On the class of projective surfaces of general type

Yoshiaki FUKUMA and Kazuhisa ITO

Abstract. Let S be a smooth complex projective surface of general type, H be a very ample divisor on S and $m(S, H)$ be the class of (S, H). In this paper, we study a lower bound for $m(S, H) - 3H^2$ and we improve an inequality obtained by Lanteri. We also study (S, H) with each value of $m(S, H) - 3H^2$ and exhibit some examples.

Key words: Class, surfaces of general type, very ample divisor, sectional genus, Δ-genus.

1. Introduction

Let S be a smooth complex projective surface, H be a very ample divisor on S, and $m(S, H)$ be its class, i.e. the degree of the dual variety of S (embedded via H). Then some relations between $m(S, H)$ and H^2 have been studied by many authors (for example, [7], [8], [12], [13] and [16]). For example, the following facts are known:

(ⅰ) $m(S, H) \geq H^2 - 1$ holds. Moreover this equality holds if and only if (S, H) is isomorphic to either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ or $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

(ⅱ) $m(S, H) = H^2$ holds if and only if (S, H) is a scroll over a smooth projective curve.

(ⅲ) If $m(S, H) \leq 3H^2 + 2$, then one of the following is satisfied.
    (a) S is ruled.
    (b) $m(S, H) = 3H^2$ and S is a minimal hyperelliptic or abelian surface.

(ⅳ) If S is a smooth elliptic surface with $\kappa(S) = 1$, then $m(S, H) \geq 3H^2 + 6$ holds.

(ⅴ) If S is of general type, then $m(S, H) \geq 3H^2 + 17$.

In this paper we consider the case when S is of general type. We improve the inequality in (ⅴ). Furthermore we study the structure of surfaces for small values of $m(S, H) - 3H^2$ and exhibit some examples. In Section 4, we study (S, H) with $m(S, H) - 3H^2 \leq 24$ first of all. Here we note that

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by using the method in this paper we can also get possible (S, H) with $m(S, H) - 3H^2 \geq 25$.

2.   Preliminaries

In this paper, we work over the field of complex numbers $\mathbb{C}$. We use the customary notation in algebraic geometry. The words "line bundles" and "(Cartier) divisors" are used interchangeably. For a smooth projective surface S and a very ample divisor H on S, let $g(S, H)$ be the sectional genus of (S, H), K_S be the canonical divisor of S, $m(S, H)$ be the class of (S, H), and $\chi(S)$ be the topological Euler characteristic. Let $q(S)$ be the irregularity of S and $p_g(S)$ be the geometric genus of S.

It is known that these invariants satisfy the following (see [12, (1.3)]):

$$\chi(S) = 4(1 - g(S, H)) + m(S, H) - H^2.$$

By using the genus formula and Noether's formula, we also have

$$
\begin{aligned}
m(S, H) &= 3H^2 + \chi(S) + 2K_S H \\
&= 3H^2 + 12\chi(\mathcal{O}_S) - K_S^2 + 2K_S H.
\end{aligned}
\tag{2.1}
$$

In this paper, we will only consider the case when S is of general type. Then we have the well-known inequalities for S, namely, $\chi(\mathcal{O}_S) \geq 1$ (Castelnuovo's inequality [1, X.4]) and $K_S^2 \leq 9\chi(\mathcal{O}_S)$ (Miyaoka-Yau inequality [18], [21]). By using these inequalities and Noether's formula, we have

$$\chi(S) = 12\chi(\mathcal{O}_S) - K_S^2 \geq 3\chi(\mathcal{O}_S) \geq 3. \tag{2.2}$$

Furthermore we see from [20] that

$$H^2 \geq 5. \tag{2.3}$$

Lemma 2.1   *Let H be a very ample divisor on a smooth projective surface S of general type. Then $K_S H \geq 3$.*

