Arithmetic identities for class regular partitions

To the memory of Yusuke Kawamoto, our friend

Hiroshi Mizukawa and Hiro-Fumi Yamada

Abstract. Extending the notion of r-(class) regular partitions, we define $(r_1, \ldots, r_m)$-class regular partitions. Partition identities are presented and described by making use of the Glaisher correspondence.

Key words: r-regular partition, r-class regular partition, Glaisher correspondence, Hall-Littlewood symmetric function, character table of symmetric group.

1. Introduction

Partitions of natural numbers are ubiquitous in representation theory. Typically, they label the ordinary irreducible representations of the symmetric groups. Turning to modular representations of the symmetric groups, some restrictions to the partitions naturally arise. Namely, for a prime r, r-modular irreducible representations are labeled by the r-regular partitions. On the other hand, the r-regular conjugacy classes correspond to the r-class regular partitions. As Euler noticed, r-regular partitions of n are equinumerous to the r-class regular partitions of n. The natural combinatorial bijection between these two sets is called the Glaisher correspondence. One of the authors studied in [1] the graded version of Glaisher correspondence and revealed an intimate role of the correspondence in modular representation theory. This suggests that the "Glaisher combinatorics" should be one of the keys in the investigation of the symmetric groups.

We are especially interested in the character tables of the symmetric groups which are regarded as a square matrix. In order to understand the results of Olsson [8] on the determinants of regular character tables, we develop some arithmetic identities for the class regular partitions (Theorem 2.1 and Theorem 2.3). In this connection, Bessenrodt et al. [3] presented

nice combinatorial identities. Our formula generalizes a part of [3].

When we look at the r-modular ($r \geq 3$, odd) representations of the covering of the symmetric group, we need to handle the partitions which are 2-class regular and r-class regular. In this note, motivated by the above, we define $\underline{r}$-regular/$\underline{r}$-class regular partitions for a mutually coprime integral sequence $\underline{r} = (r_1, \ldots, r_m)$. We give some partition identities and generating functions.

The paper is organized as follows. In Section 2, we derive our main formula (Theorem 2.1 and Theorem 2.3) which are on multiplicities of parts in $\underline{r}$-class regular partitions. Section 3 is devoted to a rephrase of the formula in terms of the Glaisher correspondence. Although this is an easy algorithm, we expect this gives a path to modular representation theory of the symmetric groups. In Section 4, the single r case is discussed. We consider the r-regular character table of the symmetric groups and provide a proof of Olsson's determinant formula [2], [3], [8]. The key of our proof is the transition matrix of the Hall-Littlewood symmetric functions at the r-th root of unity, and the Schur functions. In [3] the determinant formula is proved in a bijective way and the method using the symmetric functions is only suggested. We verify the detailed computations in this note.

The author thank the anonymous referee for his/her careful reading and helpful comments.

2. $\underline{r}$-class regular partitions

Let $\underline{r} = (r_1, r_2, \ldots, r_m)$ be a tuple of positive integers grater than 1. Throughout the paper, we assume that any two integers r_i and r_j ($i \neq j$) in $\underline{r}$ are coprime. If an integer n is not divisible by $r_1, r_2, \ldots, r_m$, then we write $n \not\equiv 0 \pmod{\underline{r}}$. A partition λ said to be $\underline{r}$-class regular if any parts of λ are not divisible by $r_1, r_2, \ldots, r_m$. Let $CP_{\underline{r},n}$ be the set of the $\underline{r}$-class regular partitions of n. We put

$$\pi_k(q) = \prod_{1 \leq i_1 < \cdots < i_k \leq m} (1 - q^{r_{i_1} \cdots r_{i_k}}).$$

and

$$\Phi_{\underline{r}}(q) = \begin{cases} \displaystyle\prod_{n\geq 1} \frac{\pi_1(q^n)\pi_3(q^n)\cdots\pi_{m-1}(q^n)}{\pi_2(q^n)\pi_4(q^n)\cdots\pi_m(q^n)} \frac{1}{1-q^n}, & m \equiv 0 \pmod 2 \\[3ex] \displaystyle\prod_{n\geq 1} \frac{\pi_1(q^n)\pi_3(q^n)\cdots\pi_m(q^n)}{\pi_2(q^n)\pi_4(q^n)\cdots\pi_{m-1}(q^n)} \frac{1}{1-q^n}, & m \equiv 1 \pmod 2. \end{cases}$$

Then the inclusion-exclusion principle gives us

$$\Phi_{\underline{r}}(q) = \prod_{n\not\equiv 0 \pmod{\underline{r}}} \frac{1}{1-q^n} = \sum_{n\geq 0} |CP_{\underline{r},n}|q^n.$$

We define, for $j \geq 1$,

$$V_{\underline{r},j,n} = \sum_{\rho\in CP_{\underline{r},n}} m_j(\rho) \quad\text{and}\quad W_{\underline{r},j,n} = \sum_{\rho\in CP_{\underline{r},n}} \big|\{i \mid m_i(\rho) \geq j\}\big|,$$

where $m_i(\rho)$ means the multiplicity of $i \geq 1$ in ρ.

Theorem 2.1 *If $j \not\equiv 0 \pmod{\underline{r}}$, then we have*

$$V_{\underline{r},j,n} = \sum_{k_1,\ldots,k_m\geq 0} W_{\underline{r},r_1^{k_1}r_2^{k_2}\cdots r_m^{k_m}j,n}.$$

Before proving this theorem, we give an example.

