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The extended zero-divisor graph of a commutative ring II

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Abstract. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The extended zero-divisor graph of R is the undirected (simple) graph $\Gamma'(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if either $Rx \cap \mathrm{Ann}(y) \neq (0)$ or $Ry \cap \mathrm{Ann}(x) \neq (0)$. In this paper, we continue our study of the extended zero-divisor graph of a commutative ring that was introduced in [4]. We show that the extended zero-divisor graph associated with an Artinian ring is weakly perfect, i.e., its vertex chromatic number equals its clique number. Furthermore, we classify all rings whose extended zero-divisor graphs are planar.

Key words: Extended zero-divisor graph, Clique number, Chromatic number, Planar graph.

1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past 20 years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, for instance, see [1], [3], [7], [8] and [10].

Throughout this paper, R denotes a unitary commutative ring which is not an integral domain. We denote by $\mathrm{Max}(R)$, $\mathrm{Min}(R)$, $\mathrm{Nil}(R)$ and $\mathrm{U}(R)$, the set of all maximal ideals of R, the set of all minimal prime ideals of R, the set of all nilpotent elements of R and the set of all invertible elements of R, respectively. Two ideals I and J of a ring R are called *coprime* if $I + J = R$. The ring R is said to be *reduced* if it has no non-zero nilpotent element. For every subset A of R, we denote the annihilator of A by $\mathrm{Ann}(A)$. Moreover, for the subset A of R we let $A^* = A \setminus \{0\}$. For any undefined notation or terminology in ring theory, we refer the reader to [2], [5].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced* subgraph by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Let

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G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Also G is called a *null graph* if it has no edge. A graph G is said to be *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. A complete bipartite graph of part sizes m, n is denoted by $K_{m,n}$. If $m = 1$, then the complete bipartite graph is called a *star graph*. Also, a complete graph of n vertices is denoted by K_n. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. For any undefined notation or terminology in graph theory, we refer the reader to [11].

The *extended zero-divisor graph* of a commutative ring R is the undirected (simple) graph $\Gamma'(R)$ with the vertex set $Z(R)^* = Z(R)\setminus\{0\}$, and two distinct vertices x and y are adjacent if and only if either $Rx \cap \mathrm{Ann}(y) \neq (0)$ or $Ry \cap \mathrm{Ann}(x) \neq (0)$. This graph was first introduced and studied in [4]. In this paper, we continue our study of the extended zero-divisor graph of a commutative ring. We show that the extended zero-divisor graph associated with an Artinian ring is weakly perfect. Moreover, we give an explicit formula for the vertex chromatic number of $\Gamma'(R)$. Furthermore, we classify all rings whose extended zero-divisor graphs are planar.

2. The Extended Zero-Divisor Graph of an Artinian Ring Is Weakly Perfect

The goal of this section is to study the coloring of the extended zero-divisor graphs of Artinian rings. For an Artinian ring R, it is shown that the graph $\Gamma'(R)$ is weakly perfect. Moreover, the exact value of the $\chi(\Gamma'(R))$ is given. Our starting result will be used frequently.

Lemma 2.1 *Let $R \cong D_1 \times \cdots \times D_n$, where $n \geq 2$ be a positive integer and D_i be an integral domain, for every $1 \leq i \leq n$. Then $\Gamma'(R)$ is a complete $(2^n - 2)$-partite graph.*