Proof.   See [12, (1.9)].   □

Lemma 2.2 *Let H be an ample divisor on a smooth projective surface S of general type. If $K_S H \leq 2$, then S is minimal.*

Proof. Suppose that S is not minimal. Let $\pi : S \to \tilde{S}$ be the minimalization of S. Set $\tilde{H} = \pi_*(H)$. Then $\tilde{H}$ is ample on $\tilde{S}$. So by the assumption we have $2 \geq K_S H > K_{\tilde{S}}\tilde{H} \geq 1$. Therefore $K_{\tilde{S}}\tilde{H} = 1$, and hence π is the blow-up at a point on $\tilde{S}$. So letting E be the exceptional curve, we have

$$H = \pi^*(\tilde{H}) - E, \quad K_S = \pi^*(K_{\tilde{S}}) + E.$$

Since $\tilde{H}^2 = 1$ by the Hodge index theorem, this implies $H^2 = \tilde{H}^2 + E^2 = 0$. But this contradicts the assumption that H is ample. $\square$

Lemma 2.3 *Let S be a smooth projective surface of general type and H be a very ample line bundle on S. If $\chi(\mathcal{O}_S) \leq 4$, then $h^0(H) \geq 5$.*

Proof. Since S is of general type, we have $h^0(H) \geq 4$. Moreover $H^2 \geq 5$ by (2.3). If $h^0(H) = 4$, then S becomes a hypersurface in $\mathbb{P}^3$ via the morphism defined by $|H|$. But then $q(S) = 0$ and $p_g(S) = \binom{H^2-1}{3} \geq 4$. Hence $\chi(\mathcal{O}_S) \geq 5$ and this contradicts the assumption that $\chi(\mathcal{O}_S) \leq 4$. So we get the assertion. $\square$

The Δ-genus We will use the Δ-genus theory in the following sections. Let (X, L) be a polarized variety, which is a pair consisting of a projective variety X of dimension n, and an ample line bundle L on it. Then the Δ-genus of (X, L) is defined to be $\Delta(X, L) = n + L^n - h^0(L)$. We will use the following theorem ([6, Chapter I, (3.5)]):

Theorem 2.1 *Let (S, H) be a polarized surface. Assume that (S, H) has a ladder and $g(S, H) \geq \Delta(S, H)$. If $H^2 \geq 2\Delta(S, H) + 1$, then $g(S, H) = \Delta(S, H)$.*

Remark 2.1 *If S is smooth and H is very ample, then (S, H) always has a ladder ([6]).*

We also use the following theorem, which is called the *double point formula*.

Theorem 2.2 *Let L be a very ample line bundle on a smooth connected projective surface S. Let $d = L^2$. Assume that $\Gamma(L)$ embeds S in $\mathbb{P}^N$ with $N \geq 4$. Then*

$$d(d-5) - 10(g(S,L) - 1) + 12\chi(\mathcal{O}_S) \geq 2K_S^2$$

with equality if $N = 4$.

Proof. See [2, Lemma 8.2.1]. □

Definition 2.1 A smooth projective surface S is called a *numerical Godeaux surface* (resp. *numerical Campedelli surface*) if S is a minimal surface of general type with $q(S) = p_g(S) = 0$ and $K_S^2 = 1$ (resp. $K_S^2 = 2$).

3. Inequality for the class of surfaces of general type

In this section we are going to prove the following.

Theorem 3.1 *Let S be a smooth projective surface of general type and H be a very ample divisor. Then the following hold.*

(i) *If $m(S,H) - 3H^2$ is odd, then $m(S,H) - 3H^2 \geq 19$.*
(ii) *If $m(S,H) - 3H^2$ is even, then $m(S,H) - 3H^2 \geq 22$.*

Proof. From (2.1) we have $m(S,H) - 3H^2 = \chi(S) + 2K_S H$. It is known that $m(S,H) - 3H^2 \geq 17$ ([12, (1.10)]).

