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Growth of meromorphic solutions of some linear differential equations

Hamid BEDDANI and Karima HAMANI

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Abstract. In this paper, we investigate the order and the hyper-order of meromorphic solutions of the linear differential equation

$$f^{(k)} + \sum_{j=1}^{k-1} (D_j + B_j e^{P_j(z)}) f^{(j)} + (D_0 + A_1 e^{Q_1(z)} + A_2 e^{Q_2(z)}) f = 0,$$

where $k \geq 2$ is an integer, $Q_1(z), Q_2(z)$, $P_j(z)$ $(j = 1, \ldots, k-1)$ are nonconstant polynomials and $A_s(z)$ $(\not\equiv 0)$ $(s = 1, 2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1, \ldots, k-1)$, $D_m(z)$ $(m = 0, 1, \ldots, k-1)$ are meromorphic functions. Under some conditions, we prove that every meromorphic solution f $(\not\equiv 0)$ of the above equation is of infinite order and we give an estimate of its hyper-order. Furthermore, we obtain a result about the exponent of convergence and the hyper-exponent of convergence of a sequence of zeros and distinct zeros of $f - \varphi$, where φ $(\not\equiv 0)$ is a meromorphic function and f $(\not\equiv 0)$ is a meromorphic solution of the above equation.

Key words: Linear Differential Equation, Meromorphic function, Hyper-order, Exponent of convergence, hyper-exponent of convergence.

1. Introduction and statement of results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [11], [15]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f, $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponent of convergece of a sequence of zeros and a sequence of distinct zeros of f. We also denote by $\sigma_2(f)$ the hyper-order of f which is defined by (see [15])

$$\sigma_2(f) = \limsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f.

The hyper-exponent of convergence of a sequence of zeros and distints

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zeros of f are respectively defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N(r,1/f)}{\log r}$$

and

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}(r,1/f)}{\log r},$$

where $N(r,1/f)$ and $\overline{N}(r,1/f)$ are respectively the counting functions of zeros and distinct zeros of f.

For the second order linear differential equation

$$f'' + e^{-z}f' + B(z)f = 0, \tag{1.1}$$

where $B(z)$ is an entire function of finite order, it is well known that every solution of (1.1) is an entire function and most solutions of (1.1) have an infinite order. Thus, a natural question is: what conditions on $B(z)$ will guarantee that every solution f ($\not\equiv 0$) of (1.1) has an infinite order? Ozawa [13], Gundersen [8], Amemiya and Ozawa [1], and Langley [12] have studied the problem, where $B(z)$ is a nonconstant polynomial or a transcendental entire function with order $\sigma(B) \neq 1$. Recently in [14], Peng and Chen have investigated the order and the hyper-order of solutions of equation (1.1) and have proved the following result:

Theorem A ([14]) *Let $A_j(z)$ ($\not\equiv 0$) ($j = 1,2$) be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1a_2 \neq 0$ and $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + e^{-z}f' + (A_1e^{a_1z} + A_2e^{a_2z})f = 0 \tag{1.2}$$

has an infinite order and $\sigma_2(f) = 1$.

Recently in [9], the authors have extended Theorem A to some higher order linear differential equations as follows:

Theorem B ([9]) *Let A_s ($\not\equiv 0$) ($s = 1,2$), B_j ($\not\equiv 0$) ($j = 1,\dots,k-1$), D_m ($m = 0,1,2,\dots,k-1$) be entire functions with* $\max\{\sigma(A_s),\sigma(B_j),\sigma(D_m)\} <$

1, b_l $(l = 1, \ldots, k-1)$ be complex numbers such that **(i)** $\arg b_l = \arg a_1$ *and* $b_l = c_l a_1$ $(0 < c_l < 1)$ $(l \in I_1)$ *and* **(ii)** b_l *is a real constant such that* $b_l \le 0$ $(l \in I_2)$*, where* $I_1 \ne \varnothing$*,* $I_2 \ne \varnothing$*,* $I_1 \cap I_2 = \varnothing$ *and* $I_1 \cup I_2 = \{1, \ldots, k-1\}$ *and* a_1*,* a_2 *are complex numbers such that* $a_1 a_2 \ne 0$ *and* $a_1 \ne a_2$ *(suppose that* $|a_1| \le |a_2|$*). If* $\arg a_1 \ne \pi$ *or* a_1 *is a real number such that* $a_1 < b/(1-c)$*, where* $c = \max\{c_l : l \in I_1\}$ *and* $b = \min\{b_l : l \in I_2\}$*, then every solution* $f(\not\equiv 0)$ *of equation*

$$f^{(k)} + \sum_{j=1}^{k-1} (D_j + B_j e^{b_j z}) f^{(j)} + (D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0 \tag{1.3}$$

satisfies $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$*.*

In this paper, we continue the research in this type of problems. So, the main purpose of this paper is to extend and improve the above results to some higher order linear differential equations. We will prove the following results:

Theorem 1.1 *Let* $k \ge 2$ *be an integer,* $Q_s(z) = \sum_{i=0}^{n} a_{i,s} z^i$ $(s = 1, 2)$ *and* $P_j(z) = \sum_{i=0}^{n} b_{i,j} z^i$ $(j = 1, \ldots, k-1)$ *be nonconstant polynomials, where* $a_{0,s}, \ldots, a_{n,s}$ $(s = 1, 2)$*,* $b_{0,j}, \ldots, b_{n,j}$ $(j = 1, \ldots, k-1)$ *are complex numbers such that* $a_{n,s} = |a_{n,s}| e^{i\theta_s} \ne 0$ $(s = 1, 2)$*,* $\theta_s \in [-\pi/2, 3\pi/2)$ *and* $a_{n,1} \ne a_{n,2}$ *(suppose that* $|a_{n,1}| \le |a_{n,2}|$*). Let* $A_s(z)$ $(\not\equiv 0)$ $(s = 1, 2)$*,* $B_j(z)$ $(\not\equiv 0)$ $(j = 1, \ldots, k-1)$ *and* $D_m(z)$ $(m = 0, 1, \ldots, k-1)$ *be meromorphic functions with* $\max\{\sigma(A_s), \sigma(B_j), \sigma(D_m)\} < n$*. Let* I *and* J *be two sets satisfying* $I \ne \varnothing$*,* $J \ne \varnothing$*,* $I \cap J = \varnothing$ *and* $I \cup J = \{1, \ldots, k-1\}$ *such that for* $j \in I$*,* $b_{n,j} = c_j a_{n,1}$ $(0 < c_j < 1)$ *and for* $j \in J$*,* $b_{n,j} < 0$*.*

If $\theta_1 \ne \pi$ *or* $a_{n,1}$ *is a real number such that* $(1-c) a_{n,1} < b$*, where* $c = \max\{c_j : j \in I\}$ *and* $b = \min\{b_{n,j} : j \in J\}$*, then every meromorphic solution* $f(\not\equiv 0)$ *of equation*

$$f^{(k)} + \sum_{j=1}^{k-1} (D_j + B_j e^{P_j(z)}) f^{(j)} + \big(D_0 + A_1 e^{Q_1(z)} + A_2 e^{Q_2(z)}\big) f = 0 \tag{1.4}$$

is of infinite order and satisfies $\sigma_2(f) \ge n$*. In addition, if* $\lambda(1/f) < +\infty$*, then* $\sigma_2(f) = n$*.*

Theorem 1.2 *Let* $k \ge 2$ *be an integer,* $Q_s(z) = \sum_{i=0}^{n} a_{i,s} z^i$ $(s = 1, 2)$ *and*

$P_j(z) = \sum_{i=0}^{n} b_{i,j} z^i$ $(j = 1, \ldots, k-1)$ be nonconstant polynomials, where $a_{0,s}, \ldots, a_{n,s}$ $(s = 1, 2)$, $b_{0,j}, \ldots, b_{n,j}$ $(j = 1, \ldots, k-1)$ are complex numbers such that $a_{n,s} = |a_{n,s}|\mathrm{e}^{i\theta_s} \neq 0$ $(s = 1, 2)$, $\theta_s \in [-\pi/2, 3\pi/2)$ and $a_{n,1} \neq a_{n,2}$. Let $A_s(z)$ $(\not\equiv 0)$ $(s = 1, 2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1, \ldots, k-1)$ and $D_m(z)$ $(m = 0, 1, \ldots, k-1)$ be meromorphic functions with $\max\{\sigma(A_s), \sigma(B_j), \sigma(D_m)\} < n$. Let I and J be two sets satisfying $I \neq \varnothing$, $J \neq \varnothing$, $I \cap J = \varnothing$ and $I \cup J = \{1, \ldots, k-1\}$ such that for $j \in I$, $b_{n,j} = \alpha_j a_{n,1}$ $(0 < \alpha_j < 1)$ and for $j \in J$, $b_{n,j} = \beta_j a_{n,2}$ $(0 < \beta_j < 1)$. Set $\alpha = \max\{\alpha_j : j \in I\}$ and $\beta = \max\{\beta_j : j \in J\}$.

