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A vector-valued estimate of multilinear Calderón-Zygmund operators in Herz-Morrey spaces with variable exponents

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Abstract. In this paper, we obtain a vector valued inequality of multilinear Calderón-Zygmund operators on products of Herz-Morrey spaces with variable exponents.

Key words: multilinear Calderón-Zygmund operator, variable exponent, Herz-Morrey space, vector valued estimate.

1. Introduction

Recent decades, variable exponent function spaces have been received more and more attention. This mainly begun with the work of Kováčik and Rákosník. In [15], the authors gave fundamental properties of the variable Lebesgue and Sobolev spaces. Then some sufficient conditions were obtained for the boundedness of Hardy-Littlewood maximal operator on variable Lebesgue space, see [21]. After that, many function spaces with variable exponents appeared, such as: Besov and Trieble-Lizorkin spaces with variable exponents, Hardy spaces with variable exponents, Morrey spaces with variable exponent, Bessel potential spaces with a variable exponent and Herz-Morrey spaces with variable exponents; see [1], [2], [3], [5], [9], [11], [14], [16], [20], [22], [24].

Recently, multilinear singular operators and their commutators are also intensively studied by a significant number of authors, for instance, Grafakos and Torres studied the boundedness of the multilinear Calderón-Zygmund operators on products of Lebesgue spaces and the endpoint weak estimates in [8], boundedness of commutators on Herz spaces with variable exponents in [13], multilinear commutators of BMO functions and multilinear singular integral operators with non-smooth kernels in [6], [17], maximal multilinear

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commutators and maximal iterated commutators generated by an multilinear operator and a Lipschitz function in [4], multilinear singular integrals and commutators in variable exponent Lebesgue spaces in [10]. In [23], Tang, Wu and the second author obtained the boundedness of a commutator generated by an multilinear Calderón-Zygmund operator and BMO functions in Herz-Morrey spaces with variable exponents.

Motivated by the previous papers, the goal of this paper is to prove a vector valued inequality of a multilinear Calderón-Zygmund operator on products of Herz-Morrey spaces with variable exponents.

2. Main result

To state the main result of this paper, we need recall some notions firstly.

A multilinear operator T is called a Calderón-Zygmund operator if it is initially defined on the m-fold product of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and can be extended bounded from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ to L^p with $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1/p$, and for $f_1, \ldots, f_m \in L^\infty_{\mathrm{C}}(\mathbb{R}^n)$ (the space of compactly supported bounded functions), $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$, $\vec{f} = (f_1, \ldots, f_m)$,

$$T\vec{f}(x) := \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) dy_1 \ldots dy_m,$$

where the kernel K is a function in $(\mathbb{R}^n)^{m+1}$ away from the diagonal $y_0 = y_1 = \cdots = y_m$ and there exist positive constants ϵ, A such that

$$|K(x, y_1, \ldots, y_m)| \leq A\left(\sum_{i=1}^m |x - y_i|\right)^{-mn},$$

$$|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \leq \frac{A|x - x'|^\epsilon}{\left(\sum_{i=1}^m |x - y_i|\right)^{mn+\epsilon}}$$

provided that $|x - x'| \leq (1/2) \max\{|x - y_1|, \ldots, |x - y_m|\}$, and for each $1 \leq i \leq m$

$$|K(x, y_1, \ldots, y_i, \ldots, y_m) - K(x, y_1, \ldots, y_i', \ldots, y_m)| \leq \frac{A|y_i - y_i'|^\epsilon}{\left(\sum_{j=1}^m |x - y_j|\right)^{mn+\epsilon}}$$

provided that $|y_i - y_i'| \leq (1/2) \max\{|x - y_1|, \ldots, |x - y_m|\}$.

Such kernels are called the $m-$linear Calderón-Zygmund kernels and the collection of such functions is denoted by $m - CZK(A, \epsilon)$ in [7]. Grafakos and Torres in [7] showed that if T is an $m-$linear Calderón-Zygmund operator then T is bounded from $L^{q_1} \times L^{q_2} \times \cdots \times L^{q_m}$ to L^q for each $1 < q_1, q_2, \ldots, q_m < \infty$ such that $1/q_1 + 1/q_2 + \cdots + 1/q_m = 1/q$. Then, Grafakos and Torres in [8] obtained weighted norm inequalities for multilinear Calderón-Zygmund operators.

Definition 2.1 Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function.

(i) The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ \begin{array}{l} f \text{ is measurable:} \\ \qquad \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \end{array} \right\}.$$

(ii) The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) := \{ f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n \},$$

where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$.

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

For brevity, we denote $\|\cdot\|_{L^{p(\cdot)}}$ by $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, we denote

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$; $\mathcal{P}^0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < \infty$. $L^{p(\cdot)}$ can be similarly defined as above for $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad \forall\, x \in \mathbb{R}^n,$$

where B is a ball. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.2 Let $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.

(i) $\alpha(\cdot)$ is locally log-Hölder continuous if there exists a constant C_1 such that

$$|\alpha(x) - \alpha(y)| \le \frac{C_1}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n\ ,\ |x-y| < \frac{1}{2};$$

(ii) $\alpha(\cdot)$ is log-Hölder continuous at the origin if

$$|\alpha(x) - \alpha(0)| \le \frac{C_2}{\log(e + 1/|x|)}, \quad \forall x \in \mathbb{R}^n;$$

(iii) $\alpha(\cdot)$ is log-Hölder continuous at the infinity if there exists $\alpha_\infty \in \mathbb{R}$ such that

$$|\alpha(x) - \alpha_\infty| \le \frac{C_3}{\log(e + |x|)}, \ \forall x \in \mathbb{R}^n;$$

(iv) $\alpha(\cdot)$ is global log-Hölder continuous if $\alpha(\cdot)$ is both locally log-Hölder continuous and log-Hölder continuous at the infinity.

We denote by $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which is log-Hölder continuous at the origin and at the infinity respectively.

To give the definitions of Herz spaces and Herz-Morrey spaces with variable exponents, we use the following notations. For each $k \in \mathbb{Z}$ we define

$$B_k := \{x \in \mathbb{R}^n : |x| \le 2^k\},\ D_k := B_k \setminus B_{k-1},\ \chi_k := \chi_{D_k}.$$

Definition 2.3 Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(i) The homogeneous Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}} \right)^{1/q};$$

(ii) The non-homogeneous Herz-Morrey space $MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents is defined by

$$MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{L\in\mathbb{N}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}} \right)^{1/q}.$$

If $\alpha(\cdot)$ is a constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ was defined in [12]. If $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constants and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{q,p}(\mathbb{R}^n)$ is the classical Herz space in [18].

