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An almost complex Castelnuovo de Franchis theorem

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Abstract. Given a compact almost complex manifold, we prove a Castelnuovo–de Franchis type theorem for it.

Key words: Almost complex structure, Castelnuovo–de Franchis theorem, Riemann surface, fundamental group.

1. Introduction

Given a smooth complex projective variety X, the classical Castelnuovo-de Franchis theorem [2, 4] associates to an isotropic subspace of $H^0(X, \Omega_X^1)$ of dimension greater than one an irrational pencil on X. The topological nature of this theorem was brought out by Catanese [3], who established a bijective correspondence between subspaces $\widetilde{U} \subset H^1(X, \mathbb{C})$ of the form $U \oplus \overline{U}$ with U being a maximal isotropic subspace of $H^1(X, \mathbb{C})$ of dimension $g \geq 1$ and irrational fibrations on X of genus g. The purpose of this note is to emphasize this topological content further. We extract topological hypotheses that allow the theorem to go through when we have only an almost complex structure.

In what follows, M will be a compact smooth manifold of dimension $2k$. Let

$$J_M : TM \longrightarrow TM$$

be an almost complex structure on M, meaning $J_M \circ J_M = -\mathrm{Id}_{TM}$.

Definition 1.1 We shall say that a collection of closed complex 1–forms $\omega_1, \ldots, \omega_n$ on M are in *general position* if

(1) the zero-sets

$$Z(\omega_i) := \{x \in M \mid \omega_i(x) = 0\} \subset M$$

are smooth embedded submanifolds, and

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(2) these submanifolds $Z(\omega_i)$ intersect transversally.

We are now in a position to state the main theorem of this note.

Theorem 1.2 *Let M be a compact smooth $2k$–manifold equipped with an almost complex structure J_M. Let $\omega_1, \ldots, \omega_g$ be closed complex 1–forms on M linearly independent over $\mathbb{C}$, with $g \geq 2$, such that*

- *each ω_i is of type $(1,0)$, meaning $\omega_i(J_M(v)) = \sqrt{-1} \cdot \omega_i(v)$ for all $v \in TM$, and*
- *ω_i are in general position with $\omega_i \wedge \omega_j = 0$ for all i, j.*

Then there exists a smooth almost holomorphic map $f : M \longrightarrow C$ to a compact Riemann surface of genus at least g, and there are linearly independent holomorphic 1–forms $\eta_1, \ldots, \eta_g$ on C, such that $\omega_i = f^*\eta_i$ for all i.

2. Leaf space and almost complex blow-up

2.1. Leaf space

Assume that ω_i are forms as in Theorem 1.2. Since $\omega_i \wedge \omega_j = 0$ for all $1 \leq i, j \leq g$, it follows that there are complex valued smooth functions $f_{i,j}$ such that

$$\omega_i = f_{i,j}\omega_j \tag{1}$$

wherever $\omega_j \neq 0$. Hence the collection

$$\mathcal{W} = \{\omega_1, \ldots, \omega_g\}$$

determines a complex line subbundle of the complexified cotangent bundle $(T^*M) \otimes \mathbb{C}$ over the open subset

$$V := M \setminus \bigcap_{i=1}^{g} Z(\omega_i) \subset M.$$

Lemma 2.1 *Let $\mathcal{F} := \{v \in TV \mid \omega_i(v) = 0 \ \forall \ 1 \leq i \leq g\} \subset TV$ be the distribution on V defined by $\mathcal{W}$. Then $\mathcal{F}$ is integrable and defines a foliation of real codimension two on V.*

Proof. For any $x \in M$ and $1 \leq i \leq g$, consider the $\mathbb{R}$–linear homomorphism

$$\omega_i^x : T_xM \longrightarrow \mathbb{C}, \quad v \longmapsto \omega_i(x)(v).$$

Since ω_i is of type $(1,0)$, if $\omega_i(x) \neq 0$, then ω_i^x is surjective. Also, ω_j^x is a scalar multiple of ω_i^x because $\omega_i \wedge \omega_j = 0$. Therefore, we conclude that the distribution $\mathcal{F}$ on V is of real codimension two.

Since $dw_i = 0$ for all i, it follows that the distribution $\mathcal{F}$ is integrable.