Suppose that one of the following statements holds:

(1) $\theta_1 \neq \pi$ *and* $\theta_1 \neq \theta_2$.
(2) $\theta_1 \neq \pi$, $\theta_1 = \theta_2$ *and* **(i)** $|a_{n,1}| < (1-\beta)|a_{n,2}|$ *or* **(ii)** $|a_{n,2}| < (1-\alpha)|a_{n,1}|$.
(3) $a_{n,1}$ *and* $a_{n,2}$ *are real number such that* **(i)** $(1-\beta)a_{n,2} < a_{n,1} < 0$ *or* **(ii)** $(1-\alpha)a_{n,1} < a_{n,2} < 0$.

Then every meromorphic solution $f (\not\equiv 0)$ of equation (1.4) *is of infinite order and satisfies $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then $\sigma_2(f) = n$.*

Theorem 1.3 *Let $k \geq 2$ be an integer, $Q_s(z) = \sum_{i=0}^{n} a_{i,s} z^i$ $(s = 1, 2)$ and $P_j(z) = \sum_{i=0}^{n} b_{i,j} z^i$ $(j = 1, \ldots, k-1)$ be nonconstant polynomials, where $a_{0,s}, \ldots, a_{n,s}$ $(s = 1, 2)$, $b_{0,j}, \ldots, b_{n,j}$ $(j = 1, \ldots, k-1)$ are complex numbers such that $a_{n,s} = |a_{n,s}|\mathrm{e}^{i\theta_s} \neq 0$ $(s = 1, 2)$, $\theta_s \in [-\pi/2, 3\pi/2)$ and $a_{n,1} \neq a_{n,2}$. Let $A_s(z)$ $(\not\equiv 0)$ $(s = 1, 2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1, \ldots, k-1)$ and $D_m(z)$ $(m = 0, 1, \ldots, k-1)$ be meromorphic functions with $\max\{\sigma(A_s), \sigma(B_j), \sigma(D_m)\} < n$. Let I and J be two sets satisfying $I \neq \varnothing$, $J \neq \varnothing$, $I \cap J = \varnothing$ and $I \cup J = \{1, \ldots, k-1\}$ such that for $j \in I$, $b_{n,j} = \alpha_j a_{n,1} + \beta_j a_{n,2}$ $(0 < \alpha_j < 1)$ $(0 < \beta_j < 1)$ and for $j \in J$, $b_{n,j} < 0$. Set $\alpha = \max\{\alpha_j : j \in I\}$, $\beta = \max\{\beta_j : j \in I\}$ and $b = \min\{b_{n,j} : j \in J\}$.*

Suppose that one of the following statements holds:

(1) $\theta_1 \neq \pi$ *and* $\theta_1 \neq \theta_2$.
(2) $\theta_1 \neq \pi$, $\theta_1 = \theta_2$ *and* **(i)** $|a_{n,1}| < (1-\beta)|a_{n,2}|$ *or* **(ii)** $|a_{n,2}| < (1-\alpha)|a_{n,1}|$.
(3) $a_{n,1}$ *and* $a_{n,2}$ *are real numbers such that* **(i)** $(1-\beta)a_{n,2} - b < a_{n,1} < 0$ *or* **(ii)** $(1-\alpha)a_{n,1} - b < a_{n,2} < 0$.

Then every meromorphic solution $f (\not\equiv 0)$ of equation (1.4) *is of infinite order and satisfies $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then $\sigma_2(f) = n$.*

Theorem 1.4 *Let $k \geq 2$ be an integer, $A_s(z)$ $(\not\equiv 0)$, $a_{n,s}$ $(s = 1, 2)$, $B_j(z)$*

($\not\equiv 0$), $b_{n,j}$ ($j = 1, \ldots, k-1$) *and* $D_m(z)$ ($m = 0, 1, \ldots, k-1$) *satisfy the additional hypotheses of Theorem* 1.1 *or Theorem* 1.2 *or Theorem* 1.3. *If* φ ($\not\equiv 0$) *is a meromorphic function of finite order, then every meromorphic solution* $f(\not\equiv 0)$ *of equation* (1.4) *satisfies* $\lambda(f-\varphi) = \overline{\lambda}(f-\varphi) = +\infty$ *and* $\lambda_2(f-\varphi) = \overline{\lambda}_2(f-\varphi) = \sigma_2(f) \geq n$. *In addition, if* $\lambda(1/f) < +\infty$, *then* $\lambda_2(f-\varphi) = \overline{\lambda}_2(f-\varphi) = \sigma_2(f) = n$.

2. Preliminary Lemmas

Lemma 2.1 ([6]) *Let* $f(z)$ *be a transcendental meromorphic function and let* $\alpha > 1$ *be a given constant. Then there exists a set* $E_1 \subset (1, +\infty)$ *having finite logarithmic measure and a constant* $B > 0$ *that depends only on* i, j (i, j *positive integers with* $0 \leq i < j \leq k$) *such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_1$, *we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{j-i}. \tag{2.1}$$

Lemma 2.2 ([5]) *Let* $g(z)$ *be a meromorphic function of order* $\sigma(g) = \sigma < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_2 \subset (1, +\infty)$ *that has finite logarithmic measure such that*

$$|g(z)| \leqslant \exp\{r^{\sigma+\varepsilon}\} \tag{2.2}$$

holds for $|z| = r \notin [0,1] \cup E_2$, $r \to +\infty$.

Lemma 2.3 ([14]) *Suppose that* $n \geq 1$ *is an integer,* $Q_j(z) = a_{jn}z^n + \cdots$ ($j = 1, 2$) *be noncostant polynomials, where* a_{jq} ($q = 1, 2, \ldots, n$) *are complex numbers and* $a_{1n}a_{2n} \neq 0$. *Set* $z = re^{i\theta}$, $a_{jn} = |a_{jn}|\mathrm{e}^{i\theta_j}$, $\theta_j \in [-\pi/2, 3\pi/2)$, $\delta(Q_j, \theta) = |a_{jn}|\cos(\theta_j + n\theta)$, *then there is a set* $E_3 \subset [-\pi/2n, 3\pi/2n)$ *that has linear measure zero. If* $\theta_1 \neq \theta_2$, *then there exists a ray* $\arg z = \theta$, $\theta \in (-\pi/2n, \pi/2n)/(E_3 \cup E_4)$ *such that*

$$\delta(Q_1, \theta) > 0, \quad \delta(Q_2, \theta) < 0 \tag{2.3}$$

or

$$\delta(Q_1, \theta) < 0, \quad \delta(Q_2, \theta) > 0, \tag{2.4}$$

where $E_4 = \{\theta \in [-\pi/2n, 3\pi/2n) : \delta(Q_j, \theta) = 0\}$ *is a finite set which has linear measure zero.*

Remark 2.1 ([14]) In Lemma 2.3, if $\theta \in (-\pi/2n, \pi/2n)/(E_3 \cup E_4)$ is replaced by $\theta \in (\pi/2n, 3\pi/2n)/(E_3 \cup E_4)$, then we obtain the same result.