Throughout this paper, $|E|$ denotes the Lebesgue measure, C will always denote a positive constant depending on the context, whose value may be different at different occurrences.

Our main result is the following.

Theorem 2.4 *Let T be a m-linear Calderón-Zygmund operator, $p_i(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ satisfying $1/p(x) = 1/p_1(x) + 1/p_2(x) + \cdots + 1/p_m(x)$, $1 < p_i^- \leq p_i^+ < n/\lambda_i$, $(p(\cdot)/p_*)' \in \mathcal{B}(\mathbb{R}^n)$ for some $0 < p_* < p_-$, $i = 1, 2, \ldots, m$. Let $0 < q_i < \infty$, $0 \leq \lambda_i < \infty$ and $\alpha_i(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ for $i = 1, 2, \ldots, m$ with*

$$2\lambda_i - n\delta_{1i} < \alpha_i^- < \alpha_i^+ < n\delta_{2i}, \tag{1}$$

where $\delta_{1i}, \delta_{2i} \in (0,1)$ *are the constants appearing in* (2) *and* (3) *for* $p_i(\cdot)$. *Suppose that* $\lambda = \sum_{i=1}^m \lambda_i$, $\alpha(x) = \sum_{i=1}^m \alpha_i(x)$, $1/q = \sum_{i=1}^m 1/q_i$. *Then*

$$\Big\| \Big(\sum_{j=1}^{\infty} |T(f_1^j, f_2^j, \ldots, f_m^j)|^r \Big)^{1/r} \Big\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \Big\| \Big(\sum_{j=1}^{\infty} |f_i^j|^{r_i} \Big)^{1/r_i} \Big\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}$$

with the constant $C > 0$ *independent of* $\vec{f^j} = (f_1^j, f_2^j, \ldots, f_m^j)$.

Remark Here we only declare our result in the homogeneous Herz-Morrey spaces with variable exponents, but there is an analogue for the non-homogeneous Herz-Morrey spaces with variable exponents, we omit the detail here.

To prove our result, we need the following Lemmas.

Lemma 2.5 (see [12, Lemma 1 and (10)]) *If* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *then there exist constants* $\delta_1, \delta_2 \in (0,1)$ *and* $C > 0$ *such that for all balls* B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \leq C \Big(\frac{|S|}{|B|} \Big)^{\delta_1}, \tag{2}$$

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} \leq C \Big(\frac{|S|}{|B|} \Big)^{\delta_2}. \tag{3}$$

Lemma 2.6 (see [19, Proposition 2]) *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in (0,\infty)$, $0 \leq \lambda < \infty$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, *then*

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} \approx \max\Bigg\{ \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \|f\chi_k\|_{L^{p(\cdot)}}^q \Big)^{1/q}, \sup_{L>0, L\in\mathbb{Z}} \Big[2^{-L\lambda} \Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f\chi_k\|_{L^{p(\cdot)}}^q \Big)^{1/q} + 2^{-L\lambda} \Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q} \|f\chi_k\|_{L^{p(\cdot)}}^q \Big)^{1/q} \Big] \Bigg\}.$$

Lemma 2.7 (see [12, Lemma 2]) *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$C^{-1} \leq \frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \leq C. \tag{4}$$

Lemma 2.8 (see [15, Theorem 2.1]) *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $r_p := 1 + 1/p_+ - 1/p_-$.

Lemma 2.9 (see [10, Theorem 2.3]) *Let $p,\ p_1,\ p_2 \in \mathcal{P}^0(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$. Then there exists a constant C_{p,p_1} independent of the functions f and g such that*

$$\|fg\|_{L^{p(\cdot)}} \leq C_{p,p_1} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}$$

holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$.

Lemma 2.10 (see [10, Corollary 2.1]) *Let T be a 2-linear Calderón-Zygmund operator and let $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ such that there exists $0 < p_* < p_-$ with $(p(\cdot)/p_*)' \in \mathcal{B}(\mathbb{R}^n)$. If $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$, then there exists a constant C independent of functions $f_i^j \in L^{p_i(\cdot)}(\mathbb{R}^n)$ for $j \in \mathbb{N}, i = 1, 2$ such that*

$$\begin{aligned} &\left\| \left(\sum_{j=1}^{\infty} |T(f_1^j, f_2^j)|^q \right)^{1/q} \right\|_{L^{p(\cdot)}} \\ &\quad \leq C \left\| \left(\sum_{j=1}^{\infty} |f_1^j|^{q_1} \right)^{1/q_1} \right\|_{L^{p_1(\cdot)}} \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{q_2} \right)^{1/q_2} \right\|_{L^{p_2(\cdot)}}, \end{aligned}$$

where $1 < q_i < \infty$ for $i = 1, 2$ and $1/q = 1/q_1 + 1/q_2$.

Proof of Theorem 2.4. In the following, we only consider 2-linear operators for simplicity. Since the set of all bounded compactly supported functions is dense in Herz-Morrey spaces with variable exponents, we let f_1^j and f_2^j be bounded compactly supported functions for $j \in \mathbb{N}$ and write

$$f_i(x) = \sum_{l_i=-\infty}^{\infty} f_i(x)\chi_{l_i}(x) =: \sum_{l_i=-\infty}^{\infty} f_{l_i}(x), \quad i = 1, 2, j \in \mathbb{N}.$$

By Lemma 2.6, we get

$$\left\| \left(\sum_{j=1}^{\infty} |T(f_1^j, f_2^j)|^r \right)^{1/r} \right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \approx \max\{E, F\},$$

where

$$E := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \Bigg(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \times \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} T(f_{l_1}^j, f_{l_2}^j) \right|^r \right)^{1/r} \chi_k \right\|_{L^{p(\cdot)}}^q \Bigg)^{1/q},$$

$$F := \sup_{L>0, L\in\mathbb{Z}} \Bigg[2^{-L\lambda} \Bigg(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \times \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} T(f_{l_1}^j, f_{l_2}^j) \right|^r \right)^{1/r} \chi_k \right\|_{L^{p(\cdot)}}^q \Bigg)^{1/q} + 2^{-L\lambda} \Bigg(\sum_{k=0}^{L} 2^{k\alpha_\infty q} \times \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} T(f_{l_1}^j, f_{l_2}^j) \right|^r \right)^{1/r} \chi_k \right\|_{L^{p(\cdot)}}^q \Bigg)^{1/q} \Bigg].$$