Example 2.2 ($\underline{r} = (2,3)$ and $n = 10$)

j	1	2	3	4	5	6	7	8	9	10
$V_{\underline{r},j,n}$	18	0	0	0	3	0	1	0	0	0
$W_{\underline{r},j,n}$	6	4	3	2	2	1	1	1	1	1

$$\begin{cases} V_{\underline{r},1,10} = W_{\underline{r},1,10} + W_{\underline{r},2,10} + W_{\underline{r},3,10} + W_{\underline{r},4,10} + W_{\underline{r},6,10} + W_{\underline{r},8,10} + W_{\underline{r},9,10} \\ V_{\underline{r},5,10} = W_{\underline{r},5,10} + W_{\underline{r},10,10} \\ V_{\underline{r},7,10} = W_{\underline{r},7,10}. \end{cases}$$

Proof of Theorem 2.1. Let $j \not\equiv 0 \pmod{\underline{r}}$. We have

$$\Phi_{\underline{r}}(q)\frac{1-q^j}{1-tq^j} = \sum_{n\geq 0}\left(\sum_{\rho\in CP_{\underline{r},n}} t^{m_j(\rho)}\right)q^n.$$

Taking the t-derivative at $t = 1$, we obtain

$$\Phi_{\underline{r}}(q)\frac{q^j}{1 - q^j} = \sum_{n \geq 0} V_{\underline{r},j,n}q^n. \tag{2.1}$$

Let $\ell \not\equiv 0 \pmod{\underline{r}}$ and $j \geq 1$. We have

$$\sum_{n \geq 0} |\{\rho \in CP_{\underline{r},n} \mid m_\ell(\rho) \geq j\}|q^n$$
$$= \Phi_{\underline{r}}(q)(1 - q^\ell)(q^{j\ell} + q^{(j+1)\ell} + q^{(j+2)\ell} + \cdots)$$
$$= \Phi_{\underline{r}}(q)q^{j\ell}$$

We take sum over $\ell \not\equiv 0 \pmod{\underline{r}}$ and obtain the generating function of $W_{\underline{r},j,n}$:

$$\sum_{n \geq 0} W_{\underline{r},j,n}q^n = \Phi_{\underline{r}}(q) \sum_{\ell \not\equiv 0 \,(\mathrm{mod}\ \underline{r})} q^{j\ell}$$

$$= \Phi_{\underline{r}}(q) \sum_{\ell \geq 1} \left\{ q^{j\ell} - \left(\sum_{i_1 = 1}^{m} q^{r_{i_1}j\ell} \right) \right.$$
$$\left. + \left(\sum_{1 \leq i_1 < i_2 \leq m} q^{r_{i_1}r_{i_2}j\ell} \right) - \cdots + (-1)^m q^{r_1 r_2 \cdots r_m j\ell} \right\}$$

$$= \Phi_{\underline{r}}(q) \left\{ \frac{q^j}{1 - q^j} + \sum_{k=1}^{m} \left((-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \frac{q^{r_{i_1}r_{i_2}\cdots r_{i_k}j}}{1 - q^{r_{i_1}r_{i_2}\cdots r_{i_k}j}} \right) \right\}.$$

We replace j by $r_1^{k_1} r_2^{k_2} \cdots r_m^{k_m} j$ in this equation and consider the summation over $k_1, \ldots, k_m \geq 0$:

$$\sum_{n \geq 0} \left(\sum_{k_1, \ldots, k_m \geq 0} W_{\underline{r}, r_1^{k_1} r_2^{k_2} \cdots r_m^{k_m} j, n} \right) q^n. \tag{2.2}$$

Now we assume that m' $(0 \leq m' \leq m)$ entries of $(k_1, \ldots, k_m)$ are not zero and the remains are zero. Look at the summation

$$\sum_{k_1,\ldots,k_m\geq 0}\left(\sum_{1\leq i_1<\cdots<i_a\leq m}\frac{q^{r_{i_1}r_{i_2}\cdots r_{i_a}r_1^{k_1}r_2^{k_2}\cdots r_m^{k_m}j}}{1-q^{r_{i_1}r_{i_2}\cdots r_{i_a}r_1^{k_1}r_2^{k_2}\cdots r_m^{k_m}j}}\right)$$

for $0\leq a\leq m'$. Then the coefficient of $(q^{r_1^{k_1}\cdots r_m^{k_m}j})/(1-q^{r_1^{k_1}\cdots r_m^{k_m}j})$ is equal to $\binom{m'}{a}$. Since $\sum_{a\geq 0}(-1)^a\binom{m'}{a}=0$, we have

$$\sum_{n\geq 0}\left(\sum_{k_1,\ldots,k_m\geq 0}W_{\underline{r},r_1^{k_1}r_2^{k_2}\cdots r_m^{k_m}j,n}\right)q^n=\Phi_{\underline{r}}(q)\frac{q^j}{1-q^j}. \qquad (2.3)$$

From (2.1) and (2.3) we obtain the formula

$$V_{\underline{r},j,n}=\sum_{k_1,\ldots,k_m\geq 0}W_{\underline{r},r_1^{k_1}r_2^{k_2}\cdots r_m^{k_m}j,n}. \qquad\qquad \square$$

We put

$$\begin{cases}a_{\underline{r},n}=\prod_{\rho\in CP_{\underline{r},n}}\prod_{i=1}^{\ell(\rho)}\rho_i,\\ b_{\underline{r},n}=\prod_{\rho\in CP_{\underline{r},n}}\prod_{i\geq 1}m_i(\rho)!,\end{cases}$$

where we write $\rho=(\rho_1,\rho_2,\ldots)$. Then the following theorem holds.

