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Kinematic expansive suspensions of irrational rotations on the circle

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Abstract. We shall show that the rotation of some irrational rotation number on the circle admits suspensions which are kinematic expansive.

Key words: kinematic expansive flows, suspensions, irrational rotations.

1. Introduction

A continuous flow $\phi = \{\phi^t\}_{t\in\mathbb{R}}$ on a compact metric space X is called *kinematic expansive* if for any $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that whenever $d(\phi^t(x), \phi^t(y)) < \delta(\epsilon)$ for any $t \in \mathbb{R}$, $y = \phi^s(x)$ for some $s \in (-\epsilon, \epsilon)$.

Given a homeomorphism f of a compact metric space Y and a continuous fucntion $T : Y \to (0, \infty)$, we shall construct the suspension flow of f with return time T as follows. Let $\tilde{\phi} = \{\tilde{\phi}^t\}$ be the flow on $Y \times \mathbb{R}$ given by

$$\tilde{\phi}^t(x, s) = (x, s + t).$$

Define a homeomorphism $F : Y \times \mathbb{R} \to Y \times \mathbb{R}$ by

$$F(x, t) = (f(x), t - T(x)).$$

The infinite cyclic group $\langle F\rangle$ acts on $Y \times \mathbb{R}$ freely, properly discontinuously and cocompactly. The action commutes with the flow $\tilde{\phi}$:

$$\tilde{\phi}^t \circ F = F \circ \tilde{\phi}^t, \quad \forall t \in \mathbb{R}.$$

Thus the flow $\tilde{\phi}$ induces a flow on the quotient space $\langle F\rangle \setminus (Y \times \mathbb{R})$, which is called the *suspension flow of f with return time T*, denoted by $\mathrm{sus}(f, T)$.

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Notice that $\mathrm{sus}(f,T)$ admits a global cross section Y', the image of $Y \times \{0\}$ by the canonical projection. The first return map of $\mathrm{sus}(f,T)$ with respect to Y' is f and the return time is T. (Notice that a point $(x,0)$ on Y' flows by time $T(x)$ to the point $(x,T(x))$, which is identified with a point $(f(x),0)$ on Y'.) Of course the kinematic expansiveness of $\mathrm{sus}(f,T)$ strongly depends upon the choice of T.

In [1], A. Artigue studies among others suspensions of homeomorphisms of the circle S^1. He obtained:

Theorem 1.1 *Let f be an orientation preserving nonminimal homeomorphism of S^1. Then f admits a kinematic expansive suspension if and only if there is a nonempty family $\{I_1,\dots,I_r\}$ of finitely many nonempty open intervals such that the wandering point set $W(f)$ of f satisfies*

$$W(f)=\bigcup_{n\in\mathbb{Z}}\bigcup_{i=1}^{r} f^n(I_i).$$

For the rotation R_α by an irrational number α, he showed that if T is absolutely continuous, then $\mathrm{sus}(R_\alpha,T)$ is not kinematic expansive, and posed the problem for T just continuous. The main result of the present paper is the following theorem.

Theorem 1.2 *There exist an irrational number α and a positive valued continuous function T on S^1 such that* $\mathrm{sus}(R_\alpha,T)$ *is kinematic expansive.*

Remark 1.3 In fact, we have shown for $\{\phi^t\}=\mathrm{sus}(R_\alpha,T)$ in the above theorem that for any $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that whenever $d(\phi^t(x),\phi^t(y))<\delta(\epsilon)$ for any *positive* t, then $y=\phi^s(x)$ for some $s\in(-\epsilon,\epsilon)$. This is slightly stronger than the kinematic expansiveness.