□

Remark 2.2 Since $Z(\omega_i)$ are smooth embedded submanifolds by hypothesis, and ω_i are of type $(1,0)$, it follows that $Z(\omega_i)$ are almost complex submanifolds of M, meaning J_M preserves the tangent subbundle $TZ(\omega_i) \subset (TM)|_{Z(\omega_i)}$. Consequently, all the intersections of the $Z(\omega_i)$'s are also almost complex submanifolds and are therefore even dimensional.

Clearly, each $Z(\omega_i)$ has complex codimension at least one (real codimension at least two) as an almost complex submanifold. Therefore, the complement $M \setminus V$ has complex codimension at least two.

2.2. Almost complex blow-up

We shall need an appropriate notion of blow-up in our context. Suppose $K \subset M$ is a smooth embedded submanifold of M of dimension $2j$ such that $J_M(TK) = TK$. By Remark 2.2, the submanifold $\bigcap_{i=1}^{g} Z(\omega_i)$ satisfies these conditions. Note that J_M induces an automorphism

$$J_{M/K} : ((TM)|_K)/TK \longrightarrow ((TM)|_K)/TK$$

of the quotient bundle over K. Since J_M is an almost complex structure it follows that $J_{M/K} \circ J_{M/K} = -\mathrm{Id}_{((TM)|_K)/TK}$. Therefore, $((TM)|_K)/TK$ is a complex vector bundle on K of rank $k - j$. We would like to replace K by the (complex) projectivized normal bundle $\mathbb{P}(((TM)|_K)/TK)$ which will be called the **almost complex blow-up** of M along K.

For notational convenience, the intersection $\bigcap_{i=1}^{g} Z(\omega_i)$ will be denoted by $\mathcal{Z}$.

We first projectivize $(TM)|_{\mathcal{Z}}$ to get a $\mathbb{CP}^{k-1}$ bundle over $\mathcal{Z}$; this $\mathbb{CP}^{k-1}$ bundle will be denoted by $\mathcal{B}$. So $\mathcal{B}$ parametrizes the space of all (real) two dimensional subspaces of $(TM)|_{\mathcal{Z}}$ preserved by J_M (such two dimensional subspaces are precisely the complex lines in $(TM)|_{\mathcal{Z}}$ equipped with the complex vector bundle structure defined by J_M). Let

$$\pi : \mathcal{B} \longrightarrow \mathcal{Z}$$

be the natural projection.

For notational convenience, the pulled back vector bundle $\pi^*((TM)|_{\mathcal{Z}})$ will be denoted by π^*TM.

Let

$$T_\pi := \text{kernel}(d\pi) \subset T\mathcal{B}$$

be the relative tangent bundle, where $d\pi : T\mathcal{B} \longrightarrow \pi^*T\mathcal{Z}$ is the differential of π. The map π is almost holomorphic, and T_π has the structure of a complex vector bundle. It is known that the complex vector bundle T_π is identified with the vector bundle $Hom_{\mathbb{C}}(\mathcal{L}, (\pi^*TM)/\mathcal{L}) = ((\pi^*TM)/\mathcal{L}) \otimes_{\mathbb{C}} \mathcal{L}^*$, where

$$\mathcal{L} \subset \pi^*TM$$

is the tautological real vector bundle of rank two; note that both $\mathcal{L}$ and $(\pi^*TM)/\mathcal{L}$ have structures of complex vector bundles given by J_M, and the above tensor product and homomorphisms are both over $\mathbb{C}$. Therefore, the pullback π^*TM splits as

$$\mathcal{L} \oplus ((\pi^*TM)/\mathcal{L}) = \mathcal{L} \oplus (T_\pi \otimes \mathcal{L}).$$

The image of the zero section of the complex line bundle $\mathcal{L} \longrightarrow \mathcal{B}$ is identified with $\mathcal{B}$, and the normal bundle of $\mathcal{B} \subset \mathcal{L}$ is identified with $\mathcal{L}$. A small deleted normal neighborhood $U_{\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{L}$ can be identified with a deleted neighborhood U of $\mathcal{Z}$ in M. Let $\overline{U}_{\mathcal{B}} := U_{\mathcal{B}} \bigcup \mathcal{B}$ be the neighborhood of $\mathcal{B}$ in $\mathcal{L}$ (it is no longer a deleted neighborhood), where $\mathcal{B}$ is again identified with the image of the zero section of $\mathcal{L}$. In the disjoint union $(M \setminus \mathcal{Z}) \sqcup \overline{U}_{\mathcal{B}}$, we may identify U with $U_{\mathcal{B}}$. The resulting topological space will be called the **almost complex blow-up** of M along $\mathcal{Z}$.