Theorem 2.3 We have $b_{\underline{r},n}=\prod_{i=1}^m r_i^{c_{r_i,n}}a_{\underline{r},n}$, where $c_{r_i,n}$ is given by

$$c_{r_i,n}=\sum_{j\not\equiv 0\,(\mathrm{mod}\,\underline{r})}\sum_{k_1,\ldots,k_m\geq 0}k_i\,W_{\underline{r},r_1^{k_1}r_2^{k_2}\cdots r_m^{k_m}j,n}.$$

Proof. Since $V_{\underline{r},j,n}=0$ unless $j\not\equiv 0\pmod{\underline{r}}$, we have

$$\prod_{\rho\in CP_{\underline{r},n}}\prod_{i\geq 1}\rho_i=\prod_{j\geq 1}j^{V_{\underline{r},j,n}}=\prod_{j\not\equiv 0\,(\mathrm{mod}\,\underline{r},)}j^{V_{\underline{r},j,n}}.$$

Let $\underline{i}=(i_1,\ldots,i_m)$ be a tuple of non-negative integers and write $\underline{r}^{\underline{i}}=r_1^{i_1}\cdots r_m^{i_m}$. We compute

$$\prod_{\rho\in CP_{\underline{r},n}}\prod_{i\geq 1}m_i(\rho)!=\prod_{j\geq 1}j^{W_{\underline{r},j,n}}=\prod_{j\not\equiv 0\,(\mathrm{mod}\,\underline{r})}\prod_{i_1,\ldots,i_m\geq 0}(\underline{r}^{\underline{i}}j)^{W_{\underline{r},\underline{r}^{\underline{i}}j,n}}$$

$$= \left(\prod_{j \not\equiv 0 \ (\mathrm{mod} \ \underline{r})} \prod_{i_1,\ldots,i_m \geq 0} (\underline{r}^{\underline{i}})^{W_{r,r^i j,n}} \right)$$

$$\times \left(\prod_{j \not\equiv 0 \ (\mathrm{mod} \ \underline{r})} \prod_{i_1,\ldots,i_m \geq 0} j^{W_{r,r^i j,n}} \right)$$

$$= \prod_{i=1}^{m} r_i^{c_{r_i,n}} \prod_{j \not\equiv 0 \ (\mathrm{mod} \ r)} j^{\sum_{i_1,\ldots,i_m \geq 0} W_{r,\underline{r}^{\underline{i}} j,n}}$$

$$= \prod_{i=1}^{m} r_i^{c_{r_i,n}} \prod_{j \not\equiv 0 \ (\mathrm{mod} \ r)} j^{V_{\underline{r},j,n}},$$

where the last equality follows from Theorem 2.1.   □

The generating functions of $c_{r_i,n}$'s are given by the following theorem.

Theorem 2.4   *For $1 \leq i \leq m$, we have*

$$\sum_{n \geq 0} c_{r_i,n} q^n = \Phi_{\underline{r}}(q) \Bigg(\sum_{n \geq 0} \frac{q^{r_i n}}{1 - q^{r_i n}}$$

$$+ \sum_{k=1}^{m-1} (-1)^k \sum_{\substack{1 \leq l_1 < \cdots < l_k \leq m \\ l_1,\ldots,l_m \neq i}} \sum_{n \geq 0} \frac{q^{r_i r_{l_1} \cdots r_{l_k} n}}{1 - q^{r_i r_{l_1} \cdots r_{l_k} n}} \Bigg)$$

$$= \Phi_{\underline{r}}(q) \sum_{n \not\equiv 0 \ (\mathrm{mod} \ \underline{r}^{(i)})} \frac{q^{r_i n}}{1 - q^{r_i n}},$$

where $\underline{r}^{(i)} = (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m)$.

Proof.   Without loss of generality, we can assume $i = 1$. We compute

$$\sum_{n \geq 0} \left(\sum_{i_1,\ldots,i_m \geq 0} i_1 W_{\underline{r},\underline{r}^{\underline{i}} j,n} \right) q^n = \sum_{i_1,\ldots,i_m \geq 0} i_1 \left(\sum_{n \geq 0} W_{\underline{r},\underline{r}^{\underline{i}} j,n} q^n \right)$$

$$= \Phi_{\underline{r}}(q) \sum_{i_2,\ldots,i_m \geq 0} \sum_{i_1 \geq 0} i_1 \left\{ \frac{q^{\underline{r}^{\underline{i}} j}}{1 - q^{\underline{r}^{\underline{i}} j}} + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq l_1 < \cdots < l_k \leq m} \frac{q^{r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}} j}}{1 - q^{r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}} j}} \right\}$$

$$= \Phi_{\underline{r}}(q) \sum_{i_2,\ldots,i_m \geq 0} \left\{ \sum_{i_1 \geq 1} i_1 \left(\frac{q^{\underline{r}^{\underline{i}}j}}{1 - q^{\underline{r}^{\underline{i}}j}} - \frac{q^{r_1 \underline{r}^{\underline{i}}j}}{1 - q^{r_1 \underline{r}^{\underline{i}}j}} \right) \right.$$