2. Proof of Theorem 1.2

We shall choose a particular irrational number $\alpha\in(0,1)$ (explained later) and construct a return time map $T: S^1\to(0,\infty)$ such that for some $\delta>0$,

$$y\neq x,\quad |y-x|<\delta \Longrightarrow \quad |T^{(n)}(y)-T^{(n)}(x)|>\delta,\quad \exists n\in\mathbb{N}, \tag{2.1}$$

where $T^{(n)}=\sum_{k=0}^{n-1}T\circ f^k$ is the n-th return time. This is sufficient for

Theorem 1.2 since we can choose the number $\delta(\epsilon)$ in the definition of the kinematic expansiveness as $\delta(\epsilon) = \min\{3^{-1}\epsilon, \delta\}$. Actually we shall construct a *real valued* continuous function T satisfying (2.1). We just need to add a positive constant in order to make it positive valued.

For $x \in \mathbb{R}$, we denote its projected image on $S^1 = \mathbb{R}/\mathbb{Z}$ by the same letter x, and the distance in S^1 to 0 by $|x|$. Notice also that S^1 is an additive group so that for example $x - y$ makes sense for $x, y \in S^1$. We first prepare a useful lemma.

Lemma 2.1 *Assume that for any $r \in (0, \delta)$, there is $x_r \in S^1$ such that for some $m \in \mathbb{N}$,*

$$|T^{(m)}(x_r + r) - T^{(m)}(x_r)| > 3\delta.$$

Then for any $x \in S^1$ and $y = x + r$, the conclusion of (2.1) *holds.*

Proof. For any $r \in (0, \delta)$, let x_r and m be as in the assumption of Lemma 2.1. For any $x \in S^1$, there exists $q \in \mathbb{N}$ such that $R^q_\alpha(x)$ is arbitrarily close to x_r (and thus $R^q_\alpha(x + r)$ to $x_r + r$). By the uniform continuity of $T^{(m)}$, one may assume

$$|T^{(m)}(R^q_\alpha(x)) - T^{(m)}(x_r)| < \delta/2, \quad |T^{(m)}(R^q_\alpha(x + r)) - T^{(m)}(x_r + r)| < \delta/2.$$

If $|T^{(q)}(x + r) - T^{(q)}(x)| > \delta$, then there is nothing to prove. Otherwise we have

$$|T^{(m+q)}(x + r) - T^{(m+q)}(x)| > \delta,$$

as is required. □

We choose the irrational number α by the continued fraction as

$$\alpha = \cfrac{1}{a + \frac{1}{a + \frac{1}{a + \cdots}}},$$

where a is an integer $\geq 10^{10}$. That is,

$$\alpha = \frac{1}{2}(-a + \sqrt{a^2 + 4}).$$

In fact, the arguments in what follows work for much smaller value of a. On the other hand, they are not applicable to Liouville numbers. So we make the assumption $a \geq 10^{10}$ in order to make various estimates easier.

Let p_n/q_n be the n-th convergent of α. The denominator q_n is obtained inductively as:

$$q_{n+2} = aq_{n+1} + q_n, \quad q_0 = 1, \quad q_1 = a.$$

Let c be the positive solution of $x^2 = ax + 1$:

$$c = 2^{-1}(a + \sqrt{a^2 + 4}).$$

Thus c is a number slightly bigger than a. We have

$$q_n = Ac^n + B(-c^{-1})^n, \quad \text{where} \quad A = \frac{a + \sqrt{a^2+4}}{2\sqrt{a^2+4}}, \quad B = \frac{-a + \sqrt{a^2+4}}{2\sqrt{a^2+4}}.$$

Thus A and B are positive numbers satisfying $A + B = 1$, and A is almost 1. Since $c > a \geq 10^{10}$, we have

$$q_n \doteq Ac^n. \tag{2.2}$$

Notation 2.2 For $a, b > 0$, $a \doteq b$ means $a/b \in (100/101, 101/100)$.

It is well known that q_n is the *closest return time* for the rotation $R_\alpha : S^1 \to S^1$. That is, $R_\alpha^{q_n}(x)$ is the closest to x among the points $R_\alpha(x), \dots, R_\alpha^{q_n - 1}(x), R_\alpha^{q_n}(x)$. More precisely (letting $x = 0$),

$$|q_n\alpha| < |i\alpha|, \quad \forall i \in \{1, 2, \dots, q_n - 1\}. \tag{2.3}$$

The point $q_n\alpha$ is very close to 0, lies on the right side of 0 if n is odd, and on the left if n is even. Let I_n be the smaller closed interval in S^1 bounded by 0 and $q_n\alpha$. Consider the first return map of R_α for the interval $I_n \cup I_{n+1}$. The part I_{n+1} returns to $I_n \cup I_{n+1}$ for the first time by the q_n iterate of R_α, and the part $I_n \setminus \{0\}$ by the q_{n+1} iterate. (All this follows from (2.3).) See Figure 1 for even n.

