \overline{\Sigma} = \overline{\sigma}_2 - \overline{\sigma}_1, \quad \Pi = \pi_2 - \pi_1, \quad \Sigma = \sigma_2 - \sigma_1. \tag{6.6}$$

The difference between the equations (5.9), (5.12) with $(\overline{\pi}_2, \overline{\sigma}_2)$ and $(\overline{\pi}_1, \overline{\sigma}_1)$ gives

$$\frac{\partial \Pi}{\partial t} + \nabla \cdot (\Pi \overline{v}) = H_{gl}\left(\overline{T}, \overline{\pi}_1, \overline{\sigma}_1\right) - H_{gl}\left(\overline{T}, \overline{\pi}_2, \overline{\sigma}_2\right), \tag{6.7}$$

$$\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial m}\left[mh_{gl}(\overline{T}, \overline{\pi}_2; m)\Sigma\right] + \nabla \cdot (\Sigma \overline{u})$$

$$= \left[h_{gl}(\overline{T}, \overline{\pi}_2; m) + B_1(\overline{\sigma}_2; m) - g_1(m)[\overline{\pi}_2 - \overline{\pi}_{vs}(\overline{T})]^-\right]\Sigma$$

$$+ \frac{\partial}{\partial m}\left\{m[h_{gl}(\overline{T}, \overline{\pi}_1; m) - h_{gl}(\overline{T}, \overline{\pi}_2; m)]\sigma_1\right\}$$

$$+ [h_{gl}(\overline{T}, \overline{\pi}_2; m) - h_{gl}(\overline{T}, \overline{\pi}_1; m)]\sigma_1 + [B_1(\overline{\sigma}_2; m) - B_1(\overline{\sigma}_1; m)]\sigma_1$$

$$+ g_1(m)\left([\overline{\pi}_1 - \overline{\pi}_{vs}(\overline{T})]^- - [\overline{\pi}_2 - \overline{\pi}_{vs}(\overline{T})]^-\right)\sigma_1$$

$$+ g_0(m)\left([N^* - \widetilde{N}(\overline{\sigma}_2)]^+[\overline{\pi}_2 - \overline{\pi}_{vs}(\overline{T})]^+ - [N^* - \widetilde{N}(\overline{\sigma}_1)]^+[\overline{\pi}_1 - \overline{\pi}_{vs}(\overline{T})]^+\right)$$

$$+ B_2(\overline{\sigma}_2; m) - B_2(\overline{\sigma}_1; m). \tag{6.8}$$

Multiplying the equations (6.7)–(6.8) respectively by $|\Pi|^{p-1}$ and $|\Sigma|^{p-1}$, integrating the first on Ω and the second on $\Omega_{\overline{M}_1}$ and taking into account the

conditions (6.1) and the estimates already known, we deduce the following inequalities

$$\frac{d}{dt}\|\Pi\|_{L^p(\Omega)}^p \leq c\|\overline{v}\|_{W_p^2(\Omega)}\|\Pi\|_{L^p(\Omega)}^p$$
$$+ c\big(1 + \|\overline{T}\|_{W_q^1(\Omega)}\big)\big(\|\overline{\Pi}\|_{L^p(\Omega)} + \|\overline{\Sigma}\|_{L^p(\Omega_{\overline{M}_1})}\big)\|\Pi\|_{L^p(\Omega)}^{p-1},$$
(6.9)

$$\frac{d}{dt}\|\Sigma\|_{L^p(\Omega_{\overline{M}_1})}^p \leq c\big(1 + \|\overline{T}\|_{W_q^1(\Omega)} + \|\overline{v}\|_{W_p^2(\Omega)}\big)\|\Sigma\|_{L^p(\Omega_{\overline{M}_1})}^p$$
$$+ c\big(1 + \|\overline{T}\|_{W_q^1(\Omega)}\big)\big(\|\overline{\Pi}\|_{L^p(\Omega)} + \|\overline{\Sigma}\|_{L^p(\Omega_{\overline{M}_1})}\big)\|\Sigma\|_{L^p(\Omega_{\overline{M}_1})}^{p-1}.$$
(6.10)

It is useful to recall that $\|\partial_m\sigma_0\|_{L^\infty(\mathbf{R}_+\times\Omega)} < \infty$, the inequality (6.10) is obtained without introducing a regularization of π. Indeed, the term that had requested a regularization of π in [14] can be treated in the following way

$$\left|\int_{\Omega_{\overline{M}_1}} |\Sigma|^{p-1}\,\partial_m\big\{m\big[h_{gl}(T,\overline{\pi}_1;m) - h_{gl}(T,\overline{\pi}_2;m)\big]\sigma_1\big\}dmdx\right|$$

$$\leq c\int_{\Omega_{\overline{M}_1}} |\Sigma|^{p-1}\,|\overline{\Pi}|\,(|\sigma_1| + |\partial_m\sigma_1|)dmdx$$

$$\leq c\big(\|\sigma_1(t,\cdot)\|_{L^\infty(\Omega_{\overline{M}_1})} + \|\partial_m\sigma_1(t,\cdot)\|_{L^\infty(\Omega_{\overline{M}_1})}\big)$$
$$\times \|\Sigma(t,\cdot)\|_{L^p(\Omega_{\overline{M}_1})}^{p-1}\,\|\overline{\Pi}(t,\cdot)\|_{L^p(\Omega)}\,.$$