$$\left. + \sum_{k=1} (-1)^k \sum_{i_1 \geq 1} i_1 \sum_{2 \leq l_1 < \cdots < l_k \leq m} \left(\frac{q^{r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}}j}}{1 - q^{r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}}j}} - \frac{q^{r_1 r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}}j}}{1 - q^{r_1 r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}}j}} \right) \right\}$$

$$= \Phi_{\underline{r}}(q) \sum_{i_2,\ldots,i_m \geq 0} \sum_{i_1 \geq 1} \left\{ \frac{q^{\underline{r}^{\underline{i}}j}}{1 - q^{\underline{r}^{\underline{i}}j}} + \sum_{k=1}^{m} (-1)^k \sum_{2 \leq l_1 < \cdots < l_k \leq m} \frac{q^{r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}}j}}{1 - q^{r_{l_1} \cdots r_{l_k} \underline{r}^{\underline{i}}j}} \right\}.$$

Now we take sum over $j \not\equiv 0 \pmod{\underline{r}}$ to have the generating function of $c_{r_1,n}$ as desired. The second equality in the theorem follows from the inclusion-exclusion principle. $\square$

3. Glaisher Combinatorics

Let $RP_{\underline{r},n}$ be the set of partitions whose parts are not divisible by r_i for any $i = 2, 3, \ldots, m$ and the multiplicity of each part is less than r_1. A partition $\lambda \in RP_{\underline{r},n}$ said to be an $\underline{r}$-regular. We rewrite $\Phi_{\underline{r}}(q)$ as follows:

$$\Phi_{\underline{r}}(q) = \prod_{n \geq 1} \frac{1 - q^{r_1 n}}{1 - q^n} \prod_{i=2}^{m} \frac{(1 - q^{r_i n})}{(1 - q^{r_1 r_i n})} \prod_{i<j} \frac{(1 - q^{r_1 r_i r_j n})}{(1 - q^{r_i r_j n})} \cdots$$

$$= \prod_{n \geq 1} \left(\sum_{k=0}^{r_1 - 1} q^{nk} \right) \prod_{i=2}^{m} \frac{1}{\sum_{k=0}^{r_1 - 1} q^{r_i nk}} \prod_{i<j} \left(\sum_{k=0}^{r_1 - 1} q^{r_i r_j nk} \right) \cdots$$

$$= \sum_{n \geq 0} |RP_{\underline{r},n}| q^n$$

Therefore we have $|CP_{\underline{r},n}| = |RP_{\underline{r},n}|$. A concrete bijection will be described in this section. The following proposition is a direct consequence of the generating function $\Phi_{\underline{r}}(q)$.

Proposition 3.1 *For any permutation* $\underline{s} = (s_1, \ldots, s_m)$ *of* $\underline{r} = (r_1, \ldots, r_m)$, *we have*

$$|RP_{\underline{r},n}| = |RP_{\underline{s},n}|.$$

For example, the number of 2-regular, 3-class regular partitions of n

is equal to the number of 3-regular, 2-class regular partitions of n. This is also equal to the number of partitions of n whose parts are of the form $6k \pm 1$ ($k \geq 0$).

There is a natural bijection between the sets $RP_{\underline{r},n}$ and $CP_{\underline{r},n}$. Take $\lambda \in RP_{\underline{r},n}$. If λ has a multiple of r_1 as a part, say kr_1, then replace kr_1 by k^{r_1}. By this step the length of the partition increases by $r_1 - 1$. Repeat these steps until the partition has come to an element $g_{r_1}(\lambda)$ of $CP_{\underline{r},n}$. The map $g_{r_1} : RP_{\underline{r},n} \to CP_{\underline{r},n}$ is called the Glaisher correspondence, and shown to be bijective.

The number of steps for obtaining $g_{r_1}(\lambda) \in CP_{\underline{r},n}$ from $\lambda \in RP_{\underline{r},n}$ equals

$$\frac{\ell(g_{r_1}(\lambda)) - \ell(\lambda)}{r - 1}.$$

Define for $\rho \in CP_{\underline{r},n}$ and $j \not\equiv 0 \pmod{r_1}$, $y_{r^k j}(\rho) = |\{i \geq 1 \mid m_i(\rho) \geq r^k j\},|$ and

$$G_j(\rho) = \sum_{k \geq 1} k y_{r_1^k j}(\rho).$$

For example, if $r_1 = 3$ and $\rho = (1^9)$, then $G_1(\rho) = 3, G_2(\rho) = 1$ and $G_j(\rho) = 0$ otherwise. Put $G(\rho) = \sum_{j \not\equiv 0 \pmod{r_1}} G_j(\rho)$. This is nothing but the times of Glaisher steps for $g^{-1}(\rho) \mapsto \rho$, and also we have the following.