Lemma 2.4 ([10]) *Let* $P(z) = (\alpha + i\beta)z^n + \cdots$ *(*α*,* β *are real numbers,* $|\alpha| + |\beta| \neq 0$*) be a polynomial with degree* $n \geq 1$ *and* $A(z)$ *be a meromorphic function with* $\sigma(A) < n$*. Set* $f(z) = A(z)e^{P(z)}$*,* $z = re^{i\theta}$*,* $\delta(P, \theta) = \alpha\cos(n\theta) - \beta\sin(n\theta)$*. Then for any given* $\varepsilon > 0$*, there exists a set* $E_5 \subset (1, +\infty)$ *having finite logarithmic measure such that for any* $\theta \in [-\pi/2, 3\pi/2)/H$ *and* $|z| = r \notin [0, 1] \cup E_5$*,* $r \to +\infty$*, we have*

(i) *if* $\delta(P, \theta) > 0$*, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} \leq |f(re^{i\theta})| \leq \exp\{(1+\varepsilon)\delta(P,\theta)r^n\}, \tag{2.5}$$

(ii) *if* $\delta(P, \theta) < 0$*, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} \leq |f(re^{i\theta})| \leq \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}, \tag{2.6}$$

where $H = \{\theta \in [-\pi/2, 3\pi/2) : \delta(P, \theta) = 0\}$*.*

Lemma 2.5 ([2]) *Suppose that* $k \geq 2$ *and* $A_0, A_1, \ldots, A_{k-1}$ *are meromorphic functions of finite order. Let* $\rho = \max\{\sigma(A_j) : j = 0, 1, \ldots, k-1\}$ *and let* f *be a transcendental meromorphic solution with* $\lambda(1/f) < +\infty$ *of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0. \tag{2.7}$$

Then $\sigma_2(f) \leq \rho$*.*

Lemma 2.6 ([2]) *Let* $P_j(z)$ $(j = 0, 1, \ldots, k)$ *be polynomials with* $\deg P_0(z) = n$ $(n \geq 1)$ *and* $\deg P_j(z) \leq n$ $(j = 1, 2, \ldots, k)$*. Let* $A_j(z)$ $(j = 0, 1, \ldots, k)$ *be meromorphic functions with finite order and* $\max\{\sigma(A_j) : j = 0, 1, \ldots, k\} < n$ *such that* $A_0(z) \not\equiv 0$*. We denote*

$$F(z) = A_k(z)e^{P_k(z)} + A_{k-1}(z)e^{P_{k-1}(z)} + \cdots + A_1(z)e^{P_1(z)} + A_0(z)e^{P_0(z)}. \tag{2.8}$$

If $\deg(P_0(z) - P_j(z)) = n$ *for all* $j = 1, \ldots, k$*, then* F *is a nontrivial mero-*

morphic function with finite order and satisfies $\sigma(F) = n$.

Lemma 2.7 ([7]) *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *and* $\psi : [0, +\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leqslant \psi(r)$ *for all* $r \notin E_6 \cup [0, 1]$, *where* $E_6 \subset (1, +\infty)$ *is a set of finite logarithmic measure. Let* $\alpha > 1$ *be a given constant. Then there exists an* $r_0 = r_0(\alpha) > 0$ *such that* $\varphi(r) \leqslant \psi(\alpha r)$ *for all* $r > r_0$.

Lemma 2.8 ([4]) *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$, $F(\not\equiv 0)$ *be finite order meromorphic functions. If* f *is an infinite order meromorphic solution of equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f + A_0(z)f = F, \tag{2.9}$$

then f *satisfies* $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = +\infty$.

Lemma 2.9 ([3]) *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$, $F(\not\equiv 0)$ *be finite order meromorphic functions. If* f *is a meromorphic solution of equation* (2.9) *with* $\sigma(f) = +\infty$ *and* $\sigma_2(f) = \sigma$, *then* f *satisfies* $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma$.

3. Proof of Theorem 1.1

First we prove that every meromorphic solution $f(\not\equiv 0)$ of equation (1.4) is transcendental. Assume that $f(\not\equiv 0)$ is a polynomial or a rational solution of equation (1.4). Then $\sigma(f) = 0$. We write equation (1.4) in the form

$$(A_1(z)f)e^{Q_1(z)} + (A_2(z)f)e^{Q_2(z)} + \sum_{j=1}^{k-1} B_j(z)f^{(j)}e^{P_j(z)} = B(z), \tag{3.1}$$

where $B(z) = -\big(f^{(k)} + \sum_{j=1}^{k-1} D_j(z)f^{(j)} + D_0(z)f\big)$, $A_s f$ $(s = 1, 2)$ and $B_j f^{(j)}$ $(j = 1, 2, \ldots, k-1)$ are meromorphic functions of finite order with $A_s f \not\equiv 0$ $(s = 1, 2)$, $\sigma(B) < n$, $\sigma(A_s f) < n$ $(s = 1, 2)$ and $\sigma(B_j f^{(j)}) < n$ $(j = 1, \ldots, k-1)$. If $\theta_1 \neq \pi$ or $a_{n,1}$ is a real number such that $(1-c)a_{n,1} < b$, it follows that $\deg(Q_1(z) - Q_2(z)) = n$ and $\deg(Q_1(z) - P_j(z)) = n$ $(j = 1, \ldots, k-1)$. Thus from (3.1) and by Lemma 2.6, we have $\sigma(B) = n$, this contradicts the fact $\sigma(B) < n$. Hence every meromorphic solution $f(\not\equiv 0)$ of equation (1.4) is transcendental.

Set $\max\{\sigma(A_s), \sigma(B_j), \sigma(D_m)\} = \rho < n$, where $(s = 1, 2)$, $(j = 1, \ldots, k-1)$ and $(m = 0, \ldots, k-1)$. Assume that $f(\not\equiv 0)$ is a meromorphic solution of equation (1.4). By Lemma 2.1, there exists a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B[T(2r, f)]^{j+1} \ (j = 1, \ldots, k). \tag{3.2}$$

By Lemma 2.2, for any given ε $(0 < \varepsilon < n - \rho)$, there exists a set $E_2 \subset (1, +\infty)$ that has finite logarithmic measure such that

$$|D_m(z)| \leq \exp\{r^{\rho+\varepsilon}\} \ (m = 0, \ldots, k-1) \tag{3.3}$$

holds for $|z| = r \notin [0, 1] \cup E_2$, $r \to +\infty$.

Case (1). $\theta_1 \neq \pi$.

(i) Suppose that $\theta_1 \neq \theta_2$. By Lemma 2.3, there exists a ray $\arg z = \theta$, such that $\theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, where E_3 and E_4 are defined as in Lemma 2.3, and satisfying

$$\delta(Q_1, \theta) > 0, \quad \delta(Q_2, \theta) < 0 \text{ or } \delta(Q_1, \theta) < 0, \quad \delta(Q_2, \theta) > 0.$$

a) When $\delta(Q_1, \theta) > 0$, $\delta(Q_2, \theta) < 0$, by Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n - \rho, (1 - c)/(2(1 + c))\})$, there exists a set $E_5 \subset [1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E_5$, $r \to +\infty$, we have

$$\left| A_1(z) e^{Q_1(z)} \right| \geq \exp\{(1 - \varepsilon)\delta(Q_1, \theta) r^n\} \tag{3.4}$$

and

$$|A_2(z) e^{Q_2(z)}| \leq \exp\{(1 - \varepsilon)\delta(Q_2, \theta) r^n\} < 1. \tag{3.5}$$

By (3.4) and (3.5), we have

$$\begin{aligned} \left| A_1(z) e^{Q_1(z)} + A_2(z) e^{Q_2(z)} \right| &\geq \left| A_1(z) e^{Q_1(z)} \right| - \left| A_2(z) e^{Q_2(z)} \right| \\ &\geq \exp\{(1 - \varepsilon)\delta(Q_1, \theta) r^n\} - 1 \end{aligned}$$

$$\geq (1-o(1))\exp\{(1-\varepsilon)\delta(Q_1,\theta)r^n\}. \tag{3.6}$$

By (1.4), we get

$$\begin{aligned}&\left|A_1(z)e^{Q_1(z)}+A_2(z)e^{Q_2(z)}\right|\\ &\quad\leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left(|D_{k-1}(z)| + \left|B_{k-1}(z)e^{P_{k-1}(z)}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right|\\ &\qquad+\cdots+\left(|D_1(z)|+\left|B_1(z)e^{P_1(z)}\right|\right)\left|\frac{f'(z)}{f(z)}\right|+|D_0(z)|.\end{aligned} \tag{3.7}$$

For $j\in I$, we have $\delta(P_j,\theta)=c_j\delta(Q_1,\theta)>0$. Thus

$$\left|B_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)c_j\delta(Q_1,\theta)r^n\} \leq \exp\{(1+\varepsilon)c\delta(Q_1,\theta)r^n\}. \tag{3.8}$$

For $j\in J$, we have $\delta(P_j,\theta)=-|b_{n,j}|\cos(n\theta)<0$. Thus

$$\left|B_j(z)e^{P_j(z)}\right| \leq \exp\{(1-\varepsilon)\delta(P_j,\theta)r^n\} < 1. \tag{3.9}$$

Substituting (3.2), (3.3), (3.6), (3.8) and (3.9) into (3.7), for all z satisfying $\arg z=\theta\in(-\{\pi/2n,\pi/2n)\backslash(E_3\cup E_4)$, $|z|=r\notin[0,1]\cup E_1\cup E_2\cup E_5$, $r\to+\infty$, we obtain

$$\begin{aligned}&(1-o(1))\exp\{(1-\varepsilon)\delta(Q_1,\theta)r^n\}\\ &\quad\leq M_1\exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)c\delta(Q_1,\theta)r^n\}[T(2r,f)]^{k+1},\end{aligned} \tag{3.10}$$

where $M_1>0$ is a constant. From (3.10) and $0<\varepsilon<(1-c)/(2(1+c))$, we get

$$(1-o(1))\exp\left\{\frac{(1-c)}{2}\delta(Q_1,\theta)r^n\right\} \leq M_1\exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}. \tag{3.11}$$