Multiplying (6.9)–(6.10) respectively by $\|\Pi\|_{L^p(\Omega)}^{1-p}$ and $\|\Sigma\|_{L^p(\Omega)}^{1-p}$ and taking into account (6.1), we deduce

$$\|\Pi(t,\cdot)\|_{L^p(\Omega)} + \|\Sigma(t,\cdot)\|_{L^p(\Omega_{\overline{M}_1})}$$

$$\leq ce^{c(t+t^{(q-1)/q}+t^{(p-1)/p})}(t + t^{(q-1)/q})$$
$$\times \big(\|\overline{\Pi}\|_{C^0([0,t];L^p(\Omega))} + \|\overline{\Sigma}\|_{C^0([0,t];L^p(\Omega_{\overline{M}_1}))}\big).$$
(6.11)

It is clear that there exists $t_3 \in]0, t_2]$ such that

$$ce^{c(t_3 + t_3^{(q-1)/q} + t_3^{(p-1)/p})}(t_3 + t_3^{(q-1)/q}) < \frac{1}{2}.$$

Therefore, remembering the definition of $G_{1,t}$ and (6.6), we deduce that the map G_{1,t_3} restricted to $A_{[t_3]}$ is a contraction with respect to the metric of $C^0([0, t_3]; L^p(\Omega)) \times C^0([0, t_3]; L^p(\Omega_{\overline{M}_1}))$. □

Lemma 6.3 *Let be $\overline{v} \in \Theta_{t_1}^{(v)}$, $\overline{T} \in \Theta_{t_1}^{(T)}$ verifying (6.1) and ϱ, π, σ the solutions of the equations (5.3)–(5.5) with the initial conditions (3.1)–(3.3). Under the same hypotheses of Lemma 6.2, there exists $t_4 \in]0, t_3]$ such that for $t \in [0, t_4]$ we have*

$$\|\varrho(t, \cdot)\|_{W_p^1}^p \le 2\|\varrho_0(\cdot)\|_{W_p^1}^p, \quad \frac{1}{2} \inf_{x' \in \Omega} \varrho_0(x') \le \varrho(t, x) \le 2 \sup_{x' \in \Omega} \varrho_0(x') \quad x \in \Omega, \tag{6.12}$$

$$0 \le \pi(t, x) \le \sup_{x' \in \Omega} \pi_0(x') + 1 \quad x \in \Omega. \tag{6.13}$$

Proof.   See [14]. □

7. Linear equation for the velocity and temperature

In this section we study the linearized equations of (2.4) and (2.5). We assume that $\overline{v} \in \Theta_{t_1}^{(v)}$ and $\overline{T} \in \Theta_{t_1}^{(T)}$ and consider the linear equations in v and T

$$(\varrho + \pi)\frac{\partial v}{\partial t} - \eta \Delta v - \left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla \cdot v)$$

$$= -(\varrho + \pi)(\overline{v} \cdot \nabla)\overline{v} - R_0 \nabla\left(\left(\frac{\varrho}{\mu_a} + \frac{\pi}{\mu_h}\right)\overline{T}\right)$$

$$- \left[\int_0^\infty (\sigma + \nu)dm + \varrho + \pi\right]\nabla\Phi, \tag{7.1}$$

$$(\varrho + \pi)c_v \frac{\partial T}{\partial t} - \kappa \Delta T = -(\varrho + \pi)c_v \sum_{j=1}^3 \overline{v}_j \frac{\partial \overline{T}}{\partial x_j} - R_0\left(\frac{\varrho}{\mu_a} + \frac{\pi}{\mu_h}\right)\overline{T}\nabla \cdot \overline{v}$$

$$+ \eta \sum_{i,j=1}^{3} \left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \overline{v} \right) \frac{\partial \overline{v}_i}{\partial x_j} + \zeta (\nabla \cdot \overline{v})^2 - \nabla \cdot \mathcal{E}$$

$$+ L_{gl} H_{gl}(\overline{T}, \pi, \sigma) + L_{ls} H_{ls}(\overline{T}, \sigma, \nu) + L_{gs} H_{gs}(\overline{T}, \pi, \nu), \quad (7.2)$$

where ϱ, π, σ are the solutions of the equations (5.3)–(5.5) with the initial conditions (3.1)–(3.3); their existence and uniqueness are proved in Lemma 6.1 and Lemma 6.2.