Proposition 3.2

$$c_{r_1,n} = \sum_{\rho \in CP_{\underline{r},n}} G(\rho).$$

Proof.   Proof is just by interchanging the order of the summation.   □

4. r-regular character table

Throughout this section, we fix a positive integer $r \geq 2$. Here we restrict our attention to the case $m = 1$. We will relate the analysis of r-(class) regular partitions with the character tables of the symmetric groups. It should be remarked that Bessenrodt et al. [3] already proved Theorem 4.1. They give a bijective proof, and also sketch a proof using generating functions. Here we supply the proof relying on the generating functions for the sake of completeness.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and $j \in \{1, 2, \dots, r-1\}$, we put

$$x_{r,j}(\lambda) = \big|\{i \mid \lambda_i \equiv j \pmod{r}\}\big| \ \text{ and } \ y_{r,j}(\lambda) = \big|\{i \mid m_i(\lambda) \geq j\}\big|.$$

We define

$$X_{r,j,n} = \sum_{\rho \in CP_{r,n}} x_{r,j}(\rho) \ \text{ and } \ Y_{r,j,n} = \sum_{\lambda \in RP_{r,n}} y_{r,j}(\lambda).$$

Theorem 4.1 ([3]) $\quad X_{r,j,n} - Y_{r,j,n} = c_{r,n} \text{ for } j = 1, 2, \dots, r-1.$

For this theorem a bijective proof is given in [3]. Here we present a proof using generating functions.

Proof. First we will compute the generating function of $X_{r,j,n}$. For $i \not\equiv 0 \pmod{r}$, we have

$$\Phi_r(q) \frac{1-q^i}{1-tq^i} = \sum_{n \geq 0} \big(t^{\sum_{\rho \in CP_{r,n}} m_i(\rho)}\big) q^n.$$

Taking the t-derivative at $t = 1$, we obtain

$$\Phi_r(q) \frac{q^i}{1-q^i} = \sum_{n \geq 0} \bigg(\sum_{\rho \in CP_{r,n}} m_i(\rho) \bigg) q^n. \tag{4.1}$$

Since $x_{r,j}(\rho) = \sum_{k \geq 0} m_{kr+j}(\rho)$, we have the following generating function of $X_{r,j,n}$.

$$\sum_{n \geq 0} X_{r,j,n} q^n = \Phi_r(q) \sum_{k \geq 0} \frac{q^{rk+j}}{1-q^{rk+j}}.$$

Second, we consider the r-regular partitions and the generating function of $Y_{r,j,n}$. We put

$$\Phi_{r,j}(q,t) = \prod_{k \geq 1} (1 + q^k + q^{2k} + \dots + q^{(j-1)k} + tq^{jk} + tq^{(j+1)k} + \dots + tq^{(r-1)k}). \tag{4.2}$$

Immediately we have

$$\Phi_{r,j}(q,t) = \sum_{n \geq 0}\left(\sum_{\lambda \in RP_{r,n}} t^{y_{r,j}(\lambda)}\right)q^n.$$

Taking the t-derivative at $t = 1$, we obtain

$$\frac{d}{dt}\Phi_{r,j}(q,t)\Big|_{t=1} = \sum_{n \geq 0}\left(\sum_{\lambda \in RP_{r,n}} y_{r,j}(\lambda)\right)q^n = \sum_{n \geq 0}Y_{r,j,n}q^n.$$

As for the equation (4.2), we have

$$\frac{d}{dt}\Phi_{r,j}(q,t)\Big|_{t=1} = \Phi_{r,j}(q,t)\sum_{k \geq 1}\frac{q^{jk} - q^{rk}}{1 - q^{rk}}.$$

The generating function of $Y_{r,j,n}$ reads

$$\sum_{n \geq 0}Y_{r,j,n}q^n = \Phi_r(q)\sum_{k \geq 1}\frac{q^{jk} - q^{rk}}{1 - q^{rk}}.$$

To complete the proof, we compute

$$\sum_{n \geq 0}X_{r,j,n}q^n - \sum_{n \geq 0}Y_{r,j,n}q^n$$

$$= \Phi_r(q)\left(\sum_{k \geq 0}(q^{rk+j} + q^{2(rk+j)} + q^{3(rk+j)} + \cdots)\right.$$

$$\left. - \sum_{m \geq 1}(q^{jm} + q^{(r+j)m} + q^{(2r+j)m} + \cdots) + \sum_{k \geq 1}\frac{q^{rk}}{1 - q^{rk}}\right)$$

$$= \Phi_r(q)\left(\sum_{k \geq 0}\sum_{m \geq 1}q^{m(rk+j)} - \sum_{m \geq 1}\sum_{k \geq 0}q^{(kr+j)m} + \sum_{k \geq 1}\frac{q^{rk}}{1 - q^{rk}}\right)$$

$$= \Phi_r(q)\sum_{k \geq 1}\frac{q^{rk}}{1 - q^{rk}} = \sum_{n \geq 0}c_{r,n}q^n. \qquad \square$$

Example 4.2   We take $r = 3$ and $n = 7$. The following table lists the 3-class regular partitions of $n = 7$:

ρ	7	52	51^2	421	41^3	2^31	2^21^3	21^5	1^7	total
$x_{3,1}(\rho)$	1	0	2	2	4	1	3	5	7	25
$x_{3,2}(\rho)$	0	2	1	1	0	3	2	1	0	10

From the table, we have $X_{3,1,7} = 25$ and $X_{3,2,7} = 10$. As for the 3-regular partitions of 7, we have

λ	7	61	52	51^2	52	421	3^21	32^2	321^2	total
$\prod_{i\geq 1} m_i(\lambda)!$	1	1·1	1·1	1·21	1·1	1·1·1	21·1	1·21	1·1·21	–
$y_{3,1}(\lambda)$	1	2	2	2	2	3	2	2	3	19
$y_{3,2}(\lambda)$	0	0	0	1	0	0	1	1	1	4