By $\delta(Q_1,\theta)>0$ and $\rho+\varepsilon<n$, then by using Lemma 2.7 and (3.11), we obtain $\sigma(f)=+\infty$ and $\sigma_2(f)\geq n$. In addition, if $\lambda(1/f)<+\infty$, then by Lemma 2.5, we have $\sigma_2(f)\leq n$. Hence $\sigma_2(f)=n$.

b) When $\delta(Q_1,\theta)<0$, $\delta(Q_2,\theta)>0$, by Lemma 2.4, for the above ε, there exists a set $E_5\subset(1,+\infty)$ having finite logarithmic measure such that for

$|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have

$$\left|A_1(z)e^{Q_1(z)}\right| \leq \exp\{(1-\varepsilon)\delta(Q_1,\theta)r^n\} < 1 \tag{3.12}$$

and

$$\left|A_2(z)e^{Q_2(z)}\right| \geq \exp\{(1-\varepsilon)\delta(Q_2,\theta)r^n\}. \tag{3.13}$$

By (3.12) and (3.13), we have

$$\left|A_1(z)e^{Q_1(z)} + A_2(z)e^{Q_2(z)}\right| \geq (1-o(1))\exp\{(1-\varepsilon)\delta(Q_2,\theta)r^n\}. \tag{3.14}$$

For $j \in I$, we have $\delta(P_j,\theta) < 0$. Thus

$$\left|B_j(z)e^{P_j(z)}\right| \leq \exp\{(1-\varepsilon)c_j\delta(Q_1,\theta)r^n\} < 1. \tag{3.15}$$

Substituting (3.2), (3.3), (3.9), (3.14) and (3.15) into (3.7), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$(1-o(1))\exp\{(1-\varepsilon)\delta(Q_2,\theta)r^n\} \leq M_2 \exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}, \tag{3.16}$$

where $M_2 > 0$ is a constant. By $\delta(Q_2,\theta) > 0$ and $\rho + \varepsilon < n$, then by using Lemma 2.7 and (3.16), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \leq n$. Hence $\sigma_2(f) = n$.

(ii) Suppose that $\theta_1 = \theta_2$. By Lemma 2.3, there is a ray $\arg z = \theta$ such that $\theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$ and $\delta(Q_1,\theta) > 0$. Since $|a_{n,1}| \leq |a_{n,2}|$, $a_{n,1} \neq a_{n,2}$ and $\theta_1 = \theta_2$, it follows that $|a_{n,1}| < |a_{n,2}|$. Thus $\delta(Q_2,\theta) > \delta(Q_1,\theta) > 0$. By Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n - \rho, (|a_{n,2}| - |a_{n,1}|)/(2(|a_{n,2}| + |a_{n,1}|))\})$, there exists a set $E_5 \subset (1,+\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.13) and

$$\left|A_1(z)e^{Q_1(z)}\right| \leq \exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\}. \tag{3.17}$$

By (3.13) and (3.17), we get

$$\begin{aligned}
&\left|A_1(z)e^{Q_1(z)} + A_2(z)e^{Q_2(z)}\right| \\
&\quad \geq \left|A_2(z)e^{Q_2(z)}\right| - \left|A_1(z)e^{Q_1(z)}\right| \\
&\quad \geq \exp\{(1-\varepsilon)\delta(Q_2,\theta)r^n\} - \exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\} \\
&\quad = \exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\}[\exp\{\gamma r^n\} - 1],
\end{aligned} \tag{3.18}$$

where

$$\gamma = (1-\varepsilon)\delta(Q_2,\theta) - (1+\varepsilon)\delta(Q_1,\theta).$$

Since $0 < \varepsilon < (|a_{n,2}| - |a_{n,1}|)/(2(|a_{n,2}| + |a_{n,1}|))$, then

$$\begin{aligned}
\gamma &= (1-\varepsilon)|a_{n,2}|\cos(\theta_2 + n\theta) - (1+\varepsilon)|a_{n,1}|\cos(\theta_1 + n\theta) \\
&= (|a_{n,2}| - |a_{n,1}| - \varepsilon(|a_{n,2}| + |a_{n,1}|))\cos(\theta_1 + n\theta) \\
&> \frac{|a_{n,2}| - |a_{n,1}|}{2}\cos(\theta_1 + n\theta) > 0.
\end{aligned}$$

Then, by $\gamma > 0$ and from (3.18), we get

$$\begin{aligned}
&\left|A_1(z)e^{Q_1(z)} + A_2(z)e^{Q_2(z)}\right| \\
&\quad \geq (1 - o(1))\exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\}\exp\{\gamma r^n\}.
\end{aligned} \tag{3.19}$$

Substituting (3.2), (3.3), (3.8), (3.9) and (3.19) into (3.7), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n)\backslash(E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$\begin{aligned}
&(1 - o(1))\exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\}\exp\{\gamma r^n\} \\
&\quad \leq M_3 \exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)c\delta(Q_1,\theta)r^n\}[T(2r,f)]^{k+1},
\end{aligned} \tag{3.20}$$

where $M_3 > 0$ is a constant. By (3.20), we have

$$\begin{aligned}
&(1 - o(1))\exp\{[(1+\varepsilon)(1-c)\delta(Q_1,\theta) + \gamma]r^n\} \\
&\quad \leq M_3 \exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}.
\end{aligned} \tag{3.21}$$

By $\delta(Q_1,\theta) > 0$, $\gamma > 0$ and $\rho + \varepsilon < n$, then by using Lemma 2.7 and (3.21), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \leq n$. Hence $\sigma_2(f) = n$.

Case (2). $a_{n,1}$ is a real number such that $(1-c)a_{n,1} < b$, which is $\theta_1 = \pi$.

(i) Assume that $\theta_1 \neq \theta_2$, then $\theta_2 \neq \pi$. By Lemma 2.3, there is a ray $\arg z = \theta$ such that $\theta \in (-\pi/2n, \pi/2n)\backslash(E_3 \cup E_4)$ and $\delta(Q_2, \theta) > 0$. Since $\cos(n\theta) > 0$, we have $\delta(Q_1, \theta) = |a_{n,1}|\cos(\theta_1 + n\theta) = -|a_{n,1}|\cos(n\theta) < 0$. By Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n-\rho, (1-c)/(2(1+c))\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.12) and (3.13). Using the same reasoning as in Case 1(i) b), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \leq n$. Hence $\sigma_2(f) = n$.

(ii) Assume that $\theta_1 = \theta_2$, then $\theta_1 = \theta_2 = \pi$. By Lemma 2.3, there is a ray $\arg z = \theta$ such that $\theta \in (\pi/2n, 3\pi/2n)\backslash(E_3 \cup E_4)$. Then $\cos(n\theta) < 0$, $\delta(Q_1, \theta) = |a_{n,1}|\cos(\theta_1 + n\theta) = -|a_{n,1}|\cos(n\theta) > 0$, $\delta(Q_2, \theta) = |a_{n,2}|\cos(\theta_2 + n\theta) = -|a_{n,2}|\cos(n\theta) > 0$. Since $|a_{n,1}| \leq |a_{n,2}|, a_{n,1} \neq a_{n,2}$ and $\theta_1 = \theta_2$, then $|a_{n,1}| < |a_{n,2}|$. Thus $\delta(Q_2, \theta) > \delta(Q_1, \theta) > 0$. By Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n-\rho, (|a_{n,2}| - |a_{n,1}|)/(2(|a_{n,2}| + |a_{n,1}|))\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.13), (3, 17) and (3.19) holds. For $j \in J$, we have $\delta(P_j, \theta) = -|b_{n,j}|\cos(n\theta) > 0$. Thus

$$\begin{aligned} |B_j(z)e^{P_j(z)}| &\leq \exp\{(1+\varepsilon)\delta(P_j, \theta)r^n\} \\ &\leq \exp\{(1+\varepsilon)br^n\cos(n\theta)\}. \end{aligned} \tag{3.22}$$

Substituting (3.2), (3.3), (3.8), (3.19) and (3.22) into (3.7), for all z satisfying $\arg z = \theta \in (\pi/2n, 3\pi/2n)\backslash(E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$\begin{aligned} &(1-o(1))\exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\}\exp\{\gamma r^n\} \\ &\quad \leq M_4\exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)c\delta(Q_1,\theta)r^n\} \\ &\qquad \times \exp\{(1+\varepsilon)br^n\cos(n\theta)\}[T(2r,f)]^{k+1}, \end{aligned} \tag{3.23}$$

where $M_4 > 0$ is a constant. Hence

$$(1-o(1))\exp\{dr^n\} \le M_4 \exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}, \tag{3.24}$$

where

$$d = (1+\varepsilon)[(1-c)\delta(Q_1,\theta) - b\cos(n\theta)] + \gamma.$$

Since $\gamma > 0$, $\cos(n\theta) < 0$, $\delta(Q_1,\theta) = -|a_{n,1}|\cos(n\theta)$, $(1-c)a_{n,1} < b$ and $b < 0$, we have

$$\begin{aligned} d &= -(1+\varepsilon)[(1-c)|a_{n,1}| + b]\cos(n\theta) + \gamma \\ &> -(1+\varepsilon)[|b| + b]\cos(n\theta) + \gamma = \gamma > 0. \end{aligned}$$

By $\rho + \varepsilon < n$ and $d > 0$, then by using Lemma 2.7 and (3.24), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \ge n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \le n$. Hence $\sigma_2(f) = n$.