Since the estimation of F is essentially similar to that of E, so it suffices to prove E is bounded in Herz-Morrey spaces with variable exponents. It is easy to see that

$$E \le C \sum_{i=1}^{9} I_i,$$

where

$$I_1 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_2 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_3 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_4 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k-1}^{k+1}\sum_{l_2=-\infty}^{k-2} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_5 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_6 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_7 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k+2}^{\infty}\sum_{l_2=-\infty}^{k-2} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_8 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k+2}^{\infty}\sum_{l_2=k-1}^{k+1} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q},$$

$$I_9 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty} T(f_{l_1}^j, f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{1/q}.$$

By the symmetry of f_1^j and f_2^j, we only need to estimate I_1, I_2, I_3, I_5, I_6 and I_9 for the estimates of I_4, I_7 and I_8 are analogous to that of I_2, I_3 and I_6 respectively.

By Lemmas 2.5 and 2.7 and the Hölder inequality, if $l_i \le k-1$, we have

$$\left\|2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_i}^j(y_i)|^{r_i}\right)^{1/r_i} dy_i\chi_k\right\|_{L^{p_i(\cdot)}}$$
$$\le C2^{-kn}\|\chi_k\|_{L^{p_i(\cdot)}}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_i}^j(y_i)|^{r_i}\right)^{1/r_i} dy_i$$

$$\leq C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}\|\chi_{l_i}\|_{L^{p_i'(\cdot)}}$$
$$\leq C2^{-kn}|B_k|\|\chi_{B_k}\|^{-1}_{L^{p_i'(\cdot)}}\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}$$
$$\leq C2^{(l_i-k)n\delta_{2i}}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}. \tag{5}$$

If $l_i = k$, then

$$\left\|2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_i}^j(y_i)|^{r_i}\right)^{1/r_i}dy_i\chi_k\right\|_{L^{p_i(\cdot)}}$$
$$\leq C2^{-kn}\|\chi_k\|_{L^{p_i(\cdot)}}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_i}^j(y_i)|^{r_i}\right)^{1/r_i}dy_i$$
$$\leq C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}\|\chi_{l_i}\|_{L^{p_i'(\cdot)}}$$
$$\leq C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}}\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}$$
$$\leq C\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}. \tag{6}$$

If $l_i \geq k+1$,then

$$\left\|2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_i}^j(y_i)|^{r_i}\right)^{1/r_i}dy_i\chi_k\right\|_{L^{p_i(\cdot)}}$$
$$\leq C2^{-kn}\|\chi_k\|_{L^{p_i(\cdot)}}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_i}^j(y_i)|^{r_i}\right)^{1/r_i}dy_i$$
$$\leq C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}\|\chi_{l_i}\|_{L^{p_i'(\cdot)}}$$

$$\leq C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}}\|\chi_{B_{l_i}}\|^{-1}_{L^{p_i(\cdot)}}\|\chi_{B_{l_i}}\|_{L^{p_i(\cdot)}}\|\chi_{B_{l_i}}\|_{L^{p'_i(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}$$

$$\leq C2^{(l_i-k)n(1-\delta_{1i})}\left\|\left(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\right)^{1/r_i}\chi_{l_i}\right\|_{L^{p_i(\cdot)}}. \tag{7}$$

Step 1: To estimate the term of I_1, we note that $l_i \leq k-2$ for $i=1,2$, and

$$|x-y_i| \geq |x|-|y_i| > 2^{k-1}-2^{l_i} \geq 2^{k-2}, \quad x \in D_k,\ y_i \in D_{l_i}.$$

Then, for $x \in D_k$, we obtain

$$|K(x,y_1,y_2)| \leq C(|x-y_1|+|x-y_2|)^{-2n} \leq C2^{-2kn}.$$

Therefore, $\forall x \in D_k, y_i \in D_{l_i}$

$$\begin{aligned}\left|T(f_{l_1}^j,f_{l_2}^j)(x)\right| &\leq \int_{\mathbb{R}^n}|K(x,y_1,y_2)|\prod_{i=1}^{2}|f_{l_i}^j(y_i)|dy_i\\ &\leq C2^{-2kn}\prod_{i=1}^{2}\int_{\mathbb{R}^n}|f_{l_i}^j(y_i)|dy_i.\end{aligned}$$

By the Hölder inequality and Minkowski inequality, we obtain

$$\begin{aligned}&\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}T(f_{l_1}^j,f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}\\
&\leq C2^{-2kn}\left\|\left(\sum_{j=1}^{\infty}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\int_{\mathbb{R}^n}|f_{l_1}^j(y_1)|dy_1\int_{\mathbb{R}^n}|f_{l_2}^j(y_2)|dy_2\right)^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}\\
&\leq C2^{-2kn}\left\|\left(\sum_{j=1}^{\infty}\left(\sum_{l_1=-\infty}^{k-2}\int_{\mathbb{R}^n}|f_{l_1}^j(y_1)|dy_1\right)^{r_1}\right)^{1/r_1}\right.\\
&\qquad\times\left.\left(\sum_{j=1}^{\infty}\left(\sum_{l_2=-\infty}^{k-2}\int_{\mathbb{R}^n}|f_{l_2}^j(y_2)|dy_2\right)^{r_2}\right)^{1/r_2}\chi_k\right\|_{L^{p(\cdot)}}\end{aligned}$$

$$\leq C2^{-2kn}\left\|\left(\sum_{j=1}^{\infty}\left(\sum_{l_1=-\infty}^{k-2}\int_{\mathbb{R}^n}|f_{l_1}^j(y_1)|dy_1\right)^{r_1}\right)^{1/r_1}\chi_k\right\|_{L^{p_1(\cdot)}}$$

$$\times\left\|\left(\sum_{j=1}^{\infty}\left(\sum_{l_1=-\infty}^{k-2}\int_{\mathbb{R}^n}|f_{l_2}^j(y_2)|dy_2\right)^{r_2}\right)^{1/r_2}\chi_k\right\|_{L^{p_2(\cdot)}}$$

$$\leq C\left\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\right)^{1/r_1}dy_1\chi_k\right\|_{L^{p_1(\cdot)}}$$

$$\times\left\|\sum_{l_2=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\right)^{1/r_2}dy_1\chi_k\right\|_{L^{p_2(\cdot)}}.$$