The intervals $I_n, R_\alpha(I_n), \dots, R_\alpha^{q_{n+1}-1}(I_n)$ and $I_{n+1}, R_\alpha(I_{n+1}), \dots, R_\alpha^{q_n - 1}(I_{n+1})$ yield a partition of S^1 (a covering of S^1 by nonoverlapping intervals.) In fact, (2.3) implies that the intervals are nonoverlapping. On

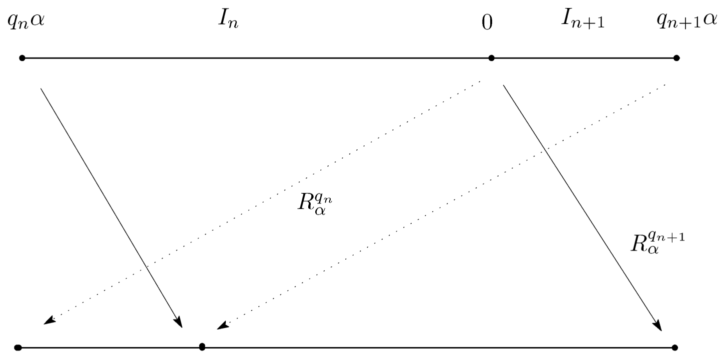

Figure 1.

the other hand, their total length is 1 by the following well-known equality:

$$\begin{aligned} q_{n+1}|I_n| + q_n|I_{n+1}| &= |q_{n+1}(q_n\alpha - p_n) + q_n(p_{n+1} - q_{n+1}\alpha)| \\ &= |-q_{n+1}p_n + q_np_{n+1}| = 1. \end{aligned}$$

Dynamically they form a Rochlin tower as is depicted in Figure 2.

Let us begin the construction of the function T. As we remarked before, T is to be positive and negative valued. We shall construct T as $T = \sum_{n=1}^\infty T_n$, where T_n is a continuous function such that

$$\sum_{n=1}^{\infty} |T_n|_\infty < \infty.$$

To describe T_n, we shall first define a function $\chi_I : S^1 \to \mathbb{R}$ for a interval $I = [r, s]$ of S^1 as follows. See Figure 3. The Lipschitz constant of χ_I is 1.

$$\chi_I(x) = \begin{cases} x - r & \text{if } r \le x \le 3^{-1}(2r+s) \\ 3^{-1}(s-r) & \text{if } 3^{-1}(2r+s) \le x \le 3^{-1}(r+2s) \\ -x+s & \text{if } 3^{-1}(r+2s) \le x \le s \\ 0 & \text{for other } x. \end{cases}$$

For each n, define $j_n = \lfloor 3^{-1}(q_n - 1)\rfloor$ and $j'_n = q_n - 2j_n$, where $\lfloor x\rfloor$ denotes the integer part of a real number x. For n even, define