4. Proof of Theorem 1.2

First we prove that every meromorphic solution $f(\not\equiv 0)$ of equation (1.4) is transcendental. Assume that $f(\not\equiv 0)$ is a polynomial or a rational solution of equation (1.4). Then $\sigma(f) = 0$. We write equation (1.4) in the form (3.1), where $B(z) = -\big(f^{(k)} + \sum_{j=1}^{k-1} D_j(z)f^{(j)} + D_0(z)f\big)$, A_sf $(s = 1,2)$ and $B_jf^{(j)}$ $(j = 1,2,\dots,k-1)$ are meromorphic functions of finite order with $A_sf \not\equiv 0$ $(s = 1,2)$, $\sigma(B) < n$, $\sigma(A_sf) < n$ $(s = 1,2)$ and $\sigma(B_jf^{(j)}) < n$ $(j = 1,\dots,k-1)$.

If case (1) or case (2)(i) or case (3)(i) holds, it follows that $\deg(Q_2(z) - Q_1(z)) = n$ and $\deg(Q_2(z) - P_j(z)) = n$ $(j = 1,\dots,k-1)$. Thus from (3.1) and by Lemma 2.6, we have $\sigma(B) = n$, this contradicts the fact $\sigma(B) < n$. Hence every meromorphic solution $f(\not\equiv 0)$ of equation (1.4) is transcendental.

If case (2)(ii) or case (3)(ii) holds, it follows that $\deg(Q_1(z)-Q_2(z)) = n$ and $\deg(Q_1(z) - P_j(z)) = n$ $(j = 1,\dots,k-1)$. Thus from (3.1) and by Lemma 2.6, we have $\sigma(B) = n$, this contradicts the fact $\sigma(B) < n$. Hence every meromorphic solution $f(\not\equiv 0)$ of equation (1.4) is transcendental.

Set $\max\{\sigma(A_s), \sigma(B_j), \sigma(D_m)\} = \rho < n$, where $(s = 1,2)$, $(j = 1,\dots,k-1)$ and $(m = 0,\dots,k-1)$. Assume that $f(\not\equiv 0)$ is a meromorphic solution of equation (1.4). By Lemma 2.1, there exist a set $E_1 \subset (1,+\infty)$

having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have (3.2). By Lemma 2.2, for any given ε $(0 < \varepsilon < n - \rho)$, there exists a set $E_2 \subset (1 + \infty)$ that has finite logarithmic measure, such that (3.3) holds for $|z| = r \notin [0,1] \cup E_2$, $r \to +\infty$.

Case (1). Suppose that $\theta_1 \neq \pi$ and $\theta_1 \neq \theta_2$. By Lemma 2.3, there exists a ray $\arg z = \theta$ such that $\theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, where E_3 and E_4 are defined as in Lemma 2.3 and satisfying

$$\delta(Q_1, \theta) > 0,\ \delta(Q_2, \theta) < 0 \text{ or } \delta(Q_1, \theta) < 0,\ \delta(Q_2, \theta) > 0.$$

a) When $\delta(Q_1, \theta) > 0$, $\delta(Q_2, \theta) < 0$, by Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n - \rho, (1 - \alpha)/(2(1 + \alpha))\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.4) and (3.5). By (3.4) and (3.5), we have (3.6).

For $j \in I$, we have $\delta(P_j, \theta) = \alpha_j \delta(Q_1, \theta) > 0$. Thus

$$\left|B_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\alpha_j\delta(Q_1, \theta)r^n\} \leq \exp\{(1+\varepsilon)\alpha\delta(Q_1, \theta)r^n\}. \quad (4.1)$$

For $j \in J$, we have $\delta(P_j, \theta) = \beta_j \delta(Q_2, \theta) < 0$. Thus

$$\left|B_j(z)e^{P_j(z)}\right| \leq \exp\{(1 - \varepsilon)\beta_j\delta(Q_2, \theta)r^n\} < 1. \quad (4.2)$$

Substituting (3.2), (3.3), (3.6), (4.1) and (4.2) into (3.7), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$\begin{aligned}&(1 - o(1)) \exp\{(1 - \varepsilon)\delta(Q_1, \theta)r^n\} \\ &\quad \leq M_1 \exp\{r^{\rho+\varepsilon}\} \exp\{(1 + \varepsilon)\alpha\delta(Q_1, \theta)r^n\}[T(2r, f)]^{k+1}, \end{aligned} \quad (4.3)$$

where $M_1 > 0$ is a constant. From (4.3) and $0 < \varepsilon < (1 - \alpha)/(2(1 + \alpha))$, we get

$$(1 - o(1)) \exp\left\{\frac{(1 - \alpha)}{2}\delta(Q_1, \theta)r^n\right\} \leq M_1 \exp\{r^{\rho+\varepsilon}\}[T(2r, f)]^{k+1}. \quad (4.4)$$

By $\delta(Q_1, \theta) > 0$ and $\rho + \varepsilon < n$, then by using Lemma 2.7 and (4.4), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by

Lemma 2.5, we have $\sigma_2(f) \le n$. Hence $\sigma_2(f) = n$.

b) When $\delta(Q_1, \theta) < 0$, $\delta(Q_2, \theta) > 0$, by Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n - \rho, (1-\beta)/(2(1+\beta))\})$, there exists a set $E_5 \subset [1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.12) and (3.13). By (3.12) and (3.13), we have we have (3.14).

For $j \in I$, we have $\delta(P_j, \theta) < 0$. Thus

$$\left|B_j(z)e^{P_j(z)}\right| \le \exp\{(1+\varepsilon)\alpha_j \delta(Q_1, \theta) r^n\} < 1. \tag{4.5}$$

For $j \in J$, we have $\delta(P_j, \theta) > 0$. Thus

$$\left|B_j(z)e^{P_j(z)}\right| \le \exp\{(1+\varepsilon)\beta_j \delta(Q_2, \theta) r^n\} \le \exp\{(1+\varepsilon)\beta \delta(Q_2, \theta) r^n\}. \tag{4.6}$$

Substituting (3.2), (3.3), (3.14), (4.5) and (4.6) into (3.7), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$\begin{aligned}(1 - o(1)) &\exp\{(1-\varepsilon)\delta(Q_2, \theta) r^n\} \\ &\le M_2 \exp\{r^{\rho+\varepsilon}\} \exp\{(1+\varepsilon)\beta\delta(Q_2, \theta) r^n\} [T(2r, f)]^{k+1},\end{aligned} \tag{4.7}$$

where $M_2 > 0$ is a constant. From (4.7) and $0 < \varepsilon < (1-\beta)/(2(1+\beta))$, we get

$$(1 - o(1)) \exp\left\{\frac{(1-\beta)}{2} \delta(Q_2, \theta) r^n\right\} \le M_2 \exp\{r^{\rho+\varepsilon}\} [T(2r, f)]^{k+1}. \tag{4.8}$$

By $\delta(Q_2, \theta) > 0$ and $\rho + \varepsilon < n$, then by using Lemma 2.7 and (4.8), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \ge n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \le n$. Hence $\sigma_2(f) = n$.

Case (2). Suppose that $\theta_1 \ne \pi$ and $\theta_1 = \theta_2$. By Lemma 2.3, there is a ray $\arg z = \theta$ such that $\theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$ and $\delta(Q_1, \theta) > 0$. Since $\theta_1 = \theta_2$, it follows that $\delta(Q_2, \theta) > 0$.

(i) If $|a_{n,1}| < (1-\beta)|a_{n,2}|$, by Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n - \rho, ((1-\beta)|a_{n,2}| - |a_{n,1}|)/(2[(1+\beta)|a_{n,2}| + |a_{n,1}|])\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.13) and (3.17).