As in [15], we introduce the following auxiliary functions

$$V_{(p,v)}(t) = \|v\|_{W_p^{2,1}(Q_t)}^p + \sup_{0 \le t' \le t} \|v(t', \cdot)\|_{W_p^{2-2/p}(\Omega)}^p, \quad (7.3)$$

$$V_{(q,T)}(t) = \|T\|_{W_q^{2,1}(Q_t)}^q + \sup_{0 \le t' \le t} \|T(t', \cdot)\|_{W_q^{2-\frac{2}{q}}(\Omega)}^q \quad (7.4)$$

(for the general theory about the concerned functional spaces, see also [16], [17], [18]).

Lemma 7.1   *Let be* $\overline{v} \in \Theta_{t_1}^{(v)}$, $\overline{T} \in \Theta_{t_1}^{(T)}$ *and* (ϱ, π, σ) *the solution of the equation system* (5.3)–(5.5) *with the initial conditions* (3.1)–(3.3) *given in Lemma 5.1 and Lemma 6.2. Then the equations* (7.1) *and* (7.2), *with the conditions* (3.4)–(3.5), *admit a unique solution*

$$v \in W_p^{2,1}(Q_{t_4}), \qquad T \in W_q^{2,1}(Q_{t_4}). \quad (7.5)$$

Moreover we have

$$V_{(p,v)}(t) \le c \left(\|v_0\|_{W_p^{2-2/p}(\Omega)}^p + \int_0^t (1 + V_{(p,\overline{v})}(t')^2) dt' + t^{(2q-p)/q} V_{(q,\overline{T})}(t) \right),$$
$$(7.6)$$

$$V_{(q,T)}(t) \le c \left[\|T_0\|_{W_q^{2-2/q}(\Omega)}^q + t^{2(p-q)/p} V_{(p,\overline{v})}(t)^{2q/p} \right.$$

$$\left. + \int_0^t \left(1 + V_{(q,\overline{T})}(t') V_{(p,\overline{v})}(t')^{q/p} + \|\nabla \cdot \mathcal{E}\|_{L^q(\Omega)}^q \right) dt' \right] \quad (7.7)$$

for $0 < t \le t_4$ (t_4 *is defined in Lemma 6.3*).

Proof.   According to Theorem 9.1 of the chapter IV of [5], the equation

(7.1) admits a unique solution $v \in W_p^{2,1}(Q_{t_5})$, and by virtue of an extension of the same theorem (see the last remark before paragraph 10, chapter IV of [5]) we obtain a unique solution $T \in W_q^{2,1}(Q_{t_5})$ of the equation (7.2).

Taking into account Lemma 6.3, we obtain (see [15] and also Lemma 3.4 of the chapter II of [5]) for $0 < t \le t_6$

$$V_{(p,v)}(t) \le c\big(\|v_0\|_{W_p^{2-2/p}(\Omega)}^p + \|F_v\|_{L^p(Q_t)}^p\big), \tag{7.8}$$

$$V_{(q,T)}(t) \le c\big(\|T_0\|_{W_q^{2-2/q}(\Omega)}^q + \|F_T\|_{L^q(Q_t)}^q\big), \tag{7.9}$$

where F_v and F_T are respectively the second member of (7.1) and that of (7.2).

As $2q > p > q > 3$ it is not difficult to see that

$$\left\| R_0 \nabla \left(\left(\frac{\varrho}{\mu_a} + \frac{\pi}{\mu_h} \right) \overline{T} \right) + \left[\int_0^\infty \sigma dm + \varrho + \pi \right] \nabla \Phi \right\|_{L^p(\Omega)}^p$$

$$\le c\big(1 + V_{(q,\overline{T})}(t)\|\overline{T}\|_{W_q^2(\Omega)}^{p-q}\big),$$

$$\int_0^t \left\| \eta \sum_{i,j=1}^3 \left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \nabla \cdot \overline{v} \right) \frac{\partial \overline{v}_i}{\partial x_j} + \zeta(\nabla \cdot \overline{v})^2 \right\|_{L^q(\Omega)}^q dt'$$

$$\le c \int_0^t \|\overline{v}\|_{W_p^2(\Omega)}^{2q-p} dt' V_{(p,\overline{v})}(t).$$

The other terms of F_v and F_T can be estimated in the usual way (see also (2.9)). Thus we deduce (7.6)–(7.7) from (7.8)–(7.9). $\qquad \square$

8. Existence and uniqueness of the local solution

To prove Theorem 3.1, we start with the following lemma.