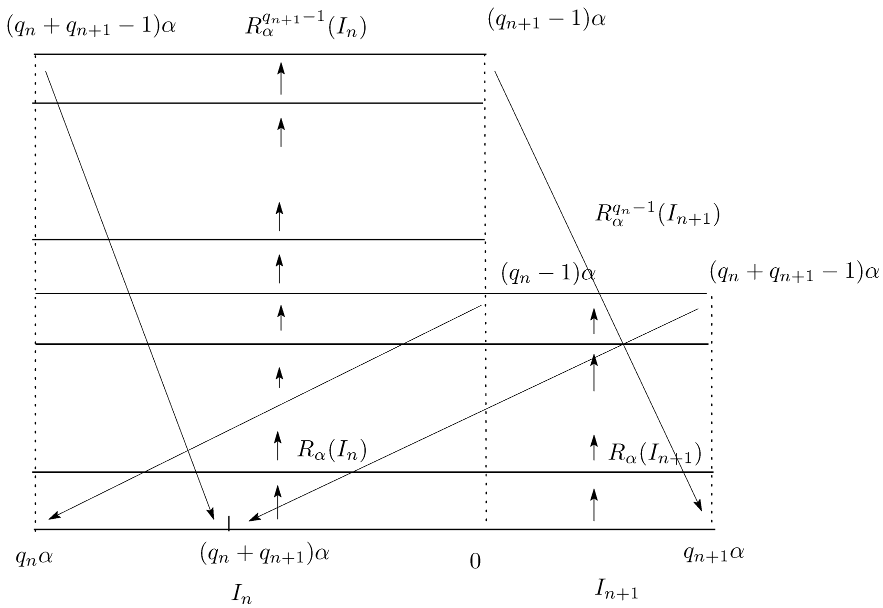

Figure 2.

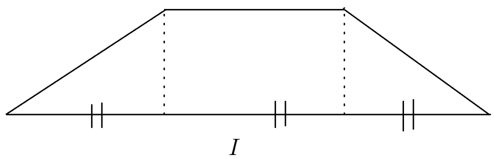

Figure 3. The graph of χ_I. The slope is 0, ± 1.

$$T_n = \sum_{i=0}^{j_{n+1}-1} \chi_{R_\alpha^i(I_n)} - 2^{-1} \sum_{i=j'_{n+1}}^{q_{n+1}-1} \chi_{R_\alpha^i(I_n)}$$

Notice that j'_{n+1} is slightly bigger than j_{n+1}: in fact, $0 \leq j'_{n+1} - j_{n+1} < 4$. The first term of T_n has j_{n+1} summands, while the second $2j_{n+1}$. See Figure 4.

For n odd, we put

$$T_n = -\sum_{i=0}^{j_{n+1}-1} \chi_{R_\alpha^i(I_n)} + 2^{-1} \sum_{i=j'_{n+1}}^{q_{n+1}-1} \chi_{R_\alpha^i(I_n)}.$$

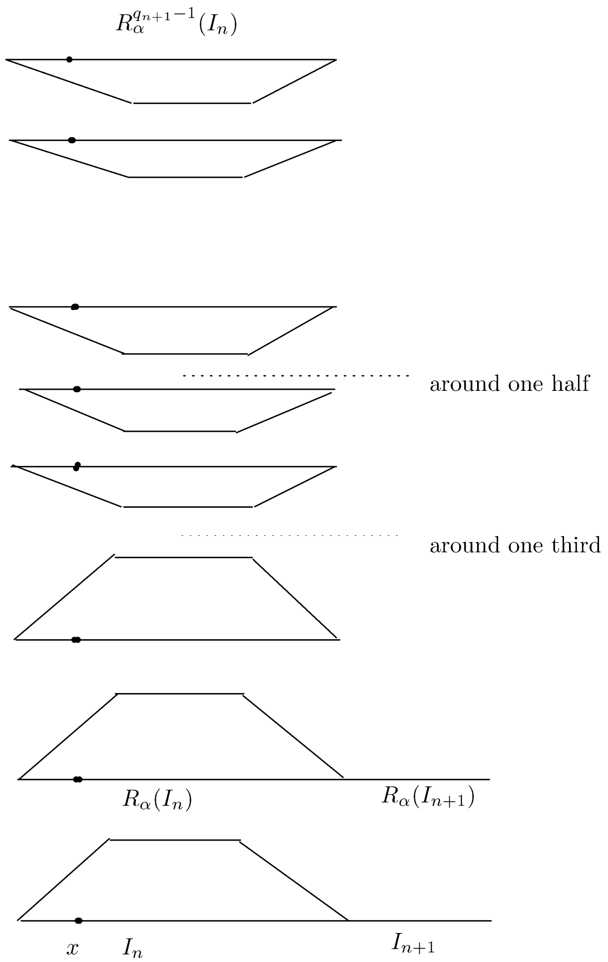

Figure 4. The function T_n. The values at dotted points sum up to 0.