Since $1/q = 1/q_1 + 1/q_2$, it follows that

$$I_1 \leq C\sup_{L\leq 0, L\in\mathbb{Z}}2^{-L\lambda}\Bigg\{\sum_{k=-\infty}^{L}2^{k\alpha(0)q}$$

$$\times\left\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\right)^{1/r_1}dy_1\chi_k\right\|_{L^{p_1(\cdot)}}^q$$

$$\times\left\|\sum_{l_2=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\right)^{1/r_2}dy_1\chi_k\right\|_{L^{p_2(\cdot)}}^q\Bigg\}^{1/q}$$

$$\leq C\sup_{L\leq 0, L\in\mathbb{Z}}2^{-L\lambda_1}\Bigg\{\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1}$$

$$\times\left\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\right)^{1/r_1}dy_1\chi_k\right\|_{L^{p_1(\cdot)}}^{q_1}\Bigg\}^{1/q_1}$$

$$\times 2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2}$$

$$\times\left\|\sum_{l_2=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\right)^{1/r_2}dy_2\chi_k\right\|_{L^{p_2(\cdot)}}^{q_2}\Bigg\}^{1/q_2}$$

$$=: C\sup_{L\leq 0, L\in\mathbb{Z}}I_{11}(L)I_{12}(L),$$

where

$$I_{1i}(L) := 2^{-L\lambda_i}\Bigg\{\sum_{k=-\infty}^{L} 2^{k\alpha_i(0)q_i}$$

$$\times \left\| \sum_{l_i=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_{l_i}^j(y_i)|^{r_i}\right)^{1/r_i} dy_i \chi_k \right\|_{L^{p_i(\cdot)}}^{q_i} \Bigg\}^{1/q_i}.$$

Therefore, from (5) we get

$$I_{1i}(L) \le C2^{-L\lambda_i}\Bigg\{\sum_{k=-\infty}^{L} 2^{k\alpha_i(0)q_i}$$

$$\times \left(\sum_{l_i=-\infty}^{k-2} 2^{(l_i-k)n\delta_2} \left\| \left(\sum_{j=1}^{\infty} |f_i^j|^{r_i}\right)^{1/r_i} \chi_{l_i} \right\|_{L^{p_i(\cdot)}}\right)^{q_i}\Bigg\}^{1/q_i}$$

$$= C2^{-L\lambda_i}\Bigg\{\sum_{k=-\infty}^{L} \Bigg(\sum_{l_i=-\infty}^{k-2} 2^{k\alpha_i(0)}$$

$$\times \left\| \left(\sum_{j=1}^{\infty} |f_i^j|^{r_i}\right)^{1/r_i} \chi_{l_i} \right\|_{L^{p_i(\cdot)}} 2^{(l_i-k)n\delta_2}\Bigg)^{q_i}\Bigg\}^{1/q_i}$$

$$= C2^{-L\lambda_i}\Bigg\{\sum_{k=-\infty}^{L} \Bigg(\sum_{l_i=-\infty}^{k-2} 2^{l_i\alpha_i(0)}$$

$$\times \left\| \left(\sum_{j=1}^{\infty} |f_i^j|^{r_i}\right)^{1/r_i} \chi_{l_i} \right\|_{L^{p_i(\cdot)}} 2^{b_i(l_i-k)}\Bigg)^{q_i}\Bigg\}^{1/q_i},$$

here $b_i := n\delta_{2i} - \alpha_i(0) > 0,\ i = 1, 2.$

If $1 < q_i < \infty$, by the Hölder inequality, we have

$$\begin{aligned} I_{1i}(L) &\le C2^{-L\lambda_i}\Bigg\{\sum_{k=-\infty}^{L}\Bigg(\sum_{l_i=-\infty}^{k-2} 2^{l_i\alpha_i(0)q_i} \\ &\qquad\qquad \times \Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Bigg)^{1/r_i}\chi_{l_i}\Bigg\|_{L^{p_i(\cdot)}}^{q_i} 2^{b_iq_i(l_i-k)/2}\Bigg) \\ &\qquad \times \Bigg(\sum_{l_i=-\infty}^{k-2} 2^{b_iq_i'(l_i-k)/2}\Bigg)^{q_i/q_i'}\Bigg\}^{1/q_i} \\ &\le C2^{-L\lambda_i}\Bigg\{\sum_{k=-\infty}^{L}\sum_{l_i=-\infty}^{k-2} 2^{l_i\alpha_i(0)q_i} \\ &\qquad \times \Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Bigg)^{1/r_i}\chi_{l_i}\Bigg\|_{L^{p_i(\cdot)}}^{q_i} \times 2^{b_iq_i(l_i-k)/2}\Bigg\}^{1/q_i} \\ &\le C2^{-L\lambda_i}\Bigg\{\sum_{l_i=-\infty}^{k-2} 2^{l_i\alpha_i(0)q_i}\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Bigg)^{1/r_i}\chi_{l_i}\Bigg\|_{L^{p_i(\cdot)}}^{q_i} \\ &\qquad \times \sum_{k=l_i+2}^{L} 2^{b_iq_i(l_i-k)/2}\Bigg\}^{1/q_i} \\ &\le C2^{-L\lambda_i}\Bigg\{\sum_{l_i=-\infty}^{k-2} 2^{l_i\alpha_i(0)q_i}\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Bigg)^{1/r_i}\chi_{l_i}\Bigg\|_{L^{p_i(\cdot)}}^{q_i}\Bigg\}^{1/q_i} \\ &\le C\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Bigg)^{1/r_i}\Bigg\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}. \end{aligned}$$