(a) First assume that $m(S,H) - 3H^2 = 17$. Then $\chi(S) + 2K_S H = 17$. Since $\kappa(S) = 2$, we have $3 \leq \chi(S) \leq 11$ and $3 \leq K_S H \leq 7$ by (2.2) and Lemma 2.1.

If $K_S H = 3$, then $\chi(S) = 11$ and $1 \leq \chi(\mathcal{O}_S) \leq 3$ by (2.2) again. If $\chi(\mathcal{O}_S) = 1$, then $K_S^2 = 1$ by Noether's formula. So we have $5 \leq H^2 \leq 9$ from (2.3) and the Hodge index theorem for H and K_S. Since $K_S H + H^2$ is even by genus formula, we get $H^2 = 5, 7$ or 9. Similarly if $\chi(\mathcal{O}_S) = 2$ or 3, then $K_S^2 = 13$ or 25 respectively. But in either cases, it is impossible by the Hodge index theorem because $H^2 \geq 5$.

In this way, we can also study the case of $4 \leq K_S H \leq 7$, and we get only the following possibilities:

$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
3	11	1	1	5, 7, 9
5	7	1	5	5
7	3	1	9	5

(a.1) Consider the case $H^2 = 5$. Then by Lemma 2.3 we get $h^0(H) \geq 5$, and $\Delta(S, H) = 7 - h^0(H) \leq 2$. Since $g(S, H) \geq 5$ for the case $H^2 = 5$ in the above table, we have $g(S, H) > \Delta(S, H)$ and $H^2 \geq 2\Delta(S, H) + 1$. This implies $g(S, H) = \Delta(S, H)$ by Theorem 2.1, but it's a contradiction.

(a.2) For the case $H^2 = 7$, we have $h^0(H) \geq 5$ by Lemma 2.3. By the same argument as in (a.1), the case of $h^0(H) \geq 6$ cannot occur. So it remains to consider the case when $h^0(H) = 5$. Since $K_S H = 3$, we can show that S is minimal. Actually assume that S is not minimal. Let $\pi : S \to \tilde{S}$ be the blow-down of a (-1)-curve E and let $\tilde{H} = \pi_*(H)$. Then

$$H = \pi^*(\tilde{H}) - aE, \quad K_S = \pi^*(K_{\tilde{S}}) + E$$

for some $a \geq 1$. In particular, we have $K_{\tilde{S}}\tilde{H} \leq 2$, $(\tilde{H})^2 \geq 8$ and $K_{\tilde{S}}^2 = 2$. But this is impossible by the Hodge index theorem. So we see that S is a minimal surface of degree 7 in $\mathbb{P}^4$. For such surfaces, there is a classification (see [9]) and we see that this case also cannot occur.

(a.3) Finally we consider the case $H^2 = 9$. Then $(K_S H)^2 = K_S^2 H^2$, so by the Hodge index theorem, H is numerically equivalent to some multiples of K_S. Clearly $H \equiv_{num} 3K_S$. Thus K_S is ample. It follows that $h^1(H) = h^1(K_S - H) = 0$ and $h^2(H) = h^0(K_S - H) = 0$ by the Kodaira vanishing theorem. Hence $\chi(H) = h^0(H)$. On the other hand, the Riemann-Roch theorem gives $\chi(H) = 1 + (1/2)(9 - 3) = 4$. We conclude that $h^0(H) = 4$. But this case cannot occur by Lemma 2.3 since $\chi(\mathcal{O}_S) = 1$.