Lemma 8.1 *There exist positive constants $\overline{R}_v$, $\overline{R}_T$ and $t_5 \in]0, t_4]$ such that, if $0 < t \le t_5$, $\overline{v} \in \Theta_t^{(v)}$, $\overline{T} \in \Theta_t^{(T)}$ and if*

$$V_{(p,\overline{v})}(t) \le \overline{R}_v, \quad V_{(q,\overline{T})}(t) \le \overline{R}_T,$$

then the solution (v, T) of the equations (7.1)–(7.2) with the conditions (3.4)–(3.5) satisfies the inequalities

$$V_{(p,v)}(t) \leq \overline{R}_v, \quad V_{(q,T)}(t) \leq \overline{R}_T.$$

Proof. The lemma follows from (7.6)–(7.7) by simple calculations. $\square$

We define

$$B_t = \left\{ (v,T) \in \Theta_t^{(v)} \times \Theta_t^{(T)} \mid V_{(p,v)}(t) \leq \overline{R}_v, \ V_{(q,T)}(t) \leq \overline{R}_T \right\}. \quad (8.1)$$

For $0 < t \leq t_5$ we define the map $G_t : B_t \to \Theta_t^{(v)} \times \Theta_t^{(T)}$ such that, for $(\overline{v}, \overline{T}) \in B_t$, $(v,T) = G_t(\overline{v}, \overline{T})$ is the solution of the equation (7.1)–(7.2) with the conditions (3.4)–(3.5). By virtue of Lemma 8.1 we have

$$G_t(B_t) \subseteq B_t, \qquad 0 < t \leq t_5. \quad (8.2)$$

Proof of Theorem 3.1. For $0 < t \leq t_5$, we define

$$Y_t = \left[L^2(0,t; H_0^1(\Omega)) \cap L^\infty(0,t; L^2(\Omega)) \right]$$
$$\times \left[L^2(0,t; H_0^1(\Omega)) \cap L^\infty(0,t; L^2(\Omega)) \right]. \quad (8.3)$$

We remark that the set B_t defined in (8.1) is a closed convex set in the space Y_t. Therefore, to prove the theorem, it is sufficient to check that there exists $\overline{t} \in \,]0, t_5]$ such that the operator $G_{\overline{t}}$ is a contraction in the natural topology of $Y_{\overline{t}}$.

Let be $(\overline{v}_1, \overline{T}_1)$, $(\overline{v}_2, \overline{T}_2) \in B_t$, $0 < t \leq t_5$. First we consider the solutions $(\varrho_1, \pi_1, \sigma_1)$ and $(\varrho_2, \pi_2, \sigma_2)$ for the equation system (5.3)–(5.5) with the conditions (3.1)–(3.3) and with the substitutions $\overline{v} = \overline{v}_1$, $\overline{T} = \overline{T}_1$ and $\overline{v} = \overline{v}_2$, $\overline{T} = \overline{T}_2$. We put

$$E^{[\varrho]} = \varrho_1 - \varrho_2, \qquad E^{[\pi]} = \pi_1 - \pi_2, \qquad E^{[\sigma]} = \sigma_1 - \sigma_2,$$
$$\overline{D}^{[v]} = \overline{v}_1 - \overline{v}_2, \qquad \overline{D}^{[T]} = \overline{T}_1 - \overline{T}_2.$$

We define

$$\overline{u}_i(t,m,x) = \overline{v}_i(t,x) - \frac{1}{\alpha_l(m)} \nabla \Phi(x),$$
$$\overline{U}_{4l,i} = \left(m h_{gl}(\overline{T}_i, \pi_i; m), \overline{u}_{i,1}, \overline{u}_{i,2}, \overline{u}_{i,3} \right)^T,$$

$$\mathcal{E}^{[i]} = (\mathcal{E}_1^{[i]}, \mathcal{E}_2^{[i]}, \mathcal{E}_3^{[i]}), \quad i = 1, 2.$$

From the difference between the equations (5.3)–(5.5) for $\varrho_1, \pi_1, \sigma_1$ and those for $\varrho_2, \pi_2, \sigma_2$ it follows that

$$\partial_t E^{[\varrho]} + \overline{v}_1 \cdot \nabla E^{[\varrho]} + \overline{D}^{[v]} \cdot \nabla \varrho_2 + E^{[\varrho]} \nabla \cdot \overline{v}_1 + \varrho_2 \nabla \cdot \overline{D}^{[v]} = 0, \qquad (8.4)$$

$$\partial_t E^{[\pi]} + \overline{v}_1 \cdot \nabla E^{[\pi]} + \overline{D}^{[v]} \cdot \nabla \pi_2 + E^{[\pi]} \nabla \cdot \overline{v}_1 + \pi_2 \nabla \cdot \overline{D}^{[v]}$$
$$= H_{gl}(\overline{T}_2, \pi_2, \sigma_2) - H_{gl}(\overline{T}_1, \pi_1, \sigma_1), \qquad (8.5)$$