Proof. Let $X, Y \in V(\Gamma'(R))$. Then $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, where $x_i, y_i \in D_i$ for every $1 \leq i \leq n$. Define the relation $\sim$ on $V(\Gamma'(R))$ as follows: $X \sim Y$, whenever "$x_i = 0$ if and only if $y_i = 0$", for every $1 \leq i \leq n$. It is easily seen that $\sim$ is an equivalence relation on $V(\Gamma'(R))$. By $[X]$, we mean the equivalence class of X. Let $X = (x_1, \ldots, x_n)$ be an arbitrary vertex of $V(\Gamma'(R))$. Then for every component x_i, $1 \leq i \leq n$, we have either $x_i = 0$ or $x_i \neq 0$. Hence, the number of all selections for X is $2^n - 2$ (the other selections for X are not vertices). This implies that the number of equivalence classes is $2^n - 2$. Suppose that $[X]$ and $[Y]$ are two distinct arbitrary equivalence classes. It is enough to show that there is no adjacency between two vertices of $[X]$ and every vertex of $[X]$ is adjacent to all vertices of $[Y]$. To see this, let X_1 and X_2 be two vertices of $[X]$ and Y_1 be a vertex of $[Y]$. So there exist elements x_i, y_i and z_i of D_i, such that $X_1 = (x_1, \ldots, x_n)$, $X_2 = (y_1, \ldots, y_n)$ and $Y_1 = (z_1, \ldots, z_n)$, for all $1 \leq i \leq n$. Thus $x_i = 0$ if and only if $y_i = 0$, for every $1 \leq i \leq n$. This implies that $\mathrm{Ann}(X_1) = \mathrm{Ann}(X_2)$, and so by Part (2) of [4, Lemma 2.2], X_1 and X_2 are not adjacent. Also, since $X_1 \nsim Y_1$, $x_i = 0$ and $z_i \neq 0$, for some $1 \leq i \leq n$. Hence $\mathrm{Ann}(X_1) \neq \mathrm{Ann}(Y_1)$. By Part (2) of [4, Lemma 2.2], X_1 is adjacent to Y_1, as desired. □

In view of Lemma 2.1, we have the following corollaries.

Corollary 2.2 *Let R be a reduced ring with $|\mathrm{Min}(R)| < \infty$ and suppose that $\mathfrak{p}, \mathfrak{q}$ are coprime, for every two distinct $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}(R)$. Then the following statements are equivalent:*
(1) $|\mathrm{Min}(R)| = n$
(2) $\Gamma'(R)$ *is a complete* $(2^n - 2)$*-partite graph.*

Proof. The proof is obtained by Lemma 2.1 and [9, Theorem 1.4]. □

Corollary 2.3 *Let R be a ring such that $R \cong D_1 \times \cdots \times D_n$, where D_i is an integral domain, for every $1 \leq i \leq n$. Then*

$$\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = 2^n - 2.$$

Proof. The proof follows from Lemma 2.1. □

By [2, Theorem 8.7], every Artinian reduced ring is a direct product of finitely many fields. Thus, we may state the following corollary.

Corollary 2.4 *Let R be a reduced Artinian ring. Then*

$$\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = 2^{|\mathrm{Max}(R)|} - 2.$$

To state our main result in this section, we need to fix some notations.

Notation Let R be a ring such that $R = R_1 \times \cdots \times R_n$, where R_i is a local ring, for every $1 \leq i \leq n$. We define an $n \times n$ matrix $M(\Gamma'(R))$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} |R_j| & \text{if } i < j, \\ |\mathrm{Nil}(R_j)| - 1 & \text{if } i = j, \\ |U(R_j)| + 1 & \text{if } i > j. \end{cases}$$

In other notation, we have

$$M(\Gamma'(R)) = \begin{bmatrix} |\mathrm{Nil}(R_1)| - 1 & |R_2| & \cdots & |R_n| \\ |U(R_1)| + 1 & |\mathrm{Nil}(R_2)| - 1 & \cdots & |R_n| \\ \vdots & \vdots & \vdots & \vdots \\ |U(R_1)| + 1 & |U(R_2)| + 1 & \cdots & |\mathrm{Nil}(R_n)| - 1 \end{bmatrix}.$$

Let $V_i = (a_{i1}, \ldots, a_{in})$ be the i-th row of $M(\Gamma'(R))$. Put $\Lambda_i = \Lambda(V_i) = a_{i1} \cdots a_{in}$ and $\Lambda(\Gamma'(R)) = \sum_{i=1}^{n} \Lambda_i$.

Now, we are ready to prove that $\omega(\Gamma'(R))$ is weakly perfect, for every Artinian ring R.