(b) Assume that $m(S, H) - 3H^2 = 18$. By the same argument as in (a) above, we get the following six possibilities:

	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
(b.1)	3	12	1	0	$2k + 5$ with $k \geq 0$
(b.2)	4	10	1	2	6
(b.3)	4	10	1	2	8
(b.4)	5	8	1	4	5
(b.5)	6	6	1	6	6
(b.6)	7	4	1	8	5

The case (b.1): Since $K_{\tilde{S}}^2 = 0$, S is not a minimal. Let $\pi : S \to \tilde{S}$ be the blow-down of a (-1)-curve E, and set $\tilde{H} = \pi_*(H)$. Then $K_{\tilde{S}}\tilde{H} \le 2$, $\tilde{H}^2 \ge 6$ and $K_{\tilde{S}}^2 = 1$. But it contradicts the Hodge index theorem for $\tilde{H}$ and $K_{\tilde{S}}$.

The case (b.4) or (b.6): Then $H^2 = 5$. The same argument as in the case (a.1) above shows that there are no surfaces satisfying each conditions in the table.

The case (b.2) or (b.5): Then $H^2 = 6$ and $g(S,H) = 6$ or 7. Suppose that $h^0(H) \ge 6$. Then $\Delta(S,H) \le 2$, and hence $H^2 > 2\Delta(S,H) + 1$ and $g(S,H) > \Delta(S,H)$. This implies $g(S,H) = \Delta(S,H)$ by Theorem 2.1, which is a contradiction. Thus $h^0(H) = 5$, so that S is a surface of degree 6 in $\mathbb{P}^4$. But there are also no surfaces of general type by the classification (see [9]). Hence the case of $h^0(H) = 5$ also cannot occur.

The case (b.3): Using the Hodge index theorem, we have $H \equiv_{num} 2K_S$. Especially K_S is ample. It follows that $h^1(H) = h^1(K_S - H) = 0$ and $h^2(H) = h^0(K_S - H) = 0$ by the Kodaira vanishing theorem. Hence $\chi(H) = h^0(H)$. On the other hand, the Riemann-Roch theorem gives $\chi(H) = 3$. We conclude that $h^0(H) = 3$, but this is impossible because $\kappa(S) = 2$.

Therefore we see that $m(S,H) - 3H^2 \ge 19$ and we get the assertion of (i).

(c) Assume that $m(S,H) - 3H^2$ is even. Then by the above argument we see that $m(S,H) - 3H^2 \ge 20$. We consider the case when $m(S,H) - 3H^2 = 20$. In this case we get only the following possibilities:

	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
(c.1)	3	14	1	-2	$2k + 5$ with $k \ge 0$
(c.2)	4	12	1	0	$2k + 6$ with $k \ge 0$
(c.3)	5	10	1	2	5, 7, 9, 11
(c.4)	6	8	1	4	6, 8
(c.5)	7	6	1	6	5, 7
(c.6)	8	4	1	8	6, 8

The case (c.1): Clearly S is not minimal. Let $\pi : S \to \tilde{S}$ be the blow-down of a (-1)-curve E and let $\tilde{H} = \pi_*(H)$. Then $3 = K_S H > K_{\tilde{S}}\tilde{H}$. Moreover since $\tilde{H}$ is ample, $\tilde{S}$ is minimal by Lemma 2.2. But since $K_{\tilde{S}}^2 = -1$, this is impossible.

The case (c.2): Clearly S is not minimal. Let $\pi : S \to \tilde{S}$ be the blow-down of a (-1)-curve E, and set $\tilde{H} = \pi_*(H)$. Then we have

$$(K_{\tilde{S}}\tilde{H})^2 \le 9, \quad \tilde{H}^2 \ge H^2 + 1 \;\; \text{and} \;\; K_{\tilde{S}}^2 = 1.$$

By the Hodge index theorem for $\tilde{H}$ and $K_{\tilde{S}}$, we see that $H^2 = 6$ or 8. Consider the case $H^2 = 6$. Then $g(S, H) = 6$. We see from Lemma 2.3 that $h^0(H) \ge 5$ since $\chi(\mathcal{O}_S) = 1$. Now we check whether this satisfies the double point formula (see Theorem 2.2). We have

$$H^2(H^2 - 5) - 10(g(S, H) - 1) + 12\chi(\mathcal{O}_S) - 2K_S^2$$
$$= 6 \cdot (6 - 5) - 10 \cdot (6 - 1) + 12 \cdot 1 - 2 \cdot 0 = -32 < 0.$$

Hence this case cannot occur. Similarly the case $H^2 = 8$ does not satisfy the double point formula. Hence (c.2) is impossible.