In any case, as is indicated in the figure, we have

$$T_n^{(q_{n+1})}(x) = \sum_{i=0}^{q_{n+1}-1} T_n(R_\alpha^i(x)) = 0 \quad \text{for any} \quad x \in I_n. \tag{2.4}$$

Let us show how the Rochlin tower for the interval $I_{n+1} \cup I_{n+2}$ is obtained from the previous one for $I_n \cup I_{n+1}$. See Figure 5 for n even. On the right side of the figure, the positive orbits of I_{n+1} form a small tower over

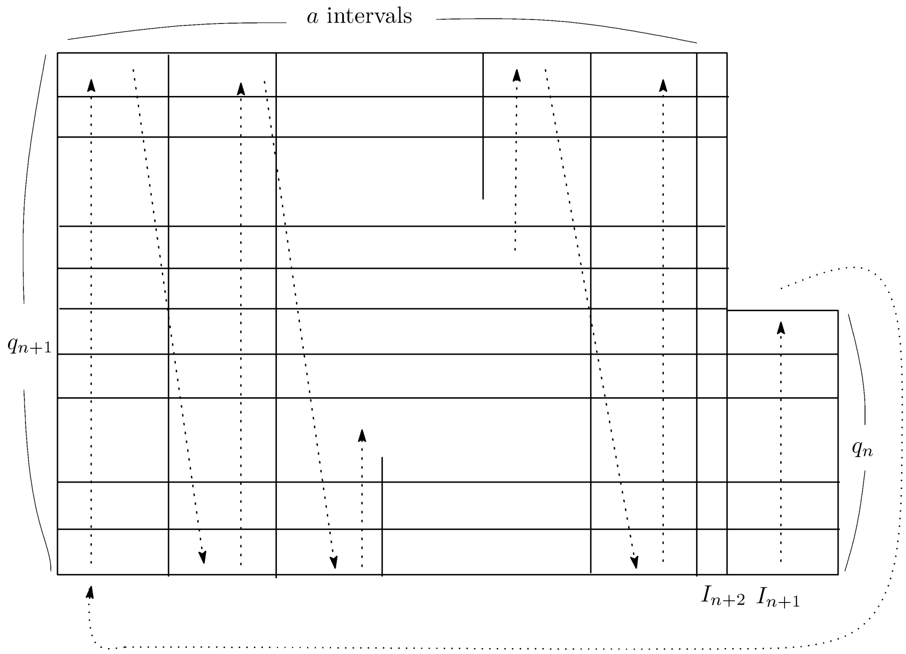

Figure 5. There are $q_{n+2} = aq_{n+1} + q_n$ intervals of size $|I_{n+1}|$.

I_{n+1} of hight q_n. Each is mapped by R_α to one floor upward. The ceiling of the tower is mapped to the leftmost small interval on the ground level. Again its orbit forms a tower, this time of hight q_{n+1}. Its ceiling is mapped to the second left interval on the ground level, and so forth.

To construct the Rochlin tower for $I_{n+1} \cup I_{n+2}$, pile up all the intervals of the size $|I_{n+1}|$ in the figure over the tower on I_{n+1} according to the dynamical order. We shall get a much taller tower over I_{n+1}. The narrow tower over I_{n+2} in the figure is left untouched. The resultant is the new Rochlin tower. Thus the function T_{n+1} looks like Figure 7.