Theorem 2.5 *Let R be an Artinian ring. Then*

$$\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \Lambda(\Gamma'(R)) + 2^{|\mathrm{Max}(R)|} - 2.$$

Proof. By [2, Theorem 8.7], one can deduce that there exists a positive integer n such that $R = R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. We put:

$A := \{(x_1, \ldots, x_n) \in R \mid x_i \in \mathrm{Nil}(R_i)^*, \text{ for some } 1 \leq i \leq n\}$ and

$B := \{(x_1, \ldots, x_n) \in R \mid (x_1, \ldots, x_n) \in Z(R)^* \text{ and } x_i \notin \mathrm{Nil}(R_i)^*, \text{ for all } 1 \leq i \leq n\}$. One may easily check that $V(\Gamma'(R)) = A \cup B$, $A \cap B = \varnothing$ and so $\{A, B\}$ is a partition of $V(\Gamma'(R))$. We Show that

$\Gamma'(R) = \Gamma'(R)[A] \vee \Gamma'(R)[B]$. Indeed, we have the following claims:

Claim 1. Every vertex of $\Gamma'(R)[A]$ is adjacent to the other vertices of $\Gamma'(R)$. Let $x = (x_1, \ldots, x_n)$ be a vertex of $\Gamma'(R)[A]$. Then $x_i \in \mathrm{Nil}(R_i)^*$ for some $1 \leq i \leq n$ and hence there exists a positive integer n, such that $x_i^n = 0$, $x_i^{n-1} \neq 0$. Since $(0, \ldots, x_i^{n-1}, \ldots, 0) \in Rx \cap \mathrm{Ann}(x)$, by Part (4) of [4, Lemma 2.2], we conclude that x is adjacent to the other vertices.

Claim 2. $\Gamma'(R)[A]$ is a complete subgraph of $\Gamma'(R)$. It is easily seen by the Claim 1.

Claim 3. $\Gamma'(R)[B]$ is a complete $(2^n - 2)$-partite subgraph of $\Gamma'(R)$. Let $(x_1, \ldots, x_n)$ be a vertex of $\Gamma'(R)[B]$. Then $x_i \in \{0\} \cup U(R_i)$, for every $1 \leq i \leq n$ and $(x_1, \ldots, x_n)$ is not unit. By a similar argument to that of proof of Lemma 2.1 one may partition $V(\Gamma'(R)[B])$ into $(2^n - 2)$ equivalence classes and show that $\Gamma'(R)[B]$ is a complete $(2^n - 2)$-partite subgraph of $\Gamma'(R)$.

Therefore, $\Gamma'(R) = \Gamma'(R)[A] \vee \Gamma'(R)[B]$ and so

$$\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \omega(\Gamma'(R)[A]) + \omega(\Gamma'(R)[B]) = |A| + 2^{|\mathrm{Max}(R)|} - 2.$$

To complete the proof, we show that $\Lambda(\Gamma'(R)) = |A|$. Let

$$A_i = \{(x_1, \ldots, x_n) \in A \mid x_i \in \mathrm{Nil}(R_i)^* \text{ and } x_j \in \{0\} \cup U(R_j) \text{ for every } j < i\},$$

for every $1 \leq i \leq n$. Clearly, $A = \bigcup_{i=1}^n A_i$ and $A_i \cap A_j = \varnothing$, for every $1 \leq i, j \leq n$ and $i \neq j$. It is not hard to check that $\Lambda_i = |A_i|$, for every $1 \leq i \leq n$. Thus $|A| = \sum_{i=1}^n |A_i| = \sum_{i=1}^n \Lambda_i = \Lambda(\Gamma'(R))$. □

The following corollary immediately follows from Theorem 2.5.

Corollary 2.6 *Let R be an Artinian ring. Then*

(1) *R is local which is not a field if and only if $\Lambda(\Gamma'(R)) \neq 0$ and $2^{|\mathrm{Max}(R)|} - 2 = 0$.*
(2) *R is reduced if and only if $\Lambda(\Gamma'(R)) = 0$.*
(3) *R is non-local and non-reduced if and only if $\Lambda(\Gamma'(R)) \neq 0$ and $2^{|\mathrm{Max}(R)|} - 2 \neq 0$.*

Following [6], we know that $Z(R)$ is finite if and only if either R is finite or an integral domain. So for an Artinian local ring R if $|\mathrm{Nil}(R)| \neq 1$, then R is finite if and only if $\mathrm{Nil}(R)$ is finite. Also, for an Artinian ring R, R is finite if and only if $U(R)$ is finite. We use these facts to prove the following theorem which characterizes Artinian rings for which the clique number of the extended zero-divisor graph is finite.