From the second table, we have $Y_{3,1,7} = 19$ and $Y_{3,2,7} = 4$. Thus we see

$$X_{3,1,7} - Y_{3,1,7} = X_{3,2,7} - Y_{3,2,7} = 6.$$

On the other hand we have

$$\Phi_3(q) \sum_{k\geq 1} \frac{q^{3k}}{1 - q^{3k}} = q^3 + q^4 + 2q^5 + 4q^6 + 6q^7 + 9q^8 + 13q^9 + 19q^{10} + \cdots .$$

4.1. Hall-Littlewood symmetric functions at root of unity

Next, we apply Theorem 4.1 to computations of some minor determinants of transition matrices and the character tables of the symmetric groups. The Hall-Littlewood P- and Q- symmetric functions ([6]) are a one parameter family of symmetric functions satisfying the orthogonality relation:

$$\langle P_\lambda(x;t), Q_\mu(x;t)\rangle_t = \delta_{\lambda\mu},$$

where the inner product $\langle,\rangle_t$ is defined by $\langle p_\lambda(x), p_\mu(x)\rangle_t = z_\lambda(t)\delta_{\lambda\mu}$ with $z_\lambda(t) = z_\lambda \prod_{i\geq 1}(1 - t^{\lambda_i})^{-1}$. Let $(a;t)_n$ be a t-shifted factorial:

$$(a;t)_n = \begin{cases} (1-a)(1-at)\cdots(1-at^{n-1}) & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

The relation between P- and Q- functions is described as

$$Q_\lambda(x) = b_\lambda(t)P_\lambda(x),$$

where $b_\lambda(t) = \prod_{i \geq 1}(t; t)_{m_i(\lambda)}$.

4.2. Q'-functions

We are interested in the case that parameter t is a primitive r-th root of unity ζ. The Hall-Littlewood symmetric functions at root of unity is studied at the first time by [7]. We remark that $\{Q_\lambda(x; \zeta) \mid \lambda \in RP_{r,n}\}$ is a $\mathbb{Q}(\zeta)$-basis for the subspace $\Lambda^{(r)} = \mathbb{Q}(\zeta)[p_s(x) \mid s \not\equiv 0 \pmod r]$ of the symmetric function ring $\Lambda = \mathbb{Q}(\zeta)[p_s(x) \mid s = 1, 2, \dots]$. This can be shown along the arguments in [6, Chapter 3–8], where the case $r = 2$ is discussed. In [5], Lascoux, Leclerc and Thibon consider the dual basis (Q'_λ) of P-functions, relative to the inner product at $t = 0$. Namely P- and Q'- functions satisfy the Cauchy identity:

$$\sum_\lambda P_\lambda(x; t)Q'_\lambda(y; t) = \prod_{i,j}(1 - x_iy_j)^{-1}.$$

When $t = \zeta$, the Q'-functions have the following nice factorization property.

Proposition 4.3 ([5])   *Let ζ be a primitive r-th root of unity. If a partition λ satisfies $m_i(\lambda) \geq r$, then we have*

$$Q'_\lambda(x; \zeta) = (-1)^{i(r-1)}Q'_{\lambda \setminus (i^r)}(x; \zeta)h_i(x^r).$$

Here $h_i(x^r) = h_i(x_1^r, x_2^r, \dots)$ and $\lambda \setminus (i^r)$ is a partition obtained by removing the rectangle (r^i) from the Young diagram λ.

We define an *r-reduction* for a symmetric function $f(x)$ by

$$f^{(r)}(x) = f(x)\big|_{p_r(x) = p_{2r}(x) = p_{3r}(x) = \cdots = 0}.$$

Proposition 4.3 leads us to the following lemma.

Lemma 4.4 $Q'^{(r)}_\lambda(x; \zeta) = 0$ *unless λ is an r-regular partition.*

We set

$$Q_\lambda(x; \zeta) = \sum_{\rho \in CP_{r,n}} Q^\lambda_\rho p_\rho(x) \text{ and } Q'^{(r)}_\lambda(x; \zeta) = \sum_{\rho \in CP_{r,n}} Q'^\lambda_\rho p_\rho(x).$$

Proposition 4.5 *Let $\lambda \in RP_{r,n}$ and $\rho \in CP_{r,n}$. We have*

$$Q'^{\lambda}_{\rho} = \prod_{i \geq 1}(1 - \zeta^{\rho_i})^{-1}Q^{\lambda}_{\rho}.$$

Proof. We compute inner products at $t = \zeta$ and $t = 0$ for r-regular partitions λ and μ. Namely, we see

$$\delta_{\lambda\mu} = \langle P_{\lambda}(x;\zeta), Q_{\mu}(x;\zeta)\rangle_{\zeta} = b_{\lambda}(\zeta)^{-1} \sum_{\rho \in CP_{r,n}} Q^{\lambda}_{\rho}Q^{\mu}_{\rho}z_{\rho}(\zeta)$$

and

$$\delta_{\lambda\mu} = \langle P_{\lambda}(x;\zeta), Q'^{(r)}_{\mu}(x;\zeta)\rangle_0 = b_{\lambda}(\zeta)^{-1} \sum_{\rho \in CP_{r,n}} Q^{\lambda}_{\rho}Q'^{\mu}_{\rho}z_{\rho}.$$