$$\partial_t E^{[\sigma]} + (\overline{U}_{4l,1} - \overline{U}_{4l,2}) \cdot \nabla_{(m,x)} E^{[\sigma]} + \overline{D}^{[U_{4l}]} \cdot \nabla_{(m,x)} \sigma_2$$
$$+ E^{[\sigma]} \nabla_{(m,x)} \cdot \overline{U}_{4l,1} + \sigma_2 \nabla_{(m,x)} \cdot \overline{D}^{[U_{4l}]}$$
$$= \left[h_{gl}(\overline{T}_1, \pi_1; m) + B_1(\sigma_1; m) - g_1(m)[\pi_1 - \overline{\pi}_{vs(l)}(\overline{T}_1)]^- \right] E^{[\sigma]}$$
$$+ \left\{ h_{gl}(\overline{T}_1, \pi_1; m) - h_{gl}(\overline{T}_2, \pi_2; m) + B_1(\sigma_1; m) - B_1(\sigma_2; m) \right.$$
$$\left. - g_1(m) \left([\pi_1 - \overline{\pi}_{vs(l)}(\overline{T}_1)]^- - [\pi_2 - \overline{\pi}_{vs(l)}(\overline{T}_2)]^- \right) \right\} \sigma_2$$
$$+ g_0(m) \left([\pi_1 - \overline{\pi}_{vs(l)}(\overline{T}_1)]^+ [N^* - N(\sigma_1)]^+ \right.$$
$$\left. - [\pi_2 - \overline{\pi}_{vs(l)}(\overline{T}_2)]^+ [N^* - N(\sigma_2)]^+ \right)$$
$$+ B_2(\sigma_1; m) - B_2(\sigma_2; m). \qquad (8.6)$$

We remember the relations

$$\int_\Omega (\overline{v}_1 \cdot \nabla E^{[\varrho]}) E^{[\varrho]} dx = -\frac{1}{2} \int_\Omega (\nabla \cdot \overline{v}_1)(E^{[\varrho]})^2 dx,$$

$$\int_\Omega (\overline{v}_1 \cdot \nabla E^{[\pi]}) E^{[\pi]} dx = -\frac{1}{2} \int_\Omega (\nabla \cdot \overline{v}_1)(E^{[\pi]})^2 dx,$$

$$\int_{\Omega_{\overline{M}_1}} (\overline{U}_{4l,1} \cdot \nabla E^{[\sigma]}) E^{[\sigma]} dm dx = -\frac{1}{2} \int_{\Omega_{\overline{M}_1}} (\nabla \cdot \overline{U}_{4l,1})(E^{[\sigma]})^2 dm dx.$$

So, multiplying (8.4)–(8.6) by $E^{[\varrho]}$, $E^{[\pi]}$, $E^{[\sigma]}$, integrating the first two on Ω and the others on $\Omega_{\overline{M}_1}$ and taking into account estimates already known, we obtain with usual calculations

$$\frac{d}{dt}\|E^{[\varrho]}\|^2_{L^2(\Omega)} \le c(1+\|\overline{v}_1\|_{W^2_p(\Omega)})\|E^{[\varrho]}\|^2_{L^2(\Omega)} + c\|\overline{D}^{[v]}\|^2_{H^1(\Omega)}, \quad (8.7)$$

$$\frac{d}{dt}\|E^{[\pi]}\|^2_{L^2(\Omega)} \le c\big(1+\|\overline{v}_1\|_{W^2_p(\Omega)} + \|\overline{T}_1\|_{W^2_q(\Omega)}\big)$$
$$\times \big(\|E^{[\pi]}\|^2_{L^2(\Omega)} + \|E^{[\sigma]}\|^2_{L^2(\Omega_{\overline{M}_1})}\big)$$
$$+ c\big(\|\overline{D}^{[v]}\|^2_{H^1(\Omega)} + \|\overline{D}^{[T]}\|^2_{L^2(\Omega)}\big), \qquad (8.8)$$

$$\frac{d}{dt}\|E^{[\sigma]}\|^2_{L^2(\Omega_{\overline{M}_1})} \le c\big(1+\|\overline{v}_1\|_{W^2_p(\Omega)} + \|\overline{T}_1\|_{W^2_q(\Omega)} + \|\overline{T}_2\|_{W^2_q(\Omega)}\big)$$
$$\times \big(\|E^{[\pi]}\|^2_{L^2(\Omega)} + \|E^{[\sigma]}\|^2_{L^2(\Omega_{\overline{M}_1})}\big)$$
$$+ c\big(\|\overline{D}^{[v]}\|^2_{H^1(\Omega)} + \|\overline{D}^{[T]}\|^2_{L^2(\Omega)}\big), \qquad (8.9)$$

From (8.7)–(8.9) and the initial conditions

$$E^{[\varrho]}(0,\cdot) = 0, \qquad E^{[\pi]}(0,\cdot) = 0, \qquad E^{[\sigma]}(0,\cdot,\cdot) = 0,$$

it follows that

$$\|E^{[\varrho]}(t)\|^2_{L^2(\Omega)} + \|E^{[\pi]}(t)\|^2_{L^2(\Omega)} + \|E^{[\sigma]}(t)\|^2_{L^2(\Omega_{\overline{M}_1})}$$
$$\le ce^{c[t+t^{(p-1)/p}+t^{(q-2)/q}]} \int_0^t \big(\|\overline{D}^{[v]}(t')\|^2_{H^1(\Omega)} + \|\overline{D}^{[T]}(t')\|^2_{L^2(\Omega)}\big)dt'.$$
$$(8.10)$$