If $0 < q_i \le 1$, since for $a_1, a_2, \cdots \ge 0$,

$$\Bigg(\sum_{s=1}^{\infty} a_s\Bigg)^{q_i} \le \sum_{s=1}^{\infty} a_s^{q_i}, \tag{8}$$

so we have

$$
\begin{aligned}
I_{1i}(L) &\le C2^{-L\lambda_i}\Bigg\{\sum_{k=-\infty}^{L}\sum_{l_i=-\infty}^{k-2} 2^{l_i\alpha_i(0)q_i}\Bigg\|\Big(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Big)^{1/r_i}\chi_{l_i}\Bigg\|_{L^{p_i(\cdot)}}^{q_i} \\
&\qquad \times 2^{b_iq_i(l_i-k)}\Bigg\}^{1/q_i} \\
&= C2^{-L\lambda_i}\Bigg\{\sum_{l_i=-\infty}^{L-2} 2^{l_i\alpha_i(0)q_i}\Bigg\|\Big(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Big)^{1/r_i}\chi_{l_i}\Bigg\|_{L^{p_i(\cdot)}}^{q_i} \\
&\qquad \times \sum_{k=l_i+2}^{L} 2^{b_iq_i(l_i-k)}\Bigg\}^{1/q_i} \\
&\le C2^{-L\lambda_i}\Bigg\{\sum_{l_i=-\infty}^{L-2} 2^{l_i\alpha_i(0)q_i}\Bigg\|\Big(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Big)^{1/r_i}\chi_{l_i}\Bigg\|_{L^{p_i(\cdot)}}^{q_i}\Bigg\}^{1/q_i} \\
&\le C\Bigg\|\Big(\sum_{j=1}^{\infty}|f_i^j|^{r_i}\Big)^{1/r_i}\Bigg\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_1(L) &\le C\sup_{L\le 0, L\in\mathbb{Z}} I_{11}(L)I_{12}(L) \\
&\le C\Bigg\|\Big(\sum_{j=1}^{\infty}|f_1^j|^{r_1}\Big)^{1/r_1}\Bigg\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha_1(\cdot),\lambda_1}(\mathbb{R}^n)}\Bigg\|\Big(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Big)^{1/r_2}\Bigg\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha_2(\cdot),\lambda_2}(\mathbb{R}^n)}.
\end{aligned}
$$

Step 2: To estimate I_2, for $x \in D_k,\ y_i \in D_{l_i},\ i = 1,\ 2$ and $l_1 \le k-2$, $k-1 \le l_2 \le k+1$, then we have

$$|x - y_2| \ge |x - y_1| \ge |x| - |y_1| > 2^{k-2}.$$

Therefore, by the Hölder inequality and Minkowski inequality, we get

$$
\Bigg\|\Big(\sum_{j=1}^{\infty}\Big|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1} T(f_{l_1}^j, f_{l_2}^j)\Big|^r\Big)^{1/r}\chi_k\Bigg\|_{L^{p(\cdot)}}
$$

$$\leq C2^{-2kn}\bigg\|\bigg(\sum_{j=1}^{\infty}\bigg(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\int_{\mathbb{R}^n}|f_{l_1}^j(y_1)|dy_1\int_{\mathbb{R}^n}|f_{l_2}^j(y_2)|dy_2\bigg)^r\bigg)^{1/r}\chi_k\bigg\|_{L^{p(\cdot)}}$$

$$\leq C2^{-2kn}\bigg\|\bigg(\sum_{j=1}^{\infty}\bigg(\sum_{l_1=-\infty}^{k-2}\int_{\mathbb{R}^n}|f_{l_1}^j(y_1)|dy_1\bigg)^{r_1}\bigg)^{1/r_1}$$

$$\times\bigg(\sum_{j=1}^{\infty}\bigg(\sum_{l_2=k-1}^{k+1}\int_{\mathbb{R}^n}|f_{l_2}^j(y_2)|dy_2\bigg)^{r_2}\bigg)^{1/r_2}\chi_k\bigg\|_{L^{p(\cdot)}}$$

$$\leq C2^{-2kn}\bigg\|\bigg(\sum_{j=1}^{\infty}\bigg(\sum_{l_1=-\infty}^{k-2}\int_{\mathbb{R}^n}|f_{l_1}^j(y_1)|dy_1\bigg)^{r_1}\bigg)^{1/r_1}\chi_k\bigg\|_{L^{p_1(\cdot)}}$$

$$\times\bigg\|\bigg(\sum_{j=1}^{\infty}\bigg(\sum_{l_2=k-1}^{k+1}\int_{\mathbb{R}^n}|f_{l_2}^j(y_2)|dy_2\bigg)^{r_2}\bigg)^{1/r_2}\chi_k\bigg\|_{L^{p_2(\cdot)}}$$

$$\leq C\bigg\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\bigg(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\bigg)^{1/r_1}dy_1\chi_k\bigg\|_{L^{p_1(\cdot)}}$$

$$\times\bigg\|\sum_{l_2=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\bigg(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\bigg)^{1/r_2}dy_1\chi_k\bigg\|_{L^{p_2(\cdot)}}.$$

Again by the Hölder inequality, we obtain,

$$I_2\leq C\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda}\bigg\{\sum_{k=-\infty}^{L}2^{k\alpha(0)q}$$

$$\times\bigg\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\bigg(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\bigg)^{1/r_1}dy_1\chi_k\bigg\|_{L^{p_1(\cdot)}}^q$$

$$\times\bigg\|\sum_{l_2=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\bigg(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\bigg)^{1/r_2}dy_2\chi_k\bigg\|_{L^{p_2(\cdot)}}^q\bigg\}^{1/q}$$

$$\leq C\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda_1}\bigg\{\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1}$$

$$\times \left\| \sum_{l_1=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_{l_1}^j(y_1)|^{r_1} \right)^{1/r_1} dy_1 \chi_k \right\|_{L^{p_1(\cdot)}}^{q_1} \Bigg\}^{1/q_1}$$

$$\times 2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2}$$

$$\times \left\| \sum_{l_2=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_{l_2}^j(y_2)|^{r_2} \right)^{1/r_2} dy_2 \chi_k \right\|_{L^{p_2(\cdot)}}^{q_2} \Bigg\}^{1/q_2}$$

$$=: C \sup_{L\le 0, L\in\mathbb{Z}} I_{21}(L) I_{22}(L).$$

It is clear that

$$I_{21}(L) = I_{11}(L) \le C \left\| \left(\sum_{j=1}^{\infty} |f_1^j|^{r_1} \right)^{1/r_1} \right\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Now we have to estimate $I_{22}(L)$. Combining the inequalities (5), (6) and (7), we have that

$$I_{22}(L) \le C 2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \sum_{l_2=k-1}^{k+1}$$

$$\times \left\| 2^{-kn} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_{l_2}^j(y_2)|^{r_2} \right)^{1/r_2} dy_2 \chi_k \right\|_{L^{p_2(\cdot)}}^{q_2} \Bigg\}^{1/q_2}$$

$$\le C 2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \sum_{l_2=k-1}^{k+1} 2^{(l_2-k)nq_2}$$