Since $\{P_{\lambda}(x;\zeta) \mid \lambda \in RP_{r,n}\}$ is also a basis of $\Lambda^{(r)}$, we have the claim. $\square$

We define $L_{\lambda\mu}(t)$ by

$$s_{\lambda}(x) = \sum_{\mu \in P_n} L_{\lambda\mu}(t)Q'_{\mu}(x;t),$$

where $s_{\lambda}(x)$ denotes the Schur function. Let $K_{\lambda\mu}(t)$ be the Kostka-Foulkes polynomial ([6]). In other words, the matrix $K(t) = (K_{\lambda\mu}(t))_{\lambda,\mu \in P_n}$ is the transition matrix $M(s, P)$ from the Schur functions to the Hall-Littelewood P-functions. It is known that $K(t)$ is an upper unitriangular matrix.

Lemma 4.6 *For partitions λ and μ, we have $L_{\lambda\mu}(t) = K^{(-1)}_{\mu\lambda}(t)$, the (λ, μ)-entry of the matrix $K(t)^{-1}$.*

Proof.

$$L_{\lambda\mu}(t) = \left\langle s_{\lambda}(x), P_{\mu}(x;t)\right\rangle_0 = \left\langle s_{\lambda}(x), \sum_{\nu \in P_n} K^{(-1)}_{\mu\nu}s_{\nu}\right\rangle_0 = K^{(-1)}_{\mu\lambda}(t). \quad \square$$

Example 4.7 ($\zeta = -1$, $n = 4$)

$$s_4(x) = Q'_4(x;-1),$$
$$s_{31}(x) = Q'_{31}(x;-1) + Q'_4(x;-1),$$

$$s_{22}(x) = Q'_{22}(x; -1) + Q'_{31}(x; -1),$$

$$s_{211}(x) = Q'_{211}(x; -1) + Q'_{22}(x; -1) + Q'_{31}(x; -1) + Q'_4(x; -1),$$

$$s_{1111}(x) = Q'_{1111}(x; -1) + Q'_{211}(x; -1) - Q'_{22}(x; -1) + Q'_4(x; -1).$$

Lemma 4.4 and 4.6 give the following expansion formula.

Proposition 4.8   Let $\lambda \in P_n$ and $\mu \in RP_{r,n}$. We have

$$s_\lambda^{(r)}(x) = \sum_{\mu \in RP_{r,n}} K_{\mu\lambda}^{(-1)}(\zeta) Q'^{(r)}_\mu(x; \zeta).$$

In particular, $(L_{\lambda\mu}(\zeta))_{\lambda,\mu \in RP_{r,n}}$ is a lower unitriangular matrix.

Example 4.9   By Proposition 4.8, we immediately see

$$s_4^{(2)}(x) = Q'^{(2)}_4(x; -1),$$

$$s_{31}^{(2)}(x) = Q'^{(2)}_{31}(x; -1) + Q'^{(2)}_4(x; -1),$$

$$s_{22}^{(2)}(x) = Q'^{(2)}_{31}(x; -1),$$

$$s_{211}^{(2)}(x) = Q'^{(2)}_{31}(x; -1) + Q'^{(2)}_4(x; -1),$$

$$s_{1111}^{(2)}(x) = Q'^{(2)}_4(x; -1).$$

From the first two equations above, we have

$$(L_{\lambda\mu}(-1))_{\lambda,\mu \in RP_{2,4}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We set

$$s^{(r)} = \{s_\lambda^{(r)}(x) \mid \lambda \in RP_{r,n}\}, \quad Q'^{(r)} = \{Q'^{(r)}_\lambda(x) \mid \lambda \in RP_{r,n}\}$$

and

$$p^{(r)} = \{p_\lambda(x) \mid \lambda \in CP_{r,n}\}.$$

For $u, v \in \{s^{(r)}, Q'^{(r)}, p^{(r)}\}$, we denote by $M(u, v)$ the transition matrix from u to v. By Lemma 4.4, we have that $M(s^{(r)}, Q'^{(r)})$ is obtained by removing

non r-regular rows and columns from the transposed inverse of $K(t)$.

Theorem 4.10 *We have* $\det M(Q'^{(r)}, p^{(r)}) \in \mathbb{R}$ *and*

$$\det M(Q'^{(r)}, p^{(r)}) = \pm \frac{1}{r^{c_{r,n}} \prod_{\rho \in CP_r(n)} \prod_{i \geq 1} \rho_i}.$$

Proof. The orthogonality relation of $P_\lambda(x; \zeta)$ and $Q'_\mu(x; \zeta)$:

$$\delta_{\lambda\mu} = \langle P_\lambda(x; \zeta), Q'^{(r)}_\mu(x; \zeta) \rangle_0 = b_\lambda(\zeta)^{-1} \sum_{\rho \in CP_{r,n}} Q'^\lambda_\rho Q'^\mu_\rho z_\rho,$$

and Proposition 4.5 give

$$\det M(Q'^{(r)}, p^{(r)})^2 = \left(\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \frac{1}{1 - \zeta^{\rho_i}} \right)^2 \frac{\prod_{\lambda \in RP_{r,n}} b_\lambda(\zeta)}{\prod_{\rho \in CP_{r,n}} z_\rho(\zeta)}$$