By (1.4), we get

$$\begin{aligned}\left|A_2(z)e^{Q_2(z)}\right| \le& \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left(|D_{k-1}(z)| + \left|B_{k-1}(z)e^{P_{k-1}(z)}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right| \\ &+ \cdots + \left(|D_1(z)| + \left|B_1(z)e^{P_1(z)}\right|\right)\left|\frac{f'(z)}{f(z)}\right| \\ &+ \left|A_1(z)e^{Q_1(z)}\right| + |D_0(z)|. \qquad (4.9)\end{aligned}$$

Substituting (3.2), (3.3), (3.13), (3.17), (4.1) and (4.6) into (4.9), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n)\backslash(E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$\begin{aligned}&\exp\{(1-\varepsilon)\delta(Q_2,\theta)r^n\} \\ &\quad \le k\exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(Q_1,\theta)r^n\} \\ &\qquad \times \exp\{(1+\varepsilon)\beta\delta(Q_2,\theta)r^n\}[T(2r,f)]^{k+1}, \\ &+ \exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\} + \exp\{r^{\rho+\varepsilon}\} \\ &\quad \le M_3\exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)\delta(Q_1,\theta)r^n\} \\ &\qquad \times \exp\{(1+\varepsilon)\beta\delta(Q_2,\theta)r^n\}[T(2r,f)]^{k+1}, \qquad (4.10)\end{aligned}$$

where $M_3 > 0$ is a constant. By (4.10), we have

$$\exp\{d_1 r^n\} \le M_3 \exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}, \qquad (4.11)$$

where

$$d_1 = (1-\varepsilon)\delta(Q_2,\theta) - (1+\varepsilon)\delta(Q_1,\theta) - (1+\varepsilon)\beta\delta(Q_2,\theta).$$

Since

$$0 < \varepsilon < \frac{(1-\beta)|a_{n,2}| - |a_{n,1}|}{2[(1+\beta)|a_{n,2}| + |a_{n,1}|]},$$

$\theta_1 = \theta_2$ and $\cos(\theta_1 + n\theta) > 0$, we have

$$\begin{aligned}
d_1 &= [1-\beta-\varepsilon(1+\beta)]\delta(Q_2,\theta)-(1+\varepsilon)\delta(Q_1,\theta)\\
&= [1-\beta-\varepsilon(1+\beta)]|a_{n,2}|\cos(\theta_1+n\theta)-(1+\varepsilon)|a_{n,1}|\cos(\theta_1+n\theta)\\
&= \{(1-\beta)|a_{n,2}|-|a_{n,1}|-\varepsilon[(1+\beta)|a_{n,2}|+|a_{n,1}|]\}\cos(\theta_1+n\theta)\\
&> \frac{(1-\beta)|a_{n,2}|-|a_{n,1}|}{2}\cos(\theta_1+n\theta)>0.
\end{aligned}$$

Since $d_1>0$ and $\rho+\varepsilon<n$, then by using Lemma 2.7 and (4.11), we obtain $\sigma(f)=+\infty$ and $\sigma_2(f)\geq n$. In addition, if $\lambda(1/f)<+\infty$, then by Lemma 2.5, we have $\sigma_2(f)\leq n$. Hence $\sigma_2(f)=n$.

(ii) If $|a_{n,2}|<(1-\alpha)|a_{n,1}|$, by Lemma 2.4, for any given ε ($0<\varepsilon<\min\{n-\rho,((1-\alpha)|a_{n,1}|-|a_{n,2}|)/(2[(1+\alpha)|a_{n,1}|+|a_{n,2}|])\}$), there exists a set $E_5\subset(1,+\infty)$ having finite logarithmic measure such that for $|z|=r\notin[0,1]\cup E_5$, $r\to+\infty$, we have (3.4) and

$$\left|A_2(z)e^{Q_2(z)}\right|\leq\exp\{(1+\varepsilon)\delta(Q_2,\theta)r^n\}. \tag{4.12}$$

By (1.4), we get

$$\begin{aligned}
\left|A_1(z)e^{Q_1(z)}\right| \leq{}& \left|\frac{f^{(k)}(z)}{f(z)}\right|+\left(|D_{k-1}(z)|+\left|B_{k-1}(z)e^{P_{k-1}(z)}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right|\\
&+\cdots+\left(|D_1(z)|+\left|B_1(z)e^{P_1(z)}\right|\right)\left|\frac{f'(z)}{f(z)}\right|\\
&+\left|A_2(z)e^{Q_2(z)}\right|+|D_0(z)|.
\end{aligned}\tag{4.13}$$

Substituting (3.2), (3.3), (3.4), (4.1), (4.6) and (4.12) into (4.13), for all z satisfying $\arg z=\theta\in(-\pi/2n,\pi/2n)\backslash(E_3\cup E_4)$, $|z|=r\notin[0,1]\cup E_1\cup E_2\cup E_5$, $r\to+\infty$, we obtain

$$\begin{aligned}
&\exp\{(1-\varepsilon)\delta(Q_1,\theta)r^n\}\\
&\quad\leq k\exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(Q_1,\theta)r^n\}\\
&\qquad\times\exp\{(1+\varepsilon)\beta\delta(Q_2,\theta)r^n\}[T(2r,f)]^{k+1},
\end{aligned}$$

$$+\exp\{(1+\varepsilon)\delta(Q_2,\theta)r^n\}+\exp\{r^{\rho+\varepsilon}\}$$
$$\le M_4\exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(Q_1,\theta)r^n\}$$
$$\times\exp\{(1+\varepsilon)\delta(Q_2,\theta)r^n\}[T(2r,f)]^{k+1}, \tag{4.14}$$

where $M_4 > 0$ is a constant. By (4.14), we have

$$\exp\{d_2 r^n\} \le M_4\exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}, \tag{4.15}$$

where

$$d_2 = (1-\varepsilon)\delta(Q_1,\theta)-(1+\varepsilon)\delta(Q_2,\theta)-(1+\varepsilon)\alpha\delta(Q_1,\theta).$$

Since

$$0<\varepsilon<\frac{(1-\alpha)|a_{n,1}|-|a_{n,2}|}{2[(1+\alpha)|a_{n,1}|+|a_{n,2}|]},$$

$\theta_1=\theta_2$ and $\cos(\theta_1+n\theta)>0$, we obtain

$$d_2 = \{(1-\alpha)|a_{n,1}|-|a_{n,2}|-\varepsilon[(1+\alpha)|a_{n,1}|+|a_{n,2}|]\}\cos(\theta_1+n\theta)$$
$$> \frac{(1-\alpha)|a_{n,1}|-|a_{n,2}|}{2}\cos(\theta_1+n\theta)>0.$$

Since $d_2>0$ and $\rho+\varepsilon<n$, then by using Lemma 2.7 and (4.15), we obtain $\sigma(f)=+\infty$ and $\sigma_2(f)\ge n$. In addition, if $\lambda(1/f)<+\infty$, then by Lemma 2.5, we have $\sigma_2(f)\le n$. Hence $\sigma_2(f)=n$.

Case (3). Suppose that $a_{n,1}$ and $a_{n,2}$ are real numbers such that $(1-\beta)a_{n,2}<a_{n,1}<0$ or $(1-\alpha)a_{n,1}<a_{n,2}<0$, which is $\theta_1=\theta_2=\pi$. By Lemma 2.3, there is a ray $\arg z=\theta$ such that $\theta\in(\pi/2n,3\pi/2n)\backslash(E_3\cup E_4)$. Then $\cos(n\theta)<0$, $\delta(Q_1,\theta)=|a_{n,1}|\cos(\theta_1+n\theta)=-|a_{n,1}|\cos(n\theta)>0$ and $\delta(Q_2,\theta)=|a_{n,2}|\cos(\theta_2+n\theta)=-|a_{n,2}|\cos(n\theta)>0$.

(i) If $(1-\beta)a_{n,2}<a_{n,1}<0$, by using the same reasoning as in case (2) (i), we get $\sigma(f)=+\infty$ and $\sigma_2(f)\ge n$. In addition, if $\lambda(1/f)<+\infty$, then by Lemma 2.5, we have $\sigma_2(f)\le n$. Hence $\sigma_2(f)=n$.

(ii) If $(1-\alpha)a_{n,1}<a_{n,2}<0$, by using the same reasoning as in case (2) (ii), we get $\sigma(f)=+\infty$ and $\sigma_2(f)\ge n$. In addition, if $\lambda(1/f)<+\infty$, then by Lemma 2.5, we have $\sigma_2(f) \le n$. Hence $\sigma_2(f) = n$.