Moreover we see from Theorem 2.2 that all of the cases in the above table cannot occur. Therefore there are no surfaces (S, H) with $m(S, H) - 3H^2 = 20$. This completes the proof. $\qquad\square$

4. Polarized surfaces (S, H) with $19 \le m(S, H) - 3H^2 \le 24$

Let S be a smooth projective surface of general type and H be a very ample line bundle on S. In this section, we study (S, H) with $19 \le m(S, H) - 3H^2 \le 24$.

4.1. The case of $m(S, H) - 3H^2 = 19$

By the same arguments as in the proof of Theorem 3.1, we first get the following possibilities:

	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
(d.1)	3	13	1	-1	$2k+5$ with $k \geq 0$
(d.2)	4	11	1	1	6
(d.3)	4	11	1	1	8
(d.4)	4	11	1	1	10
(d.5)	4	11	1	1	12
(d.6)	4	11	1	1	14
(d.7)	4	11	1	1	16
(d.8)	5	9	1	3	5
(d.9)	5	9	1	3	7
(d.10)	6	7	1	5	6
(d.11)	7	5	1	7	5
(d.12)	7	5	1	7	7
(d.13)	8	3	1	9	6

The case (d.1): This is impossible by the same argument as (c.1) in the proof of Theorem 3.1.

The case (d.2): We see from Lemma 2.3 that $h^0(H) \geq 5$ since $\chi(\mathcal{O}_S) = 1$. We check whether this case satisfies the double point formula. We have

$$H^2(H^2 - 5) - 10(g(S, H) - 1) + 12\chi(\mathcal{O}_S) - 2K_S^2$$
$$= 6 \cdot (6 - 5) - 10 \cdot (6 - 1) + 12 \cdot 1 - 2 \cdot 1 = -34 < 0.$$

Hence this case cannot occur.

Similarly all of the case except for (d.5), (d.6) and (d.7) are impossible.

For the cases (d.5), (d.6) and (d.7), we see that S is minimal by the Hodge index theorem. If $q(S) > 0$, then $K_S^2 \geq 2p_g(S) \geq 2q(S) \geq 2$ holds by [4, Théorème 6.1]. Hence $K_S^2 = 1$ implies $q(S) = 0$. Therefore S is a numerical Godeaux surface because $\chi(\mathcal{O}_S) = 1$.

4.2. The case of $m(S, H) - 3H^2 = 21$

The same argument as in 4.1 above shows that there are the following possibilities:

	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
(e.1)	3	15	1	-3	$2k + 5$ with $k \geq 0$
(e.2)	4	13	1	-1	$2k + 6$ with $k \geq 0$
(e.3)	5	11	1	1	$2k + 13$ with $0 \leq k \leq 6$
(e.4)	6	9	1	3	12

The case (e.1): As in (c.1), this is impossible.

The case (e.2): Since $K_S^2 = -1$, S can be blown down at least twice. But then the Hodge index theorem shows that this case cannot occur.

The case (e.3): We see that S is a numerical Godeaux surface.

The case (e.4): We have $H \equiv_{num} 2K_S$. Then $h^0(H) = \chi(H) = 4$, so that S is in $\mathbb{P}^3$. But this is impossible by Lemma 2.3 since $\chi(\mathcal{O}_S) = 1$.

4.3. The case of $m(S, H) - 3H^2 = 22$

We get the following possibilities:

	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
(f.1)	3	16	1	-4	$2k + 5$ with $k \geq 0$
(f.2)	4	14	1	-2	$2k + 6$ with $k \geq 0$
(f.3)	5	12	1	0	$2k + 5$ with $k \geq 0$
(f.4)	6	10	1	2	12, 14, 16, 18

The case (f.1): As in (c.1), this case cannot occur.