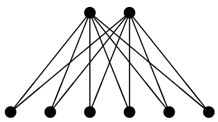
Theorem 2.7 *Let R be an Artinian ring. Then $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) < \infty$ if and only if one of the following statements holds*:
(1) *R is a reduced ring.*
(2) *R is a finite ring.*

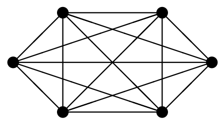
Proof. One side is clear. To prove the converse, suppose that $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) < \infty$. By Theorem 2.5, $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \Lambda(\Gamma'(R)) + 2^{|\mathrm{Max}(R)|} - 2$. So $0 \leqslant \Lambda(\Gamma'(R)) < \infty$. Now, if $\Lambda(\Gamma'(R)) = 0$, then by Corollary 2.6, R is a reduced ring. Also, if $0 < \Lambda(\Gamma'(R)) < \infty$, then we show that $|R| < \infty$. By [2, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring for every $1 \leq i \leq n$. Therefore, it suffices to show that $|R_i| < \infty$, for every $1 \leq i \leq n$. Since $0 < \Lambda(\Gamma'(R)) < \infty$, we deduce that $0 < \Lambda_i < \infty$, for some $1 \leq i \leq n$. Since $\Lambda_i = (|U(R_1)| + 1) \cdots (|U(R_{i-1})| + 1)(|\mathrm{Nil}(R_i)| - 1)|R_{i+1}| \cdots |R_n|$, one may easily check that $|R_i| < \infty$, for every $1 \leq i \leq n$. □

In what follow we describe our method to find the clique number and the chromatic number of the extended zero-divisor graphs, for some finite rings. Recall that the notations are the same as the notations in the proof of Theorem 2.5.

Example 2.8 (1). Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_9$. Then $A = \{(\overline{0},\overline{3}), (\overline{0},\overline{6}), (\overline{1},\overline{3}), (\overline{1},\overline{6}), (\overline{2},\overline{3}), (\overline{2},\overline{6})\}$ and $B = \{(\overline{1},\overline{0}), (\overline{2},\overline{0}), (\overline{0},\overline{1}), (\overline{0},\overline{2}), (\overline{0},\overline{4}), (\overline{0},\overline{5}), (\overline{0},\overline{7}), (\overline{0},\overline{8})\}$.

We have $V(\Gamma'(R)) = A \cup B$ and $A \cap B = \varnothing$.


$\Gamma'(R)[B]$

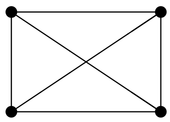

$\Gamma'(R)[A]$

Then $\Gamma'(R) = \Gamma'(R)[A] \vee \Gamma'(R)[B]$ and $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = 8$. On the other hand, by Theorem 2.5,

$$M(\Gamma'(R)) = \begin{bmatrix} 0 & 9 \\ 3 & 2 \end{bmatrix}.$$

So $\Lambda_1 = 0$, $\Lambda_2 = 6$ and $\Lambda(\Gamma'(R)) = \sum_{i=1}^{2} \Lambda_i = 6$. Thus we have $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \Lambda(\Gamma'(R)) + 2^{|\mathrm{Max}(R)|} - 2 = 6 + 2^2 - 2 = 8$.

(2). Let $R \cong \mathbb{Z}_{25}$. Then $A = \{\overline{5}, \overline{10}, \overline{15}, \overline{20}\}$ and $B = \varnothing$. So $\Gamma'(R) = \Gamma'(R)[A]$ and so $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = 4$.



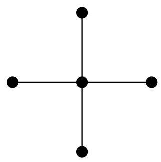
$\Gamma'(\mathbb{Z}_{25})$

Also, by Theorem 2.5, we have

$$M(\Gamma'(R)) = [\,4\,].$$

So $\Lambda_1 = 4$ and $\Lambda(\Gamma'(R)) = \Lambda_1 = 4$. Thus we again obtain $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \Lambda(\Gamma'(R)) + 2^{|\mathrm{Max}(R)|} - 2 = 4 + 2^1 - 2 = 4$.