We consider the difference between the equations (7.1)–(7.2) for $(v_1, T_1) = G_t(\overline{v}_1, \overline{T}_1)$ and those for $(v_2, T_2) = G_t(\overline{v}_2, \overline{T}_2)$, so that if we put $D^{[v]} = v_1 - v_2$ and $D^{[T]} = T_1 - T_2$ we obtain

$$(\varrho_1 + \pi_1)\partial_t D^{[v]} - \eta\Delta D^{[v]} - \left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla \cdot D^{[v]})$$

$$= -(E^{[\varrho]} + E^{[\pi]})\partial_t v_2 - (\varrho_1 + \pi_1)(\overline{v}_1 \cdot \nabla)\overline{D}^{[v]} - (E^{[\varrho]} + E^{[\pi]})(\overline{v}_1 \cdot \nabla)\overline{v}_2$$

$$- (\varrho_2 + \pi_2)(\overline{D}^{[v]} \cdot \nabla)\overline{v}_2 - R_0\nabla\left(\left(\frac{E^{[\varrho]}}{\mu_a} + \frac{E^{[\pi]}}{\mu_h}\right)\overline{T}_1\right)$$

$$- R_0 \nabla \left(\left(\frac{\varrho_2}{\mu_a} + \frac{\pi_2}{\mu_h} \right) \overline{D}^{[T]} \right) - \left[\int_0^\infty E^{[\sigma]} dm + E^{[\varrho]} + E^{[\pi]} \right] \nabla \Phi, \quad (8.11)$$

$$(\varrho_1 + \pi_1) c_v \partial_t D^{[T]} - \kappa \Delta D^{[T]}$$

$$= -(E^{[\varrho]} + E^{[\pi]}) c_v \partial_t T_2 - (\varrho_1 + \pi_1) c_v \sum_{i=1}^3 \overline{v}_{1,i} \frac{\partial \overline{D}^{[T]}}{\partial x_i}$$

$$- (\varrho_1 + \pi_1) c_v \sum_{i=1}^3 \overline{D}_i^{[v]} \frac{\partial \overline{T}_2}{\partial x_i} - (E^{[\varrho]} + E^{[\pi]}) c_v \sum_{i=1}^3 \overline{v}_{2,i} \frac{\partial \overline{T}_2}{\partial x_i}$$

$$- R_0 \left(\frac{\varrho_1}{\mu_a} + \frac{\pi_1}{\mu_h} \right) \overline{T}_1 \nabla \cdot \overline{D}^{[v]} - R_0 \left(\frac{\varrho_1}{\mu_a} + \frac{\pi_1}{\mu_h} \right) \overline{D}^{[T]} \nabla \cdot \overline{v}_2$$

$$- R_0 \left(\frac{E^{[\varrho]}}{\mu_a} + \frac{E^{[\pi]}}{\mu_h} \right) \overline{T}_2 \nabla \cdot \overline{v}_2$$

$$+ \eta \sum_{i,j=1}^3 \left(\frac{\partial \overline{D}_i^{[v]}}{\partial x_j} + \frac{\partial \overline{D}_j^{[v]}}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \overline{D}^{[v]} \right) \frac{\partial \overline{v}_{1,i}}{\partial x_j}$$

$$+ \eta \sum_{i,j=1}^3 \left(\frac{\partial \overline{v}_{2,i}}{\partial x_j} + \frac{\partial \overline{v}_{2,j}}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \overline{v}_2 \right) \frac{\partial \overline{D}_i^{[v]}}{\partial x_j}$$

$$+ \zeta \left[(\nabla \cdot \overline{v}_1)^2 - (\nabla \cdot \overline{v}_2)^2 \right] - \nabla \cdot (\mathcal{E}^{[1]} - \mathcal{E}^{[2]})$$

$$+ L_{gl} \left[H_{gl}(\overline{T}_1, \pi_1, \sigma_1) - H_{gl}(\overline{T}_2, \pi_2, \sigma_2) \right]. \quad (8.12)$$