The case (f.2): In view of Lemma 2.2, S can be blown down at most twice. But then the self-intersection number of the canonical divisor on the blow-down surface is not positive. Hence this case is impossible.

The case (f.3): Using (2.3), the Hodge index theorem and Theorem 2.2, we get $H^2 = 13$ or 15 by the same argument as in (c.2). We investigate the structure of S. Clearly S is not minimal. Let $\pi : S \to \tilde{S}$ be the blow-down of a (-1)-curve E and put $\tilde{H} = \pi_*(H)$. We see that $\tilde{S}$ is minimal by the Hodge index theorem. Since $\chi(\mathcal{O}_{\tilde{S}}) = 1$ and $K_{\tilde{S}}^2 = 1$, we get $q(\tilde{S}) = p_g(\tilde{S}) = 0$. Therefore $\tilde{S}$ is a numerical Godeaux surface.

The case (f.4): We see that S is minimal by the Hodge index theorem. Moreover, if $q(S) > 0$, then $K_S^2 \geq 2p_g(S) \geq 2q(S)$ holds by [4, Théorème

6.1]. So we get $p_g(S) = 1$ and $q(S) = 1$ if $q(S) > 0$. Hence S is a numerical Campedelli surface or minimal surface with $q(S) = p_g(S) = 1$.

4.4.   The case of $m(S, H) - 3H^2 = 23$

We get the following possibilities:

	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
(g.1)	3	17	1	-5	$2k + 5$ with $k \geq 0$
(g.2)	4	15	1	-3	$2k + 6$ with $k \geq 0$
(g.3)	5	13	1	-1	$2k + 5$ with $k \geq 0$
(g.4)	6	11	1	1	$2k + 12$ with $0 \leq k \leq 12$
(g.5)	7	9	1	3	13 or 15

The case (g.1) (resp. (g.2)): As in (c.1) (resp. (f.2)), this is impossible.

The case (g.3): In this case, S can be blown down twice. Then we have $H^2 = 5$ or 7 by the Hodge index theorem. But both cases are impossible by Theorem 2.2.

The case (g.4): We see that S is a numerical Godeaux surface.

The case (g.5): S is minimal by the Hodge index theorem.

4.5.   The case of $m(S, H) - 3H^2 = 24$

We get the following possibilities:

	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2
(h.1)	3	18	1	-6	$2k + 5$ with $k \geq 0$
(h.2)	4	16	1	-4	$2k + 6$ with $k \geq 0$
(h.3)	5	14	1	-2	$2k + 5$ with $k \geq 0$
(h.4)	6	12	1	0	$2k + 6$ with $k \geq 0$
(h.5)	7	10	1	2	$2k + 13$ with $0 \leq k \leq 5$
(h.6)	8	8	1	4	14 or 16

The case (h.1) (resp. (h.2) and (h.3)): As in (c.1) (resp. (f.2) and (e.2)), this is impossible.

The case (h.4): As in (c.2), we get $H^2 = 2k + 12$ with $0 \leq k \leq 6$. The structure of S is the same as (f.3).

The case (h.5): We see that S is a numerical Campedelli surface or minimal surface with $q(S) = p_g(S) = 1$ as in (f.4).

The case (h.6): Consider the case of $H^2 = 16$. Then we have $H \equiv_{num} 2K_S$. It follows that $h^0(H) = \chi(H) = 5$, so that S is in $\mathbb{P}^4$. But this is impossible by Theorem 2.2.

For the case of $H^2 = 14$, we see that S is minimal.

4.6. Summary

From the above results, we get the following (Table).