(3). Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$. Then $A = \varnothing$ and $B = \{(\overline{0}, \overline{1}), (\overline{0}, \overline{2}), (\overline{0}, \overline{3}), (\overline{0}, \overline{4}), (\overline{1}, \overline{0})\}$. So $\Gamma'(R) = \Gamma'(R)[B]$ and so $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = 2$.



$\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_5)$

On the other hand, by Theorem 2.5, we have

$$M(\Gamma'(R)) = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}.$$

Thus, we again obtain $\Lambda_1 = 0$, $\Lambda_2 = 0$, $\Lambda(\Gamma'(R)) = \sum_{i=1}^{2} \Lambda_i = 0$ and $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \Lambda(\Gamma'(R)) + 2^{|\mathrm{Max}(R)|} - 2 = 0 + 2^2 - 2 = 2$.

3. When The Extended Zero-Divisor Graph is Planar?

As the planarity is an important invariant in graph theory, our concentration in this section is on the planarity of the extended zero-divisor graphs.

Indeed, we characterize all rings whose extended zero-divisor graphs are planar. We first study the case when R is reduced.

We begin with the following lemma.

Lemma 3.1 *Let R be a reduced ring. If $x^n = x^m$ for some $x \in Z(R)^*$, where n, m are distinct positive integers, then R is decomposable.*

Proof. With no loss of generality, assume that $n < m$. Since $x^n = x^m$ for some $x \in Z(R)^*$, we have $x^n(1 - x^{m-n}) = 0$. Thus $(1 - x^{m-n}) \in \mathrm{Ann}(x^n)$. This implies that $Rx + \mathrm{Ann}(x) = R$. As R is reduced, we conclude that R is decomposable. $\square$

To prove our results in this section, we need a celebrated theorem due to Kuratowski.

Theorem 3.2 ([11, Theorem 6.2.2]) *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

Theorem 3.3 *Let R be a ring. Then the following statements hold*:
(1) *If $R \cong R_1 \times R_2 \times R_3$, then $\Gamma'(R)$ is not planar.*
(2) *If $R \cong R_1 \times R_2$ and $|\mathrm{Min}(R)| \geq 3$, then $\Gamma'(R)$ is not planar.*

Proof. (1) Obviously, the vertices $(1,0,0), (0,1,0), (0,0,1), (1,1,0)$ and $(1,0,1)$ forms K_5 as a subgraph of $\Gamma'(R)$.

(2) Let $R \cong R_1 \times R_2$. With no loss of generality, we may assume that $|\mathrm{Min}(R_2)| \geq 2$ and so $Z(R_2) \neq (0)$. If R_2 is decomposable, then by Part (1) the proof is complete. If R_2 is indecomposable, then by Lemma 3.1, for every $x \in Z(R_2)^*$ $x^n \neq x^m$ where n, m are distinct positive integers. So $|Rx| = |\mathrm{Ann}(x)| = \infty$, for every $x \in Z(R_2)^*$. Since $Rx \cap \mathrm{Ann}(x) = (0)$, we deduce that $\Gamma'(R)$ is not planar. $\square$

Theorem 3.4 *Let R be a reduced ring. Then the following statements are equivalent*:

(1) $\Gamma'(R)$ *is planar.*
(2) $|\mathrm{Min}(R)| = 2$ *and one of minimal prime ideals of R has at most three distinct elements.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\Gamma'(R)$ is planar. We show that $|\mathrm{Min}(R)| = 2$. Suppose to the contrary, $|\mathrm{Min}(R)| \geq 3$. Let $a \in Z(R)^*$. If $|Ra| = |\mathrm{Ann}(a)| = \infty$, then $\Gamma'(R)$ is not planar, a contradiction. Otherwise, R

has a minimal ideal and so it is decomposable which contradicts Part (2) of Theorem 3.3. So $|\mathrm{Min}(R)| = 2$. Now, let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two distinct minimal prime ideals of R such that $|\mathfrak{p}_1| \geq 4, |\mathfrak{p}_2| \geq 4$. Since $\mathfrak{p}_1\mathfrak{p}_2 = (0)$, we find $K_{3,3}$ as a subgraph of $\Gamma'(R)$,which is impossible. Thus one of the minimal prime ideal of R must has at most three distinct elements.