$$\times \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_{l_2} \right\|_{L^{p_2(\cdot)}}^{q_2} \Bigg\}^{1/q_2}$$

$$\le C 2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_k \right\|_{L^{p_2(\cdot)}}^{q_2} \Bigg\}^{1/q_2}$$

$$\le C \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \right\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha_2(\cdot),\lambda_2}(\mathbb{R}^n)},$$

here we used $2^{-n\delta_{2i}} < 1$ and $2^{(l_2-k)n(1-\delta_{1i})} < 2^{(l_2-k)n}$ in (5), (7) respectively. Therefore, we get

$$\begin{aligned} I_2 &\le C \sup_{L\le 0, L\in\mathbb{Z}} I_{21}(L) I_{22}(L) \\ &\le C \Big\| \Big(\sum_{j=1}^{\infty} |f_1^j|^{r_1} \Big)^{1/r_1} \Big\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)} \Big\| \Big(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \Big)^{1/r_2} \Big\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Step 3: To estimate I_3, for $x \in D_k,\ y_i \in D_{l_i},\ i = 1,\ 2$ and $l_1 \le k-2$, $l_2 \ge k+2$, then we have

$$|x-y_1| \ge |x| - |y_1| > 2^{k-2},\ |x-y_2| \ge |y_2| - |x| > 2^{l_2-2}.$$

$$|K(x,y_1,y_2)| \le C(|x-y_1| + |x-y_2|)^{-2n} \le C2^{-kn}2^{-l_2 n}.$$

$$\begin{aligned} \big|T(f^j_{l_1}, f^j_{l_2})(x)\big| &\le \int_{\mathbb{R}^n} |K(x,y_1,y_2)| \prod_{i=1}^{2} |f^j_{l_i}(y_i)| dy_i \\ &\le C2^{-kn}2^{-l_2 n} \prod_{i=1}^{2} \int_{\mathbb{R}^n} |f^j_{l_i}(y_i)| dy_i. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Big\| \Big(\sum_{j=1}^{\infty} \Big| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=k+2}^{\infty} T(f^j_{l_1}, f^j_{l_2}) \Big|^r \Big)^{1/r} \chi_k \Big\|_{L^{p(\cdot)}} \\ &\quad \le C \Big\| \Big(\sum_{j=1}^{\infty} \Big(\sum_{l_1=-\infty}^{k-2} \sum_{l_2=k+2}^{\infty} 2^{-kn} \\ &\qquad\qquad \times \int_{\mathbb{R}^n} |f^j_{l_1}(y_1)| dy_1 2^{-l_2 n} \int_{\mathbb{R}^n} |f^j_{l_2}(y_2)| dy_2 \Big)^r \Big)^{1/r} \chi_k \Big\|_{L^{p(\cdot)}} \\ &\quad \le C \Big\| \Big(\sum_{j=1}^{\infty} \Big(\sum_{l_1=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f^j_{l_1}(y_1)| dy_1 \Big)^{r_1} \Big)^{1/r_1} \\ &\qquad \times \Big(\sum_{j=1}^{\infty} \Big(\sum_{l_2=k+2}^{\infty} 2^{-l_2 n} \int_{\mathbb{R}^n} |f^j_{l_2}(y_2)| dy_2 \Big)^{r_2} \Big)^{1/r_2} \chi_k \Big\|_{L^{p(\cdot)}} \end{aligned}$$

$$\leq C\Big\|\Big(\sum_{j=1}^{\infty}\Big(\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{l_1}^j(y_1)|dy_1\Big)^{r_1}\Big)^{1/r_1}\chi_k\Big\|_{L^{p_1(\cdot)}}$$
$$\times\Big\|\Big(\sum_{j=1}^{\infty}\Big(\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}|f_{l_2}^j(y_2)|dy_2\Big)^{r_2}\Big)^{1/r_2}\chi_k\Big\|_{L^{p_2(\cdot)}}$$
$$\leq C\Big\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\Big)^{1/r_1}dy_1\chi_k\Big\|_{L^{p_1(\cdot)}}$$
$$\times\Big\|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\Big)^{1/r_2}dy_1\chi_k\Big\|_{L^{p_2(\cdot)}}.$$

Then,

$$I_3\leq C\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha(0)q}$$
$$\times\Big\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\Big)^{1/r_1}dy_1\chi_k\Big\|_{L^{p_1(\cdot)}}^q$$
$$\times\Big\|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\Big)^{1/r_2}dy_2\chi_k\Big\|_{L^{p_2(\cdot)}}^q\Big\}^{1/q}$$
$$\leq C\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda_1}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1}$$
$$\times\Big\|\sum_{l_1=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\Big)^{1/r_1}dy_1\chi_k\Big\|_{L^{p_1(\cdot)}}^{q_1}\Big\}^{1/q_1}$$
$$\times 2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2}$$
$$\times\Big\|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\Big)^{1/r_2}dy_2\chi_k\Big\|_{L^{p_2(\cdot)}}^{q_2}\Big\}^{1/q_2}$$
$$=: C\sup_{L\leq 0,L\in\mathbb{Z}}I_{31}(L)I_{32}(L).$$

It is easy to see that

$$I_{31}(L) = I_{21}(L) \le C\left\|\left(\sum_{j=1}^{\infty}|f_1^j|^{r_1}\right)^{1/r_1}\right\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}.$$

So we only need to estimate the term $I_{32}(L)$.

$$\begin{aligned}
&\left\|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\right)^{1/r_2}dy_2\chi_k\right\|_{L^{p_2(\cdot)}}\\
&\quad\le C\sum_{l_2=k+2}^{\infty}2^{-l_2n}\|\chi_k\|_{L^{p_2(\cdot)}}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\right)^{1/r_2}dy_2\\
&\quad\le C\sum_{l_2=k+2}^{\infty}2^{-l_2n}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\right)^{1/r_2}\chi_{l_2}\right\|_{L^{p_2(\cdot)}}\|\chi_{l_2}\|_{L^{p_2'(\cdot)}}\\
&\quad\le C\sum_{l_2=k+2}^{\infty}2^{-l_2n}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\|\chi_{B_{l_2}}\|_{L^{p_2'(\cdot)}}\left\|\left(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\right)^{1/r_2}\chi_{l_2}\right\|_{L^{p_2(\cdot)}}\\
&\quad\le C\sum_{l_2=k+2}^{\infty}2^{-l_2n}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\|\chi_{B_{l_2}}\|^{-1}_{L^{p_2(\cdot)}}|B_{l_2}|\left\|\left(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\right)^{1/r_2}\chi_{l_2}\right\|_{L^{p_2(\cdot)}}\\
&\quad\le C\sum_{l_2=k+2}^{\infty}2^{(k-l_2)n\delta_{12}}\left\|\left(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\right)^{1/r_2}\chi_{l_2}\right\|_{L^{p_2(\cdot)}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
I_{32}(L) &\le C2^{-L\lambda_2}\left\{\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2}\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)n\delta_{12}}\right.\right.\\
&\qquad\left.\left.\times\left\|\left(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\right)^{1/r_2}\chi_{l_2}\right\|_{L^{p_2(\cdot)}}\right)^{q_2}\right\}^{1/q_2}\\
&= C2^{-L\lambda_2}\left\{\sum_{k=-\infty}^{L}\left(\sum_{l_2=k+2}^{\infty}2^{k\alpha_2(0)}\right.\right.
\end{aligned}$$