Remark 2.3

(1) Consider the foliation on the deleted neighborhood U of $\mathcal{Z}$ in M defined by $\mathcal{F}$. Let $\mathbb{L}_U$ denote the leaf space for it. After identifying $\mathcal{B}$ with the image of the zero section of π^*TM, we obtain locally, from a neighborhood of $\mathcal{B}$, a map to the leaf space $\mathbb{L}_U$.

(2) Note that the notion of an almost complex blow-up above is well-defined up to the choice of an identification of U with $U_{\mathcal{B}}$. Therefore, the construction yields a well-defined almost complex manifold, up to an isomorphism.

3. Proof of Theorem 1.2

We will first show that the leaves of $\mathcal{F}$ in Section 2 are proper embedded submanifolds of $V = M \setminus \mathcal{Z}$.

Suppose some leaf $\mathbb{L}_0$ does not satisfy the above property. Then there exists $x \in V$ and a neighborhood U_x of x such that $U_x \bigcap \mathbb{L}_0$ contains infinitely many leaves (of $\mathcal{F}$) accumulating at x. Let σ be a smooth path starting at x transverse to $\mathcal{F}$. Thus for every $\epsilon > 0$ there exist

(1) distinct (local) leaves $F_1, F_2 \subset U_x \bigcap \mathcal{F}$, such that (globally) $F_1, F_2 \subset \mathbb{L}_0$,
(2) points $y_j \in F_j$, $j = 1, 2$,
(3) a sub-path $\sigma_{12} \subset \sigma \subset U_x$ joining y_1 and y_2, and
(4) a path $\tau_{12} \subset L_0$ joining y_1 and y_2 such that

$$\left| \int_{\sigma_{12}} \omega_i \right| < \epsilon \ \ \forall\, i.$$

Since $\mathcal{F}$ is in the kernel of each ω_i,

$$\left| \int_{\tau_{12}} \omega_i \right| = 0, \quad \forall\, i.$$

Setting ϵ smaller than the absolute value of any non-zero period of the ω_i's, it follows that

$$\int_{\tau_{12} \cup \sigma_{12}} \omega_i = 0 \ \ 1 \leq i \leq g.$$

Hence

$$\int_{\sigma_{12}} \omega_i = 0 \ \ \forall\, 1 \leq i \leq g.$$

Taking limits we obtain that $\omega_i(\sigma'(0)) = 0$; but this contradicts the choice that σ is transverse to $\mathcal{F}$. Therefore, the leaves of $\mathcal{F}$ are proper embedded submanifolds of V.

Consequently, the leaf space

$$D = V / \langle \mathcal{F} \rangle$$

for the foliation $\mathcal{F}$ is a smooth 2-manifold. Let

$$q : V \longrightarrow D$$

be the quotient map.

Remark 3.1 The referee kindly pointed out to us the following considerably simpler proof of the fact that leaves of $\mathcal{F}$ are closed in V:

By equation (1), we have $\omega_i = f_{i,j}\omega_j$. Since ω_i are closed,

$$d(f_{i,j}) \wedge \omega_j = 0, \quad \forall\ i, j.$$

The functions $f_{i,j}$ define a map $f : V \longrightarrow \mathbb{C}P^1$. Since $d(f_{i,j}) \wedge \omega_j = 0$, it follows that f is constant on the leaves of the foliation $\mathcal{F}$ defined by the forms ω_i, $1 \leq i \leq n$.

Next, let

$$\xi := TV/\mathcal{F}$$

be the quotient complex line bundle on V. Then ξ carries a natural flat partial connection $\mathcal{D}$ along the leaves (cf. [5]); this $\mathcal{D}$ is known as the Bott partial connection. It is straightforward to check that $\mathcal{D}$ preserves the complex structure on ξ. Indeed, this follows immediately from the fact that ω_i are of type $(1,0)$. Hence ξ induces an almost complex structure on D. Since D is two–dimensional, this almost complex structure is integrable giving D the structure of a Riemann surface. Further, there exist closed integral $(1,0)$ forms η_i, $1 \leq i \leq g$, on D such that

$$\omega_i = f^*\eta_i.$$

3.1. Removing indeterminacy

Since $Z(\omega_i)$ are mutually transverse, we can construct the almost complex blow-ups of M along $\mathcal{Z}$ successively along the μ–fold intersections, $\mu = 2, \ldots, g$, to obtain $\widehat{M}$ such that

(1) there is an extension of the line bundle ξ to all of $\widehat{M}$,
(2) there is a well-defined smooth map

$$\widehat{q} : \widehat{M} \longrightarrow D$$

extending q, and

(3) the blown-up locus has the structure of complex analytic $\mathbb{CP}^1$–bundles as usual (due to transversality).