Since $a \geq 10^{10}$, the left rectangle in Figure 2 occupies almost all portion of the circle S^1. In fact the total length of the intervals contained in the left rectangle is $|I_n| \cdot q_{n+1}$, while the right $|I_{n+1}| \cdot q_n$. We have $q_{n+1} = aq_n + q_{n-1} > 10^{10} q_n$ and $|I_n| > 10^{10}|I_{n+1}|$ (Notice the number a in Figure 5). This shows in particular

$$|I_n| = |q_n \alpha| \doteq q_{n+1}^{-1}. \tag{2.5}$$

Proposition 2.3 *We have $\sum_{n=1}^{\infty} |T_n|_\infty < \infty$. The series $\sum_{n=1}^{\infty} T_n$ converges uniformly to a continuous function T.*

Proof. By construction, $|T_n|_\infty \le |I_n| = |q_n\alpha| \doteq q_{n+1}^{-1}$. On the other hand, by (2.2), $q_n \doteq Ac^n$. □

To show that T satisfies the required property, we make use of Lemma 2.1. Given any sufficiently small r, say $0 < r < 10^{-100}$, we only need to compare the value $T^{(i)}(0)$ with $T^{(i)}(x)$ for a suitably chosen x such that $|x| = r$. Let J_n be a subinterval of I_n bounded by $2^{-1}q_n\alpha$ and $-2^{-1}q_{n+1}\alpha$. See Figure 6.

For $|2^{-1}q_{n+1}\alpha| \le r \le |2^{-1}q_n\alpha|$, we choose x such that $|x| = r$ from the interval J_n, and compare the value $T^{(i)}(0)$ with $T^{(i)}(x)$ for $i = \lfloor 2^{-1}q_{n+1} \rfloor$. To do this, we divide T into five terms

$$T = \sum_{\nu=1}^{n-2} T_\nu + T_{n-1} + T_n + T_{n+1} + \sum_{\nu=n+2}^{\infty} T_\nu, \tag{2.6}$$

and for each of these five terms, say S, we estimate the value of $S^{(i)}(x) - S^{(i)}(0)$. Thus in the rest, we assume the following.

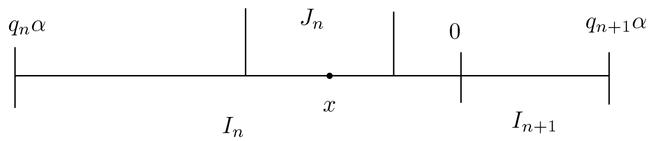

Figure 6.

Assumption 2.4 $x \in J_n$ and $i = \lfloor 2^{-1}q_{n+1} \rfloor$.

First of all, let us study the middle term of (2.6).

Proposition 2.5 *The sign of $T_n^{(i)}(x) - T_n^{(i)}(0)$ alternates according to n, and we have*

$$|T_n^{(i)}(x) - T_n^{(i)}(0)| > 100^{-1}c^{-1},$$

where c is the constant in (2.2).

Proof. To fix the idea, assume n is even. See Figure 4. We are summing up the value of T along one half of the vertical orbit which starts at the bottom line. Clearly it sums up to 0 for the initial value 0: $T_n^{(i)}(0) = 0$. For the initial value $x \in J_n$, the first terms up to hight one third are positive, while the rest nonpositive. We are summing up the value up to one half the hight of the tower of Figure 4, since $i = \lfloor 2^{-1} q_{n+1} \rfloor$. Now $x \in J_n$ implies $|x| \geq 2^{-1}|q_{n+1}\alpha|$. The value of each first one third term is the same and bigger than or equal to $2^{-1}|q_{n+1}\alpha|$. On the other hand, the rest terms are either zero or minus half of this value. Moreover these opposing terms are smaller in number since we are summing up until one half the hight. Therefore we have the following very safe estimate:

$$T_n^{(i)}(x) \geq 50^{-1} q_{n+1}|q_{n+1}\alpha| \doteq 50^{-1} q_{n+1} q_{n+2}^{-1} \doteq 50^{-1} c^{-1}. \qquad \square$$