If we remember (2.6), (2.8) (3.15), we have

$$\nabla \cdot (\mathcal{E}^{[1]} - \mathcal{E}^{[2]})$$

$$= - \int_0^\infty \left(\left(a_\lambda^{(1)} + r_\lambda^{(1)} \right) \varrho_1 + \left(a_\lambda^{(2)} + r_\lambda^{(2)} \right) \pi_1 \right.$$

$$\left. + \int_0^\infty \left(a_\lambda^{(3)}(m) + r_\lambda^{(3)}(m) \right) \sigma_1(m) dm \right)$$

$$\times \int_{S^2} \left(I_\lambda^{(1)}(x, q_1) - I_\lambda^{(2)}(x, q_1) \right) dq_1 d\lambda$$

$$- \int_0^\infty \left(\left(a_\lambda^{(1)} + r_\lambda^{(1)} \right) E^{[\varrho]} + \left(a_\lambda^{(2)} + r_\lambda^{(2)} \right) E^{[\pi]} \right.$$

$$\left. + \int_0^\infty \left(a_\lambda^{(3)}(m) + r_\lambda^{(3)}(m) \right) E^{[\sigma]} dm \right) \int_{S^2} I_\lambda^{(2)}(x, q_1) dq_1 d\lambda$$

$$+ \int_0^\infty \left(r_\lambda^{(1)} \varrho_1 + r_\lambda^{(2)} \pi_1 + \int_0^\infty r_\lambda^{(3)} \sigma_1(m) dm \right)$$

$$\times \int_{S^2} \left(I_\lambda^{(1)}(x, q_1') - I_\lambda^{(2)}(x, q_1') \right) dq_1' d\lambda$$

$$+ \int_0^\infty \left(r_\lambda^{(1)} E^{[\varrho]} + r_\lambda^{(2)} E^{[\pi]} + \int_0^\infty r_\lambda^{(3)}(m) E^{[\sigma]} dm \right) \int_{S^2} I_\lambda^{(2)}(x, q_1') dq_1' d\lambda$$

$$+ 4\pi \int_0^\infty \left(a_\lambda^{(1)} \varrho_1 + a_\lambda^{(2)} \pi_1 + \int_0^\infty a_\lambda^{(3)}(m) \sigma_1(m) dm \right)$$

$$\times \left(B(\lambda, \overline{T}_1) - B(\lambda, \overline{T}_2) \right) d\lambda$$

$$+ 4\pi \int_0^\infty \left(a_\lambda^{(1)} E^{[\varrho]} + a_\lambda^{(2)} E^{[\pi]} + \int_0^\infty a_\lambda^{(3)}(m) E^{[\sigma]} dm \right) B(\lambda, \overline{T}_2) d\lambda. \tag{8.13}$$

We multiply the equations (8.11) and (8.12) respectively by $D^{[v]}/(\varrho_1 + \pi_1)$ and $D^{[T]}/(\varrho_1 + \pi_1)$ and integrate them on Ω. We remember the inequalities

$$\eta \left| \int_\Omega \frac{\sum_{i,j}^3 \frac{\partial(\varrho_1 + \pi_1)}{\partial x_j} \frac{\partial D_i^{[v]}}{\partial x_j} D_i^{[v]}}{(\varrho_1 + \pi_1)^2} dx \right| + \left(\zeta + \frac{\eta}{3} \right) \left| \int_\Omega \frac{\sum_{i,j}^3 \frac{\partial(\varrho_1 + \pi_1)}{\partial x_i} \frac{\partial D_j^{[v]}}{\partial x_j} D_i^{[v]}}{(\varrho_1 + \pi_1)^2} dx \right|$$

$$\leq \overline{c} \left(\|\varrho_1\|_{W_p^1(\Omega)} + \|\pi_1\|_{W_p^1(\Omega)} \right) \|D^{[v]}\|_{H^1(\Omega)}^{1+3/p} \|D^{[v]}\|_{L^2(\Omega)}^{1-3/p}$$

$$\leq \varepsilon \|D^{[v]}\|_{H^1(\Omega)}^2 + C_\varepsilon \left(\|\varrho_1\|_{W_p^1(\Omega)}^{2p/(p-3)} + \|\pi_1\|_{W_p^1(\Omega)}^{2p/(p-3)} \right) \|D^{[v]}\|_{L^2(\Omega)}^2,$$

where C_ε is a constant determined by an arbitrary constant $\varepsilon > 0$. To estimate the term

$$\left| \int_\Omega \frac{D^{[T]}}{\rho_1 + \pi_1} \nabla \cdot (\mathcal{E}^{[1]} - \mathcal{E}^{[2]}) dx \right| \leq \frac{2}{\inf \rho_0} \left\| \nabla \cdot (\mathcal{E}^{[1]} - \mathcal{E}^{[2]}) \right\|_{L^2(\Omega)} \left\| D^{[T]} \right\|_{L^2(\Omega)},$$

we deduce, from Lemma 4.5

$$
\|\nabla \cdot (\mathcal{E}^{[1]} - \mathcal{E}^{[2]})\|_{L^2(\Omega)}
$$
$$
\leq c\big(1 + \|\varrho_1\|_{W_p^1(\Omega)} + \|\pi_1\|_{W_p^1(\Omega)} + \|\sigma_1\|_{W_p^1(\Omega_{\overline{M}_1})}\big)
$$
$$
\times \big(\|E^{[\varrho]}\|_{L^2(\Omega)} + \|E^{[\pi]}\|_{L^2(\Omega)} + \|E^{[\sigma]}\|_{L^2(\Omega_{\overline{M}_1})} + \|\overline{D}^{[T]}\|_{L^2(\Omega)}\big). \quad (8.14)
$$