$$\times \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_{l_2} \right\|_{L^{p_2(\cdot)}} \times 2^{(k-l_2)n\delta_{12}} \Bigg)^{q_2} \Bigg\}^{1/q_2}$$

$$= C2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} \Bigg(\sum_{l_2=k+2}^{\infty} 2^{l_2 \alpha_2(0)} \times \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_{l_2} \right\|_{L^{p_2(\cdot)}} 2^{d(k-l_2)} \Bigg)^{q_2} \Bigg\}^{1/q_2},$$

where $d := n\delta_{12} + \alpha_2(0) > 0$. Again we consider it into two cases. If $1 < q_2 < \infty$, then by the Hölder inequality, we obtain

$$\begin{aligned}
I_{32}(L) &\le C2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} \Bigg(\sum_{l_2=k+2}^{\infty} 2^{l_2 \alpha_2(0) q_2} \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_{l_2} \right\|_{L^{p_2(\cdot)}}^{q_2} \\
&\qquad \times 2^{dq_2(k-l_2)/2} \Bigg) \times \Bigg(\sum_{l_2=k+2}^{\infty} 2^{dq_2'(k-l_2)/2} \Bigg)^{q_2/q_2'} \Bigg\}^{1/q_2} \\
&\le C2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} \sum_{l_2=k+2}^{\infty} 2^{l_2 \alpha_2(0) q_2} \\
&\qquad \times \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_{l_2} \right\|_{L^{p_2(\cdot)}}^{q_2} \times 2^{dq_2(k-l_2)/2} \Bigg\}^{1/q_2} \\
&\le C \Bigg\{ 2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} \sum_{l_2=k+2}^{L+2} 2^{l_2 \alpha_2(0) q_2} \\
&\qquad \times \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_{l_2} \right\|_{L^{p_2(\cdot)}}^{q_2} \times 2^{dq_2(k-l_2)/2} \Bigg\}^{1/q_2} \\
&\quad + 2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{L} \sum_{l_2=L+3}^{\infty} 2^{l_2 \alpha_2(0) q_2} \\
&\qquad \times \left\| \left(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \right)^{1/r_2} \chi_{l_2} \right\|_{L^{p_2(\cdot)}}^{q_2} \times 2^{dq_2(k-l_2)/2} \Bigg\}^{1/q_2} \Bigg\} \\
&=: C\{J_1(L) + J_2(L)\}
\end{aligned}$$

Now we consider $J_1(L)$ and $J_2(L)$ respectively. Since $d > 0$, it is easy to see that

$$\begin{aligned} J_1(L) &= 2^{-L\lambda_2}\Bigg\{\sum_{l_2=-\infty}^{L+2} 2^{l_2\alpha_2(0)q_2}\Big\|\Big(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Big)^{1/r_2}\chi_{l_2}\Big\|_{L^{p_2(\cdot)}}^{q_2} \\ &\qquad \times \sum_{k=-\infty}^{l_2-2} 2^{dq_2(k-l_2)/2}\Bigg\}^{1/q_2} \\ &\le C2^{-L\lambda_2}\Bigg\{\sum_{l_2=-\infty}^{L+2} 2^{l_2\alpha_2(0)q_2}\Big\|\Big(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Big)^{1/r_2}\chi_{l_2}\Big\|_{L^{p_2(\cdot)}}^{q_2}\Bigg\}^{1/q_2} \\ &\le C\Big\|\Big(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Big)^{1/r_2}\Big\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha_2(\cdot),\lambda_2}(\mathbb{R}^n)}. \end{aligned}$$

Since $\lambda_2 - d/2 < 0$, we obtain

$$\begin{aligned} J_2(L) &\le C2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L}\sum_{l_2=L+3}^{\infty} 2^{dq_2(k-l_2)/2} \\ &\qquad \times \Bigg[\sum_{m=-\infty}^{l_2} 2^{m\alpha_2(0)q_2}\Big\|\Big(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Big)^{1/r_2}\chi_m\Big\|_{L^{p_2(\cdot)}}^{q_2}\Bigg]\Bigg\}^{1/q_2} \\ &\le C2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L}\sum_{l_2=L+3}^{\infty} 2^{dq_2(k-l_2)/2}2^{l_2q_2\lambda_2} \\ &\qquad \times \Big\|\Big(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Big)^{1/r_2}\Big\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha_2(\cdot),\lambda_2}(\mathbb{R}^n)}^{q_2}\Bigg\}^{1/q_2} \\ &= C2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L} 2^{dq_2k/2}\sum_{l_2=L+3}^{\infty} 2^{l_2q_2(\lambda_2-d/2)} \\ &\qquad \times \Big\|\Big(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Big)^{1/r_2}\Big\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha_2(\cdot),\lambda_2}(\mathbb{R}^n)}^{q_2}\Bigg\}^{1/q_2} \end{aligned}$$

$$\leq C\Bigg\{2^{-Lq_2\lambda_2}\times 2^{dq_2L/2}2^{(\lambda_2-d/2)q_2L} \times\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Bigg)^{1/r_2}\Bigg\|^{q_2}_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}\Bigg\}^{1/q_2}$$

$$\leq C\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Bigg)^{1/r_2}\Bigg\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}.$$