Proposition 2.6 *We have $T_{n-1}^{(i)}(x) - T_{n-1}^{(i)}(0) = 0$.*

Proof. We are going to show that if $x \in J_n$ and $i = \lfloor 2^{-1} q_{n+1} \rfloor$, then $T_{n-1}^{(i)}(x) - T_{n-1}^{(i)}(0) = 0$. In order to utilize the previous figures, we shift the number by one. So we assume $x \in J_{n+1}$ and $i = \lfloor 2^{-1} q_{n+2} \rfloor$, and show $T_n^{(i)}(x) - T_n^{(i)}(0) = 0$. Thus x as well as 0 lies in I_{n+1} in Figures 5 and 6. We shall compare their orbits up to $\lfloor 2^{-1} q_{n+2} \rfloor$, half of the number of the intervals of size $|I_{n+1}|$ in Figure 5. Their first q_n orbits are above I_{n+1} on that figure. The values of T_n sum up to 0, since T_n vanishes there. Then they come to the leftmost small interval in the bottom line. They climb the tower and from the top falls down to the 2nd leftmost interval. At this moment, the values of T_n of both orbits sum up to 0, by (2.4). They repeat these processes until they come to the middle of I_n. At the last stage, both orbits climb up from some points in the middle part and stop at certain (same) hight. But there the function T_n is flat (Figure 4). Therefore the sums for 0 and x are exactly the same. $\square$

Proposition 2.7 *Let x and i be as in Assumption 2.4. The number $T_{n+1}^{(i)}(x) - T_{n+1}^{(i)}(0)$ has the same sign as $T_n^{(i)}(x) - T_n^{(i)}(0)$.*

Proof. To fix the idea, assume n is even. Then by the construction of T_n, we have

$$T_n^{(i)}(x) - T_n^{(i)}(0) = T_n^{(i)}(x) > 0.$$

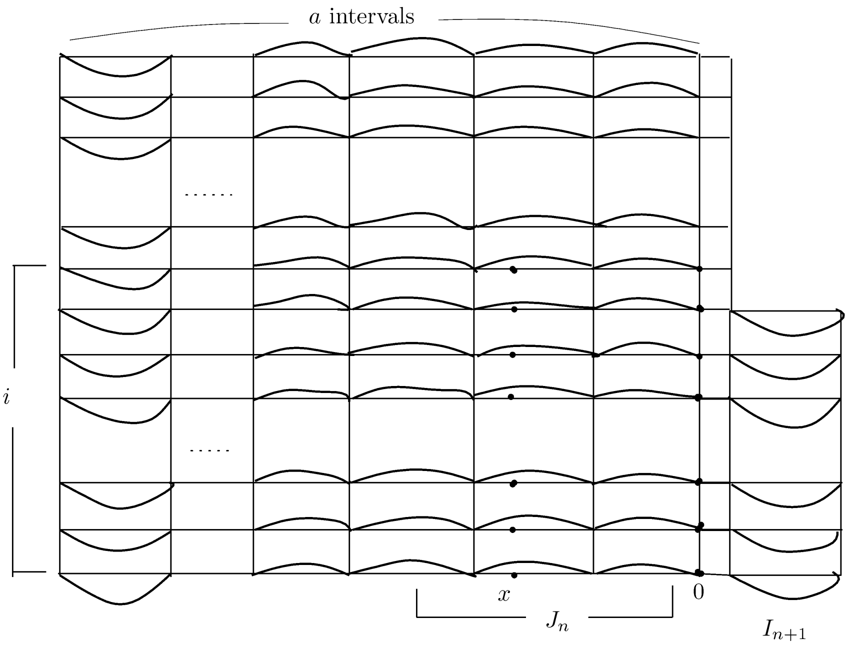

Figure 7. $T^{(i)}_{n+1}(x)$ and $T^{(i)}_{n+1}(0)$ are the sum of the function at the dotted points.

Now the graph of T_{n+1} is indicated in Figure 7. Since $n+1$ is odd, it takes negative value on $I_{n+1}, R_\alpha(I_{n+1}), \ldots$, until at around one third of the way, it changes the sign, to positive. It is clear from the figure that $T^{(i)}_{n+1}(0) = 0$ and $T^{(i)}_{n+1}(x) > 0$. $\square$

Corollary 2.8 *For $S = T_{n-1} + T_n + T_{n+1}$, we have $|S^{(i)}(x) - S^{(i)}(0)| > 100^{-1}c^{-1}$.*

Now let us consider the remaining two terms in (2.6).