(2) $\Rightarrow$ (1) It is obtained by proof of [4, Theorem 4.1]. □

In the rest of this paper, we assume that R is non-reduced. But we first note that by Part (5) of [4, Lemma 2.2], the subgraph induced by $\mathrm{Nil}(R)^*$ is complete in $\Gamma'(R)$. Therefore, to seek for planar extended zero-divisor graphs associated with non-reduced rings we study non-reduced rings with at most five nilpotent elements.

Lemma 3.5 *Let R be a non-reduced ring and $\Gamma'(R)$ is planar. Then the following statements hold*:
(1) *If $\Gamma'(R)$ is an infinite graph, then R is indecomposable.*
(2) *If $R \cong R_1 \times R_2$, then $|\mathrm{Nil}(R)| = 2$.*

Proof. (1) Assume to the contrary, $R \cong R_1 \times R_2$, for some rings R_1 and R_2. With no loss of generality, assume that $a \in \mathrm{Nil}(R_1)^*$. If $|Z(R_2)| = \infty$, then the vertices of the set $\{(1,0),(a,0),(a,1)\}$ and the vertices of the set $\{(0,1),(0,b),(0,c)\}$ form $K_{3,3}$, where $b,c \in Z(R_2)^*$, a contradiction. If $|Z(R_1)| = \infty$, then the vertices of the set $\{(0,1),(a,0),(a,1)\}$ and the vertices of the set $\{(1,0),(b,0),(c,0)\}$ form $K_{3,3}$, where $b,c \in Z(R_1)^*$, a contradiction.

(2) Suppose to the contrary, and with no loss of generality, $|\mathrm{Nil}(R)| \geq 3$, $a \in \mathrm{Nil}(R)^*, b \in \mathrm{Nil}(R_1)^*$ and $a \neq b$. It is easily seen that $\{(1,0),(0,1),$ $(b,1),(b,0),a\}$ forms K_5 in $\Gamma'(R)$, a contradiction. □

Theorem 3.6 *Let R be a non-reduced ring and $|\mathrm{Nil}(R)| = 2$. Suppose that R is not ring-isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Then*

(1) *If $|Z(R)| < \infty$, then the following statements are equivalent*:
   (i) *$\Gamma'(R)$ is planar.*
   (ii) *R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.*
(2) *If $|Z(R)| = \infty$, then the following statements are equivalent*:
   (i) *$\Gamma'(R)$ is planar.*
   (ii) *$\mathrm{Ann}(Z(R))$ is a prime ideal of R.*

Proof. (1) (i) $\Rightarrow$ (ii) Since $|Z(R)| < \infty$, R is an Artinian ring (Indeed R

is finite). By [2, Theorem 8.7], there exists a positive integer n such that $R = R_1 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is an Artinian local ring. The planarity of $\Gamma'(R)$ implies that $n = 2$, by Part (1) of Theorem 3.3. With no lost of generality, assume that $|\mathrm{Nil}(R_2)| = 2$. Thus $|U(R_2)| \geq 2$. We show that $|U(R_2)| = 2$. If $|U(R_2)| > 2$, then the vertices of the set $\{(1,0),(0,a),(1,a)\}$ and the vertices of the set $\{(0,1),(0,u),(0,v)\}$ form $K_{3,3}$, where $a \in \mathrm{Nil}(R_2)^*$ and $1 \neq u, 1 \neq v, u \neq v, u, v \in U(R_2)$, a contradiction. Thus $|U(R_2)| = 2$. Similarly, the cases $|U(R_1)| \geq 2$ and $|Z(R_1)| \geq 2$ lead to a contradiction. Thus $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_4$ or $R_2 \cong \mathbb{Z}_2[X]/(X^2)$.

(ii) $\Rightarrow$ (i) is clear.

(2) (ii) $\Rightarrow$ (i) is clear, by [4, Corollary 4.5].