The Riemann surface D is compact because $\widehat{M}$ is so. Further, D has genus greater than one as $g > 1$. So any complex analytic map from $\mathbb{CP}^1$ to D must be constant. Therefore, $\widehat{q}$ actually induces a smooth map from M to D, and we may assume that the indeterminacy locus of q was empty to start off with, or equivalently that q extends to a smooth map $\overline{q} : M \longrightarrow D$. This furnishes the required conclusion and completes the proof of Theorem 1.2.

4. Refinements and consequences

4.1. Stein factorization

We now proceed as in the proof of the classical Castelnuovo-de Franchis Theorem, [1, p. 24, Theorem 2.7], to deduce *a posteriori* that D is a Stein factorization of a map to a compact Riemann surface. Define

$$h : V \longrightarrow \mathbb{CP}^{g-1}, \quad m \longmapsto [\omega_1(m) : \cdots : \omega_g(m)].$$

Suppose $\omega_i(m) \neq 0$. Then there exists a small neighborhood $U(m)$ such that

$$h(x) = [f_{1,i}(x) : \cdots : f_{i-1,i}(x) : 1 : f_{i+1,i}(x) : \cdots : f_{g,i}(x)], \quad \forall\, x \in U(m),$$

where $f_{l,j}$ are defined in equation (1). Since $\omega_j \wedge \omega_i = 0$ and $d\omega_j = 0$ for all i, j, it follows that $df_{j,i} \wedge \omega_i = 0$. Consequently, each $f_{j,i}$ is constant on the leaves of $\mathcal{F}$. Hence h has a (complex) one dimensional image, so the image of h is a Riemann surface C.

Further, h induces a holomorphic map

$$h_1 : D \longrightarrow C.$$

We note that D may be thought of as the Stein factorization of h, and h factors as $h = h_1 \circ q$.

4.2. Further generalizations

Let M be a compact manifold, and let $\mathcal{S} \subset TM$ be a nonsingular foliation such that M has a flat transversely almost complex structure. This means that we have an automorphism

$$J : TM/\mathcal{S} \longrightarrow TM/\mathcal{S}$$

such that

(1) $J \circ J = -\mathrm{Id}_{TM/\mathcal{S}}$, and
(2) J is flat with respect to the Bott partial connection on $TM/\mathcal{S}$.

Let ω_i, $1 \leq i \leq g$, be linearly independent smooth sections of $(TM/\mathcal{S})^* \otimes \mathbb{C}$ such that

(1) each ω_i is flat with respect to the connection on $(TM/\mathcal{S})^* \otimes \mathbb{C}$ induced by the Bott partial connection on $TM/\mathcal{S}$,
(2) each ω_i is of type $(1,0)$, meaning the corresponding homomorphism $TM/\mathcal{S} \longrightarrow M \times \mathbb{C}$ is $\mathbb{C}$–linear,
(3) each ω_i is closed when considered as a complex 1–form on M using the composition

$$TM \otimes \mathbb{C} \longrightarrow (TM/\mathcal{S}) \otimes \mathbb{C} \xrightarrow{\omega_i} M \times \mathbb{C}$$

(4) $\omega_i \wedge \omega_j = 0$ for all i, j, and
(5) ω_j are in general position.

Theorem 1.2 can be generalized to this set–up.

Remark 4.1 The only real use we have made of the hypothesis in Theorem 1.2 that ω_i's are in general position is to ensure that $\mathcal{Z}$ has complex codimension at least two. This is what allows us to remove indeterminacies in the smooth category (as opposed to the algebraic or complex analytic categories, where complex codimension greater than one follows naturally). Any hypothesis that ensures that the indeterminacy locus has complex codimension greater than one would suffice.

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