Proposition 2.9 *Assume x and i be as in Assumption 2.4. For $S = \sum_{\nu=n+2}^{\infty} T_\nu$, we have $|S^{(i)}(x) - S^{(i)}(0)| < 4c^{-2}$.*

Proof. Recall by (2.2) and (2.5) that

$$|T_\nu|_\infty \leq |q_\nu \alpha| \doteq q_{\nu+1}^{-1} \doteq A^{-1}c^{-\nu-1}.$$

This shows that

$$|S|_\infty \leq 2A^{-1}c^{-n-3}.$$

Since $i = \lfloor 2^{-1}q_{n+1} \rfloor \doteq 2^{-1}Ac^{n+1}$, we have $|S^{(i)}|_\infty \leq 2c^{-2}$, showing the proposition. □

Proposition 2.10 *Assume x and i be as in Assumption* 2.4. *For $S = \sum_{\nu=1}^{n-2} T_\nu$, we have $|S^{(i)}(x) - S^{(i)}(0)| < 5c^{-2}$.*

Proof. For each $\nu \leq n-2$, the points $0, x \in J_n$ lie on the interval $I_\nu \cup I_{\nu+1}$. For a point y of $I_\nu \cup I_{\nu+1}$, the sum $T_\nu^{(k)}(y) = 0$ whenever $R_\alpha^k(y)$ is contained in $I_\nu \cup I_{\nu+1}$, by (2.4). Let $\mathcal{J}$ be the interval $I_\nu \cup I_{\nu+1}$ with the $2^{-1}|q_n\alpha|$-neighbourhoods of the two boundary points removed. Then if $R_\alpha^k(0)$ is contained in $\mathcal{J}$, $R_\alpha^k(x)$ is contained in $I_\nu \cup I_{\nu+1}$, since

$$|R_\alpha^k(x) - R_\alpha^k(0)| = |x| \leq 2^{-1}|q_n\alpha|.$$

In that case, we have $T_\nu^{(k)}(x) = T_\nu^{(k)}(0) = 0$.

Now it is easy to show that the first return time of R_α for $\mathcal{J}$ is at most $2q_{\nu+1}$. Let k be the largest integer in $\{1, 2, \ldots, i\}$ such that $R_\alpha^k(0) \in \mathcal{J}$, and let $l = i - k$. Then

$$\begin{aligned} T_\nu^{(i)}(x) - T_\nu^{(i)}(0) &= (T_\nu^{(k)}(x) - T_\nu^{(k)}(0)) + (T_\nu^{(l)}(R_\alpha^k(x)) - T_\nu^{(l)}(R_\alpha^k(0))) \\ &= T_\nu^{(l)}(R_\alpha^k(x)) - T_\nu^{(l)}(R_\alpha^k(0))), \end{aligned}$$

where $l \leq 2q_{\nu+1}$. Since the Lipshitz constant of T_ν is 1, we have

$$|T_\nu^{(i)}(x) - T_\nu^{(i)}(0)| \leq l|x| \leq 2q_{\nu+1}|q_{n+1}\alpha| \leq 4c^{-n+\nu}.$$

Summing up for $1 \leq \nu \leq n-2$, we get Proposition 2.10. □

End of the proof of Theorem 1.2. For x and i as in Assumption 2.4, we have shown that

$$|T^{(i)}(x) - T^{(i)}(0)| > 100^{-1}c^{-1} - 4c^{-2} - 5c^{-2} > 200^{-1}c^{-1},$$

where the last inequality follows from $c > 10^{10}$. Now we have shown that the assumption of Lemma 2.1 is met for $\delta = 3^{-1}\min\{10^{-100}, 200^{-1}c^{-1}\}$. This shows Theorem 1.2. In fact, given $\epsilon > 0$. one can choose the $\delta(\epsilon)$ in the definition of the kinematic expansiveness as $\delta(\epsilon) = 3^{-1}\min\{\epsilon, 10^{-100}, 200^{-1}c^{-1}\}$.

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