(i) $\Rightarrow$ (ii). Let $a \in \mathrm{Nil}(R)^*$. We show that $xy \neq 0$ for every $x, y \in Z(R) \setminus \{0, a\}$. Suppose to the contrary, $xy = 0$ for some $x, y \in Z(R) \setminus \{0, a\}$. By Lemma 3.5, R is indecomposable and so $x^i \neq x^{i+1}$ and $y^i \neq y^{i+1}$, for $0 \leq i \leq 2$. Thus the vertices of the set $\{x, x^2, x^3\}$ and the vertices of the set $\{y, y^2, y^3\}$ form $K_{3,3}$, a contradiction. Therefore, $\mathrm{Ann}(Z(R)) = \mathrm{Nil}(R)$ and $\mathrm{Ann}(x) = \{0, a\}$, for every $x \in Z(R) \setminus \{0, a\}$. Hence $\Gamma(R) = \Gamma'(R) = K_1 \vee \overline{K}_m$ and so by [4, Corollary 4.5], the result holds. □

To prove Theorem 3.8, the following lemma is needed.

Lemma 3.7 *Let R be a ring and $x - y$ be an edge of $\Gamma'(R)$. Then either $x - z$ is an edge or $y - z$ is an edge of $\Gamma'(R)$, for every $z \in Z(R) \setminus \{0, x, y\}$.*

Proof. Suppose that $x - y$ is an edge of $\Gamma'(R)$. If $x - z$ is not an edge of $\Gamma'(R)$, then by Part (2) of [4, Lemma 2.2], $\mathrm{Ann}(x) = \mathrm{Ann}(z)$. If $\mathrm{Ann}(y) \neq \mathrm{Ann}(z)$, then $y - z$ is an edge of $\Gamma'(R)$, by Part (3) of [4, Lemma 2.2], and if $\mathrm{Ann}(y) = \mathrm{Ann}(z)$, $y - z$ is an edge of $\Gamma'(R)$. □

Theorem 3.8 *Let R be a non-reduced ring and $|\mathrm{Nil}(R)| = 3$. Then the following statements are equivalent*:
(1) $\mathrm{Ann}(Z(R))$ *is a prime ideal of* R.
(2) $\Gamma'(R)$ *is planar.*

Proof. (1) $\Rightarrow$ (2) is clear, by [4, Corollary 4.5].

(2) $\Rightarrow$ (1) Suppose that $\Gamma'(R)$ is planar. If $Z(R) = \mathrm{Nil}(R)$, there is nothing to prove. So let $Z(R) \neq \mathrm{Nil}(R)$. If $|Z(R)| < \infty$, then R is an Artinian ring and so by Part (2) of Lemma 3.5, $Z(R) = \mathrm{Nil}(R)$, a contradiction. Thus $|Z(R)| = \infty$. We now show that $\Gamma'(R)[Z(R) \setminus \mathrm{Nil}(R)]$ is null.

Suppose to the contrary, $x - y$ is an edge of $\Gamma'(R)[Z(R) \setminus \mathrm{Nil}(R)]$. Thus every vertex of $Z(R) \setminus \mathrm{Nil}(R)$ is adjacent to x or y, by Lemma 3.7. This together with Part (4) of [4, Lemma 2.2] imply that $\Gamma'(R)$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $\Gamma'(R)[Z(R) \setminus \mathrm{Nil}(R)]$ is null. Now, we easily see that $\Gamma(R) = \Gamma'(R) = K_2 \vee \overline{K}_m$. Now, by [4, Corollary 4.5], the result holds. □

We close this paper with the following result.

Theorem 3.9 *Let R be a non-reduced ring. Then*
(1) *If* $|\mathrm{Nil}(R)| \geq 6$, *then* $\Gamma'(R)$ *is not planar.*
(2) *If* $4 \leq |\mathrm{Nil}(R)| \leq 5$, *then* $\Gamma'(R)$ *is planar if and only if* $Z(R) = \mathrm{Nil}(R)$.

Proof. (1) It is clear by Part (5) of [4, Lemma 2.2].

(2) Let $\Gamma'(R)$ be planar and suppose that $4 \leq |\mathrm{Nil}(R)| \leq 5$. By Part (4) of [4, Lemma 2.2], $|Z(R)| < \infty$ and hance R is an Artinian ring. Now, by Part (2) of Lemma 3.5, R is local and so $Z(R) = \mathrm{Nil}(R)$. The converse is clear. □

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