$$= \frac{\prod_{\lambda \in RP_{r,n}} b_\lambda(\zeta)}{\prod_{\rho \in CP_{r,n}} z_\rho (\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} (1 - \zeta^{\rho_i}))}$$

$$= \frac{\prod_{\lambda \in RP_{r,n}} \prod_{i \geq 1} (1 - \zeta^{m_i(\lambda)})}{\prod_{\rho \in CP_{r,n}} z_\rho (\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} (1 - \zeta^{\rho_i}))}$$

$$= \frac{\prod_{\lambda \in RP_{r,n}} \prod_{j=1}^{r-1} (1 - \zeta^j)^{y_{r,j}(\lambda)}}{\prod_{\rho \in CP_{r,n}} z_\rho (\prod_{\rho \in CP_{r,n}} \prod_{j=1}^{r-1} (1 - \zeta^j)^{x_{r,j}(\rho)})}$$

$$= \frac{\prod_{j=1}^{r-1} (1 - \zeta^j)^{Y_{r,j,n}}}{\prod_{\rho \in CP_{r,n}} z_\rho \prod_{j=1}^{r-1} (1 - \zeta^j)^{X_{r,j,n}}}$$

$$= \frac{1}{\prod_{\rho \in CP_{r,n}} z_\rho \prod_{j=1}^{r-1} (1 - \zeta^j)^{X_{r,j,n} - Y_{r,j,n}}}$$

$$= \frac{1}{\prod_{\rho \in CP_{r,n}} z_\rho \prod_{j=1}^{r-1} (1 - \zeta^j)^{c_{r,n}}}.$$

We apply Theorem 4.1 to the last equality above. By noticing $\prod_{i=1}^{r-1} (1 - \zeta^i) = r$ and using Theorem 2.3, we obtain the formula. $\qquad \square$

4.3. Regular character tables of the symmetric groups

Let $T_n = (\chi_\rho^\lambda)_{\lambda,\rho \in P_n}$ be the ordinary character table of the symmetric group S_n. The orthogonality relation of the characters implies

$$(\det T_n)^2 = \prod_{\rho \in P_n} z_\rho.$$

From James's book [4, Corollary 6.5], this formula can be simplified as

$$(\det T_n)^2 = \prod_{\rho \in P_n} \prod_{i \geq 1} \rho_i^2.$$

Olsson considers the r-regular character table $T_n^{(r)} = (\chi_\rho^\lambda)_{\substack{\lambda \in RP_{r,n} \\ \rho \in CP_{r,n}}}$ and computes its determinant. He proves the following theorem.

Theorem 4.11 ([8])

$$\det T_n^{(r)} = \pm \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i.$$

Proof.   Theorem 4.10 and Proposition 4.8 enable us to compute the determinant of the regular character table as follows:

$$\det M(s^{(r)}, p^{(r)})^2 = \det M(s^{(r)}, Q'^{(r)})^2 \det M(Q'^{(r)}, p^{(r)})^2$$

$$= 1 \times \frac{1}{r^{2c_{r,n}} \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i^2}$$

By Theorem 2.3 we have

$$\det M(s^{(r)}, p^{(r)})^2 = (\det T_n^{(r)})^2 \times \prod_{\rho \in CP_{r,n}} z_\rho^{-2}$$

$$= (\det T_n^{(r)})^2 \times r^{-2c_{r,n}} \prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i^{-4}.$$

This leads to

$$(\det T_n^{(r)})^2 = \left(\prod_{\rho \in CP_{r,n}} \prod_{i \geq 1} \rho_i \right)^2. \qquad \qquad \square$$

References

[1] Ando M., Suzuki T. and Yamada H.-F., *Combinatorics for graded Cartan matrices of the Iwahori-Hecke algebra of type A*. Ann. Comb. **17** (2013), 427–442.

[2] Bessenrodt C. and Olsson J. B., *Submatrices of character tables and basic sets*. J. Combinatorial Th. Ser. A no. 119 (2012), 1774–1788.

[3] Bessenrodt C., Olsson J. B. and Stanley R. P., *Properties of some character tables related to the symmetric groups*. J. Algebraic Comb. **21** (2005), 163–177.

[4] James G., *The Representation Theory of the Symmetric Groups*. Lecture Notes in Mathematics, 682, Springer-Verlag, 1978.

[5] Lascoux A., Leclerc B. and Thibon J.-Y., *Fonctions de Hall-Littlewood et polynômes de Kostka-Foulkes aux racines de l'unité*. C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 1–6.

[6] Macdonald I. G., *Symmetric Functions and Hall Polynomials, 2nd. ed.*, Oxford, Clarendon Press, 1995.

[7] Morris A. O., *On an algebra of symmetric functions*. Quart. J. Math. Oxford Ser. (2) no. 16 (1965), 53–64.

[8] Olsson J. B., *Regular character tables of symmetric groups*. Electron. J. Combin. **10** (2003), MR1975776.

Hiroshi MIZUKAWA
Department of Mathematics
National Defense Academy of Japan
Yokosuka 239-8686, Japan
E-mail: mzh@nda.ac.jp

Hiro-Fumi YAMADA
Department of Mathematics
Okayama University
Okayama 700-8530, Japan
E-mail: yamada@math.okayama-u.ac.jp

Current address
Department of Mathematics
Kumamoto University
Kumamoto 860-8555, Japan
E-mail: hfyamada@kumamoto-u.ac.jp