Thus using repeatedly the Sobolev, Hölder and Cauchy-Schwartz inequalities, we obtain

$$
\frac{d}{dt}\|D^{[v]}\|_{L^2(\Omega)}^2 + \overline{c}_0\|D^{[v]}\|_{H^1(\Omega)}^2
$$
$$
\leq c(1 + \|\overline{v}_2\|_{W_p^2(\Omega)^2})\big(\big\|D^{[v]}\big\|_{L^2(\Omega)}^2 + \big\|E^{[\varrho]}\big\|_{L^2(\Omega)}^2 + \big\|E^{[\pi]}\big\|_{L^2(\Omega)}^2
$$
$$
+ \big\|E^{[\sigma]}\big\|_{L^2(\Omega_{\overline{M}_1})}^2 + \big\|\overline{D}^{[v]}\big\|_{L^2(\Omega)}^2 + \big\|\overline{D}^{[T]}\big\|_{L^2(\Omega)}^2\big), \quad (8.15)
$$

$$
\frac{d}{dt}\|D^{[T]}\|_{L^2(\Omega)}^2 + \overline{c}_0\|D^{[T]}\|_{H^1(\Omega)}^2
$$
$$
\leq c\big(1 + \|T_2\|_{W_q^2(\Omega)}^2 + \|\overline{v}_2\|_{W_p^2(\Omega)}^2\big)\big(\big\|D^{[T]}\big\|_{L^2(\Omega)}^2 + \big\|E^{[\varrho]}\big\|_{L^2(\Omega)}^2
$$
$$
+ \big\|E^{[\pi]}\big\|_{L^2(\Omega)}^2 + \big\|E^{[\sigma]}\big\|_{L^2(\Omega_{\overline{M}_1})}^2 + \big\|\overline{D}^{[v]}\big\|_{L^2(\Omega)}^2 + \big\|\overline{D}^{[T]}\big\|_{L^2(\Omega)}^2\big). \quad (8.16)
$$

From (8.10), (8.15)–(8.16) we obtain

$$
\|D^{[v]}(t)\|_{L^2(\Omega)}^2 + \|D^{[T]}(t)\|_{L^2(\Omega)}^2
$$
$$
+ \overline{c}_0 \int_0^t \big(\|D^{[v]}(t')\|_{H^1(\Omega)}^2 + \|D^{[T]}(t')\|_{H^1(\Omega)}^2 dt'\big)
$$
$$
\leq c e^{c(t + t^{(q-2)/q} + t^{(p-2)/p})}\big(t + t^{(q-2)/q} + t^{(p-2)/p}\big)
$$
$$
\times \Big[\|\overline{D}^{[T]}\|_{L^\infty(0,t;L^2(\Omega))}^2 + \|\overline{D}^{[v]}\|_{L^\infty(0,t;L^2(\Omega))}^2
$$
$$
+ c e^{c[t + t^{(p-1)/p} + t^{(q-2)/q}]} \int_0^t \big(\|\overline{D}^{[v]}(t')\|_{H^1(\Omega)}^2 + \|\overline{D}^{[T]}(t')\|_{L^2(\Omega)}^2\big) dt'\Big].
$$
$$
(8.17)
$$

The inequality (8.17) allows us to find a $\bar{t} \in [0, t_5]$ such that

$$
\|D^{[v]}\|^2_{L^\infty(0,\bar{t};L^2(\Omega))} + \|D^{[T]}\|^2_{L^\infty(0,\bar{t};L^2(\Omega))}
$$
$$
+ \|D^{[v]}\|^2_{L^2(0,\bar{t};H^1(\Omega))} + \|D^{[T]}\|^2_{L^2(0,\bar{t};H^1(\Omega))}
$$
$$
\leq \kappa\big(\|\overline{D}^{[v]}\|^2_{L^\infty(0,\bar{t};L^2(\Omega))} + \|\overline{D}^{[T]}\|^2_{L^\infty(0,\bar{t};L^2(\Omega))}
$$
$$
+ \|\overline{D}^{[v]}\|^2_{L^2(0,\bar{t};H^1(\Omega))} + \|\overline{D}^{[T]}\|^2_{L^2(0,\bar{t};H^1(\Omega))}\big)
$$

with $0 < \kappa < 1$. It means that the operator $G_{\bar{t}} : B_{\bar{t}} \to B_{\bar{t}}$ is a contraction. This, also thanks to the Lemma 4.1, allows us to conclude the proof of the existence and uniqueness of the solution on the interval $[0, \bar{t}]$. $\qquad \square$

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