5. Proof of Theorem 1.3

Using the same reasoning as in the proof of Theorem 1.2, we obtain that every meromorphic solution $f(\not\equiv 0)$ of equation (1.4) is transcendental.

Set $\max\{\sigma(A_s), \sigma(B_j), \sigma(D_m)\} = \rho < n$, where $(s = 1, 2)$, $(j = 1, \dots, k-1)$ and $(m = 0, \dots, k-1)$. Assume that $f(\not\equiv 0)$ is a meromorphic solution of equation (1.4). By Lemma 2.1, there exist a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$,we have (3.2). By Lemma 2.2, for any given ε $(0 < \varepsilon < n - \rho)$, there exists a set $E_2 \subset (1, +\infty)$ that has finite logarithmic measure, such that (3.3) holds for $|z| = r \notin [0, 1] \cup E_2$, $r \to +\infty$.

Case (1). Suppose that $\theta_1 \neq \pi$ and $\theta_1 \neq \theta_2$. By Lemma 2.3, there exists a ray $\arg z = \theta$, such that $\theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, where E_3 and E_4 are defined as in Lemma 2.3 and satisfying

$$\delta(Q_1, \theta) > 0,\ \delta(Q_2, \theta) < 0 \text{ or } \delta(Q_1, \theta) < 0,\ \delta(Q_2, \theta) > 0.$$

a) When $\delta(Q_1, \theta) > 0$, $\delta(Q_2, \theta) < 0$, by Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n - \rho, (1 - \alpha)/(2(1 + \alpha))\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E_5$, $r \to +\infty$, we have (3.4) and (3.5). By (3.4) and (3.5), we have (3.6).

For $j \in I$, we have

$$\delta(\alpha_j a_{n,1} z^n, \theta) = \alpha_j \delta(Q_1, \theta) > 0 \text{ and } \delta(P_j(z) - \alpha_j a_{n,1} z^n, \theta) = \beta_j \delta(Q_2, \theta) < 0.$$

Thus

$$\begin{aligned} \left|B_j(z) e^{\alpha_j a_{n,1} z^n}\right| &\le \exp\{(1 + \varepsilon)\alpha_j \delta(Q_1, \theta) r^n\} \\ &\le \exp\{(1 + \varepsilon)\alpha \delta(Q_1, \theta) r^n\} \end{aligned} \tag{5.1}$$

and

$$\left|e^{P_j(z) - \alpha_j a_{n,1} z^n}\right| \le \exp\{(1 - \varepsilon)\beta_j \delta(Q_2, \theta) r^n\} < 1. \tag{5.2}$$

By (5.1) and (5.2), we obtain

$$\begin{aligned}\left|B_j(z)e^{P_j(z)}\right| &= \left|B_j(z)e^{\alpha_j a_{n,1}z^n}\right|\left|e^{P_j(z)-\alpha_j a_{n,1}z^n}\right| \\ &\leq \exp\{(1+\varepsilon)\alpha\delta(Q_1,\theta)r^n\}. \end{aligned} \tag{5.3}$$

Substituting (3.2), (3.3), (3.6), (3.9) and (5.3) into (3.7), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n)\backslash(E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain (4.3). From (4.3) and $0 < \varepsilon < (1-\alpha)/(2(1+\alpha))$, we get (4.4). By $\delta(Q_1,\theta) > 0$ and $\rho + \varepsilon < n$, then by using Lemma 2.7 and (4.4), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \leq n$. Hence $\sigma_2(f) = n$.

b) When $\delta(Q_1,\theta) < 0$, $\delta(Q_2,\theta) > 0$, by Lemma 2.4, for any given ε ($0 < \varepsilon < \min\{n-\rho, (1-\beta)/(2(1+\beta))\}$), there exists a set $E_5 \subset (1,+\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.12) and (3.13). By (3.12) and (3.13), we have (3.14).

For $j \in I$, we have

$$\delta(\beta_j a_{n,2}z^n,\theta) = \beta_j\delta(Q_2,\theta) > 0 \text{ and } \delta(P_j(z)-\beta_j a_{n,2}z^n,\theta) = \alpha_j\delta(Q_1,\theta) < 0.$$

Thus

$$\begin{aligned}\left|B_j(z)e^{\beta_j a_{n,2}z^n}\right| &\leq \exp\{(1+\varepsilon)\beta_j\delta(Q_2,\theta)r^n\} \\ &\leq \exp\{(1+\varepsilon)\beta\delta(Q_2,\theta)r^n\} \end{aligned} \tag{5.4}$$

and

$$\left|e^{P_j(z)-\beta_j a_{n,2}z^n}\right| \leq \exp\{(1-\varepsilon)\alpha_j\delta(Q_1,\theta)r^n\} < 1. \tag{5.5}$$

By (5.4) and (5.5), we obtain

$$\begin{aligned}\left|B_j(z)e^{P_j(z)}\right| &= \left|B_j(z)e^{\beta_j a_{n,2}z^n}\right|\left|e^{P_j(z)-\beta_j a_{n,2}z^n}\right| \\ &\leq \exp\{(1+\varepsilon)\beta\delta(Q_2,\theta)r^n\}. \end{aligned} \tag{5.6}$$

Substituting (3.2), (3.3), (3.9), (3.14) and (5.6) into (3.7), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n)\backslash(E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain (4.7). From (4.7) and $0 < \varepsilon < (1-\beta)/(2(1+\beta))$, we get (4.8). By $\delta(Q_2,\theta) > 0$ and $\rho + \varepsilon < n$, then by using Lemma 2.7 and (4.8), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma

2.5, we have $\sigma_2(f) \le n$. Hence $\sigma_2(f) = n$.

Case (2). Suppose that $\theta_1 \neq \pi$ and $\theta_1 = \theta_2$. By Lemma 2.3, there is a ray $\arg z = \theta$ such that $\theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$ and $\delta(Q_1, \theta) > 0$. Since $\theta_1 = \theta_2$, it follows that $\delta(Q_2, \theta) > 0$.

(i) If $|a_{n,1}| < (1-\beta)|a_{n,2}|$, by Lemma 2.4, for any given ε ($0 < \varepsilon < \min\{n-\rho, ((1-\beta)|a_{n,2}| - |a_{n,1}|)/(2[(1+\beta)|a_{n,2}| + |a_{n,1}|])\}$), there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.13) and (3.17).

For $j \in I$, we have

$$\delta(\alpha_j a_{n,1} z^n, \theta) = \alpha_j \delta(Q_1, \theta) > 0 \text{ and } \delta(P_j(z) - \alpha_j a_{n,1} z^n, \theta) = \beta_j \delta(Q_2, \theta) > 0.$$

Thus (5.1) holds and

$$\begin{aligned} \left| e^{P_j(z) - \alpha_j a_{n,1} z^n} \right| &\le \exp\{(1+\varepsilon)\beta_j \delta(Q_2, \theta) r^n\} \\ &\le \exp\{(1+\varepsilon)\beta \delta(Q_2, \theta) r^n\}. \end{aligned} \tag{5.7}$$

By (5.1) and (5.7), we get

$$\left| B_j(z) e^{P_j(z)} \right| \le \exp\{(1+\varepsilon)\alpha \delta(Q_1, \theta) r^n\} \exp\{(1+\varepsilon)\beta \delta(Q_2, \theta) r^n\}. \tag{5.8}$$

Substituting (3.2), (3.3), (3.9), (3.13), (3.17) and (5.8) into (4.9), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain (4.10). By (4.10), we have (4.11). Using similar reasoning as in case (2) (i) in the proof of Theorem 1.2, we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \ge n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \le n$. Hence $\sigma_2(f) = n$.

(ii) If $|a_{n,2}| < (1-\alpha)|a_{n,1}|$, by Lemma 2.4, for any given ε ($0 < \varepsilon < \min\{n-\rho, ((1-\alpha)|a_{n,1}| - |a_{n,2}|)/(2[(1+\alpha)|a_{n,1}| + |a_{n,2}|])\}$), there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.4) and (4.12).

Substituting (3.2), (3.3), (3.4), (3.9), (4.12) and (5.8) into (4.13), for all z satisfying $\arg z = \theta \in (-\pi/2n, \pi/2n) \backslash (E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain (4.14). By (4.14), we have (4.15). Using similar reasoning as in case (2) (ii) in the proof of Theorem 1.2, we obtain

$\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \leq n$. Hence $\sigma_2(f) = n$.