$m(S,H) - 3H^2$	$K_S H$	$\chi(S)$	$\chi(\mathcal{O}_S)$	K_S^2	H^2	structure of S
19	4	11	1	1	12, 14, 16	numerical Godeaux surface
21	5	11	1	1	$2k + 13$ $(0 \le k \le 6)$	numerical Godeaux surface
22	5	12	1	0	13, 15	blow-up of a numerical Godeaux surface at a point
	6	10	1	2	$2k + 12$ $(0 \le k \le 3)$	numerical Campedelli surface or minimal surface with $q(S) = p_g(S) = 1$
23	6	11	1	1	$2k + 12$ $(0 \le k \le 12)$	numerical Godeaux surface
	7	9	1	3	13, 15	minimal surface
24	6	12	1	0	$2k + 12$ $(0 \le k \le 6)$	blow-up of a numerical Godeaux surface at a point
	7	10	1	2	$2k + 13$ $(0 \le k \le 5)$	numerical Campedelli surface or minimal surface with $q(S) = p_g(S) = 1$
	8	8	1	4	14	minimal surface

5.　Some examples

5.1.　Examples of the case where $m(S,H) - 3H^2$ is odd

First we will give some examples of (S,H) with odd $m(S,H) - 3H^2$.

(5.1.1)　Let S be a numerical Godeaux surface which is torsion free and has an ample canonical bundle (see e.g. [5], [14, Example 7.4]). Then we can show that $4K_S$ is very ample by Reider's theorem ([19]). In fact, if $4K_S$ is not very ample, then there exists an effective divisor Z on S such that $Z^2 = 1$ and $3K_S \equiv_{num} 3Z$. Then $K_S \equiv_{num} Z$ and since S is torsion free, we have $K_S = Z$, which is a contradiction because $p_g(S) = 0$. In this case we get

$$m(S, 4K_S) - 3 \cdot (4K_S)^2 = 12\chi(\mathcal{O}_S) - K_S^2 + 2K_S(4K_S) = 19.$$

(5.1.2) Let S be a numerical Godeaux surface with K_S ample (e.g. the Godeaux surface [1, Examples X.3 (4)]). It is known that nK_S is very ample for any $n \geq 5$ ([3]). So we have

$$m(S, nK_S) - 3 \cdot (nK_S)^2 = 11 + 2n \geq 21.$$

5.2.　Examples of the case where $m(S,H) - 3H^2$ is even

Next we shall show that there exists an example of (S,H) such that $m(S,H) - 3H^2$ is even. First we prove the following proposition.

Proposition 5.1　*Let M be a numerical Godeaux surface with K_M ample. Then there exists the blow-up $\pi : S \to M$ at a point with (-1)-curve E such that $H := \pi^*(mK_M) - E$ is very ample for $m \geq 6$. Moreover if M is torsion free, then $\pi^*(5K_M) - E$ is also very ample.*

Proof.　First we see from [17, Lemma 6] that $|2K_M|$ has no fixed part and has an irreducible and reduced member. We also note that $(2K_M)^2 > 2$ and $g(M, 2K_M) = 4 > 0 = q(M)$. Hence by [22, (1.8)], there is the blow-up $\pi : S \to M$ at a point on M with (-1)-curve E such that $2\pi^*(K_M) - E$ is ample. Since K_M is ample, $\pi^*(K_M)$ is nef and big. Here we set $H := \pi^*(mK_M) - E$. Then we have

$$H = \pi^*(mK_M) - E$$
$$= \pi^*(K_M) + E + (m-1)\pi^*(K_M) - 2E$$
$$= K_S + (m-1)\pi^*(K_M) - 2E.$$

Let $D = (m-1)\pi^*(K_M) - 2E$. Then $D^2 > 9$ for every $m \geq 5$ and we have

$$D = (m-1)\pi^*(K_M) - 2E = (m-5)\pi^*(K_M) + 2(2\pi^*(K_M) - E).$$

Therefore D is ample for $m \geq 5$.

(a) First assume that $m \geq 6$. For any effective divisor Z on S, we have

$$DZ = (m-5)\pi^*(K_M)Z + 2(2\pi^*(K_M) - E)Z \geq 2.$$

If H is not very ample, then there exists an effective divisor Z such that $DZ = 2$ and $Z^2 = 0$ by Reider's theorem. Then $K_M \pi_*(Z) = \pi^*(K_M)Z = 0$ and $(2\pi^*(K_M) - E)Z = 1$. It follows that $\pi_*(Z) = 0$ and Z is irreducible and reduced. This implies that Z is (-1)-curve, but $Z^2 = 0$ and this is a contradiction. So we conclude that $H = \pi^*(mK_M) - E$ is very ample for $m \geq 6$.

(b) Finally we consider the case $m = 5$. Then $D = 2(2\pi^*(K_M) - E)$. For any effective divisor Z on S, we have $DZ \geq 2$. Suppose that there exists an effective divisor Z on S such that $DZ = 2$ and $Z^2 = 0$. Then $(2\pi^*(K_M) - E)Z = 1$ and hence Z is irreducible and reduced but not (-1)-curve. This implies $\pi^*(K_M)Z = K_M \pi_*(Z) > 0$. Since $2\pi^*(K_M)Z = 1 + EZ$, we have $EZ \geq 1$ and we can write $Z = \pi^*(\pi_*(Z)) - aE$ with $a \geq 1$. Thus $EZ = a$ and $2\pi^*(K_M)Z = 1 + EZ = 1 + a$. It follows that

$$K_M \pi_*(Z) = \pi^*(K_M)Z = \frac{1+a}{2}, \quad \pi_*(Z)^2 = Z^2 + a^2 = a^2.$$

By using the Hodge index theorem for K_M and $\pi_*(Z)$, we have $((1+a)/2)^2 \geq 1 \cdot a^2$. This implies $a = 1$ and therefore $K_M \equiv_{num} \pi_*(Z)$.

By the assumption that M is torsion free, $K_M = \pi_*(Z)$ holds, but this contradicts the assumption that $p_g(M) = 0$.

Thus $H = \pi^*(5K_M) - E$ is very ample by Reider's theorem. $\qquad \square$

(5.2.1)   By [10] there exists an example of a minimal surface S of general type such that $q(S) = p_g(S) = 0$, $\pi_1^{\mathrm{alg}}(S) \cong \mathbb{Z}/2\mathbb{Z}$, $K_S^2 = 2$ and K_S is ample. So we see from [11, Theorem 0.4] that $3K_S$ is very ample. Then we get

$$m(S, 3K_S) - 3 \cdot (3K_S)^2 = 22.$$

(5.2.2)   Let S be a numerical Campedelli surface with K_S ample. It is known that such a surface exists (see e.g. [10], [15, Proposition 4.1]). Then nK_S is very ample if $n \geq 4$ (see [11, Corollary 0.1]). Since $\chi(\mathcal{O}_S) = 1$ and $K_S^2 = 2$, we have

$$m(S, nK_S) - 3 \cdot (nK_S)^2 = 10 + 4n \geq 26.$$

(5.2.3)   Again let M be the numerical Godeaux surface as (5.1.1). Then there exists the blow-up $\pi : S \to M$ at a point of M such that $H = \pi^*(mK_M) - E$ is very ample for $m \geq 5$ by Proposition 5.1, where E is the (-1)-curve. Since $\chi(\mathcal{O}_S) = 1$, $K_S^2 = 0$ and $K_S H = mK_M^2 + 1 = m + 1$, we get

$$m(S, H) - 3H^2 = 12 + 2(m + 1) \geq 24.$$

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Yoshiaki FUKUMA
Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@kochi-u.ac.jp

Kazuhisa ITO
Studies in Science
Graduate School of Integrated Arts and Sciences
Kochi University
Akebono-cho, Kochi 780-8520
Japan