Case (3). Suppose that $a_{n,1}$ and $a_{n,2}$ are real numbers such that $(1-\beta)a_{n,2} - b < a_{n,1} < 0$ or $(1-\alpha)a_{n,1} < a_{n,2} < 0$, which is $\theta_1 = \theta_2 = \pi$. By Lemma 2.3, there is a ray $\arg z = \theta$ such that $\theta \in (\pi/2n, 3\pi/2n)\backslash(E_3 \cup E_4)$. Then $\cos(n\theta) < 0$, $\delta(Q_1,\theta) = |a_{n,1}|\cos(\theta_1 + n\theta) = -|a_{n,1}|\cos(n\theta) > 0$ and $\delta(Q_2,\theta) = |a_{n,2}|\cos(\theta_2 + n\theta) = -|a_{n,2}|\cos(n\theta) > 0$. For $j \in J$, we have $\delta(P_j,\theta) = -|b_{n,j}|\cos(n\theta) > 0$.

(i) If $(1-\beta)a_{n,2} - b < a_{n,1} < 0$, by Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n-\rho, ((1-\beta)|a_{n,2}| - |a_{n,1}| + b)/(2[(1+\beta)|a_{n,2}| + |a_{n,1}| - b])\})$, there exists a set $E_5 \subset (1,+\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.13) and (3.17).

Substituting (3.2), (3.3), (3.13), (3.17), (3.22) and (5.8) into (4.9), for all z satisfying $\arg z = \theta \in (\pi/2n, 3\pi/2n)\backslash(E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$\begin{aligned}&\exp\{(1-\varepsilon)\delta(Q_2,\theta)r^n\}\\ &\quad\leq M_5\exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)[\delta(Q_1,\theta) + \beta\delta(Q_2,\theta)\\ &\qquad\qquad + b\cos(n\theta)]r^n\}[T(2r,f)]^{k+1},\end{aligned} \tag{5.9}$$

where $M_5 > 0$ is a constant. By (5.9), we have

$$\exp\{d_3 r^n\} \leq M_5\exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}, \tag{5.10}$$

where

$$d_3 = (1-\varepsilon)\delta(Q_2,\theta) - (1+\varepsilon)[\delta(Q_1,\theta) + \beta\delta(Q_2,\theta) + b\cos(n\theta)].$$

From $0 < \varepsilon < ((1-\beta)|a_{n,2}| - |a_{n,1}| + b)/(2[(1+\beta)|a_{n,2}| + |a_{n,1}| - b])$, $\theta_1 = \theta_2 = \pi$ and $\cos(n\theta) < 0$, we obtain

$$\begin{aligned}d_3 &= [1-\beta-\varepsilon(1+\beta)]\delta(Q_2,\theta) - (1+\varepsilon)[\delta(Q_1,\theta) + b\cos(n\theta)]\\ &= -[1-\beta-\varepsilon(1+\beta)]|a_{n,2}|\cos(n\theta) + (1+\varepsilon)[|a_{n,1}| - b]\cos(n\theta)\\ &= -\{(1-\beta)|a_{n,2}| - |a_{n,1}| + b - \varepsilon[(1+\beta)|a_{n,2}| + |a_{n,1}| - b]\}\cos(n\theta)\end{aligned}$$

$$> -\frac{[(1-\beta)|a_{n,2}| - |a_{n,1}| + b]}{2}\cos(n\theta) > 0.$$

Since $d_3 > 0$ and $\rho+\varepsilon < n$, then by using Lemma 2.7 and (5.10), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \leq n$. Hence $\sigma_2(f) = n$.

(ii) If $(1-\alpha)a_{n,1} - b < a_{n,2} < 0$, by Lemma 2.4, for any given ε $(0 < \varepsilon < \min\{n-\rho, ((1-\alpha)|a_{n,1}| - |a_{n,2}| + b)/(2[(1+\alpha)|a_{n,1}| + |a_{n,2}| - b)])\})$, there exists a set $E_5 \subset (1, +\infty)$ having finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have (3.4) and (4.12).

Substituting (3.2), (3.3), (3.4), (3.22), (4.12) and (5.8) into (4.13), for all z satisfying $\arg z = \theta \in (\pi/2n, 3\pi/2n)\backslash(E_3 \cup E_4)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_5$, $r \to +\infty$, we obtain

$$\begin{aligned}\exp\{(1-\varepsilon)&\delta(Q_1,\theta)r^n\}\\ &\leq M_6 \exp\{r^{\rho+\varepsilon}\}\exp\{(1+\varepsilon)[\delta(Q_2,\theta) + \alpha\delta(Q_1,\theta)\\ &\qquad + b\cos(n\theta)]r^n\}[T(2r,f)]^{k+1}, \end{aligned}\tag{5.11}$$

where $M_6 > 0$ is a constant. By (5.11), we have

$$\exp\{d_4 r^n\} \leq M_6 \exp\{r^{\rho+\varepsilon}\}[T(2r,f)]^{k+1}, \tag{5.12}$$

where

$$d_4 = (1-\varepsilon)\delta(Q_1,\theta) - (1+\varepsilon)[\delta(Q_2,\theta) + \alpha\delta(Q_1,\theta) + b\cos(n\theta)].$$

From $0 < \varepsilon < ((1-\alpha)|a_{n,1}| - |a_{n,2}| + b)/(2[(1+\alpha)|a_{n,1}| + |a_{n,2}| - b])$, $\theta_1 = \theta_2 = \pi$ and $\cos(n\theta) < 0$, we obtain

$$\begin{aligned} d_3 &= [1-\alpha-\varepsilon(1+\alpha)]\delta(Q_1,\theta) - (1+\varepsilon)[\delta(Q_2,\theta) + b\cos(n\theta)]\\ &= -[1-\alpha-\varepsilon(1+\alpha)]|a_{n,1}|\cos(n\theta) + (1+\varepsilon)[|a_{n,2}| - b]\cos(n\theta)\\ &= -\{(1-\alpha)|a_{n,1}| - |a_{n,2}| + b - \varepsilon[(1+\alpha)|a_{n,1}| + |a_{n,2}| - b]\}\cos(n\theta)\\ &> -\frac{[(1-\alpha)|a_{n,1}| - |a_{n,2}| + b]}{2}\cos(n\theta) > 0. \end{aligned}$$

Since $d_4 > 0$ and $\rho+\varepsilon < n$, then by using Lemma 2.7 and (5.12), we obtain

$\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. In addition, if $\lambda(1/f) < +\infty$, then by Lemma 2.5, we have $\sigma_2(f) \leq n$. Hence $\sigma_2(f) = n$.

6. Proof of Theorem 1.4

Assume that f ($\not\equiv 0$) is a solution of equation (1.4). Set $g = f - \varphi$. Then we have $\sigma(g) = \sigma(f) = +\infty$. Substituting $f = g + \varphi$ into (1.4), we obtain

$$g^{(k)} + \sum_{j=1}^{k-1}(D_j + B_j e^{P_j(z)})g^{(j)} + \big(D_0 + A_1 e^{Q_1(z)} + A_2 e^{Q_2(z)}\big)g = H, \quad (6.1)$$

where $H = -\big[\varphi^{(k)} + \sum_{j=1}^{k-1}(D_j + B_j e^{P_j(z)})\varphi^{(j)} + (D_0 + A_1 e^{Q_1(z)} + A_2 e^{Q_2(z)})\varphi\big]$. Now we prove that $H \not\equiv 0$. In fact if $H \equiv 0$, then

$$\varphi^{(k)} + \sum_{J=1}^{k-1}\big(D_j + B_j e^{P_j(z)}\big)\varphi^{(j)} + \big(D_0 + A_1 e^{Q_1(z)} + A_2 e^{Q_2(z)}\big)\varphi = 0. \quad (6.2)$$

Hence φ ($\not\equiv 0$) is a solution of equation (1.4). Thus $\sigma(\varphi) = +\infty$ by the hypotheses of Theorem 1.4, which is a contradiction. Hence $H \not\equiv 0$. By Lemma 2.8 and Lemma 2.9, we have

$$\lambda(g) = \overline{\lambda}(g) = \sigma(g) = \sigma(f) = +\infty \text{ and } \lambda_2(g) = \overline{\lambda}_2(g) = \sigma_2(f) \geq n,$$

i.e.,

$$\lambda(f-\varphi) = \overline{\lambda}(f-\varphi) = \sigma(f) = +\infty \text{ and } \lambda_2(f-\varphi) = \overline{\lambda}_2(f-\varphi) = \sigma_2(f) \geq n.$$

In addition, if $\lambda(1/f) < +\infty$, then $\lambda_2(f - \varphi) = \overline{\lambda}_2(f - \varphi) = \sigma_2(f) = n$.

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Hamid Beddani
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem, Algeria
E-mail: beddanihamid@gmail.com

Karima Hamani
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem, Algeria
E-mail: karima.hamani@univ-mosta.dz
hamanikarima@yahoo.fr