If $0<q_2\leq 1$, in $I_{32}(L)$, we use (8) to get

$$I_{32}(L)\leq C2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L}\sum_{l_2=k+2}^{\infty}2^{l_2\alpha_2(0)q_2} \times\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Bigg)^{1/r_2}\chi_{l_2}\Bigg\|^{q_2}_{L^{p_2(\cdot)}}2^{d(k-l_2)q_2}\Bigg\}^{1/q_2}$$

$$\leq C\Bigg\{2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L}\sum_{l_2=k+2}^{L+2}2^{l_2\alpha_2(0)q_2} \times\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Bigg)^{1/r_2}\chi_{l_2}\Bigg\|^{q_2}_{L^{p_2(\cdot)}}2^{d(k-l_2)q_2}\Bigg\}^{1/q_2}$$

$$+2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L}\sum_{l_2=L+3}^{\infty}2^{l_2\alpha_2(0)q_2} \times\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Bigg)^{1/r_2}\chi_{l_2}\Bigg\|^{q_2}_{L^{p_2(\cdot)}}2^{d(k-l_2)q_2}\Bigg\}^{1/q_2}\Bigg\}$$

$$=: C\{J_3(L)+J_4(L)\}.$$

It is clear that the estimates of $J_3(L)$ and $J_4(L)$ are similar to the process of case $1<q_2<\infty$, we can conclude that

$$I_{32}(L)\leq C\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Bigg)^{1/r_2}\Bigg\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}.$$

Therefore,

$$
\begin{aligned}
I_3 &\le C \sup_{L\le 0, L\in\mathbb{Z}} I_{31}(L) I_{32}(L)\\
&\le C \Big\| \Big(\sum_{j=1}^{\infty} |f_1^j|^{r_1} \Big)^{1/r_1} \Big\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)} \Big\| \Big(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \Big)^{1/r_2} \Big\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Step 4: Now it goes to the estimate of I_5. Using the Hölder inequality and Lemma 2.10, we get

$$
\begin{aligned}
I_5 &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{l_1=k-1}^{k+1} \sum_{l_2=k-1}^{k+1} \\
&\qquad \times \Big\| \Big(\sum_{j=1}^{\infty} \big| T(f_{l_1}^j, f_{l_2}^j) \big|^r \Big)^{1/r} \Big\|_{L^{p(\cdot)}}^q \Big)^{1/q}\\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{l_1=k-1}^{k+1} \sum_{l_2=k-1}^{k+1} \Big\| \Big(\sum_{j=1}^{\infty} |f_{l_1}^j|^{r_1} \Big)^{1/r_1} \Big\|_{L^{p_1(\cdot)}}^q \\
&\qquad \times \Big\| \Big(\sum_{j=1}^{\infty} |f_{l_2}^j|^{r_2} \Big)^{1/r_2} \Big\|_{L^{p_2(\cdot)}}^q \Big)^{1/q}\\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1} \sum_{l_1=k-1}^{k+1} \Big\| \Big(\sum_{j=1}^{\infty} |f_{l_1}^j|^{r_1} \Big)^{1/r_1} \Big\|_{L^{p_1(\cdot)}}^{q_1} \Big)^{1/q_1}\\
&\quad \times 2^{-L\lambda_2} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \sum_{l_2=k-1}^{k+1} \Big\| \Big(\sum_{j=1}^{\infty} |f_{l_2}^j|^{r_2} \Big)^{1/r_2} \Big\|_{L^{p_2(\cdot)}}^{q_2} \Big)^{1/q_2}\\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1} \Big\| \Big(\sum_{j=1}^{\infty} |f_1^j|^{r_1} \Big)^{1/r_1} \chi_k \Big\|_{L^{p_1(\cdot)}}^{q_1} \Big)^{1/q_1}\\
&\quad \times 2^{-L\lambda_2} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \Big\| \Big(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \Big)^{1/r_2} \chi_k \Big\|_{L^{p_2(\cdot)}}^{q_2} \Big)^{1/q_2}\\
&\le C \Big\| \Big(\sum_{j=1}^{\infty} |f_1^j|^{r_1} \Big)^{1/r_1} \Big\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)} \Big\| \Big(\sum_{j=1}^{\infty} |f_2^j|^{r_2} \Big)^{1/r_2} \Big\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}
\end{aligned}
$$

Step 5: Now we do the estimate of I_6. Since $k-1 \le l_1 \le k+1$ and $l_2 \ge k+2$, then for $x \in D_k, y_i \in D_{l_i}$.

$$|K(x,y_1,y_2)| \le C2^{-kn} \times 2^{-l_2 n}$$

$$\begin{aligned}\left|T(f_{l_1}^j, f_{l_2}^j)(x)\right| &\le \int_{\mathbb{R}^n} |K(x,y_1,y_2)| \prod_{i=1}^{2} |f_{l_i}^j(y_i)| dy_i \\ &\le C2^{-kn}2^{-l_2 n} \prod_{i=1}^{2} \int_{\mathbb{R}^n} |f_{l_i}^j(y_i)| dy_i.\end{aligned}$$

Applying the Hölder inequality and Minkowski inequality, we get

$$\begin{aligned}&\left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{\infty} T(f_{l_1}^j, f_{l_2}^j) \right|^r \right)^{1/r} \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} f_{l_1}^j(y_1) dy_1 2^{-l_2 n} \int_{\mathbb{R}^n} f_{l_2}^j(y_2) dy_2 \right|^r \right)^{1/r} \chi_k \right\|_{L^{p(\cdot)}} \\ &\le C \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_1=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} f_{l_1}^j(y_1) dy_1 \right|^{r_1} \right)^{1/r_1} \right. \\ &\qquad \left. \times \left(\sum_{j=1}^{\infty} \left| \sum_{l_2=k+2}^{\infty} 2^{-l_2 n} \int_{\mathbb{R}^n} f_{l_2}^j(y_2) dy_2 \right|^{r_2} \right)^{1/r_2} \chi_k \right\|_{L^{p(\cdot)}} \\ &\le C \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_1=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} f_{l_1}^j(y_1) dy_1 \right|^{r_1} \right)^{1/r_1} \chi_k \right\|_{L^{p_1(\cdot)}} \\ &\qquad \times \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l_2=k+2}^{\infty} 2^{-l_2 n} \int_{\mathbb{R}^n} f_{l_2}^j(y_2) dy_2 \right|^{r_2} \right)^{1/r_2} \chi_k \right\|_{L^{p_2(\cdot)}} \\ &\le C \left\| \sum_{l_1=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_{l_1}^j(y_1)|^{r_1} \right)^{1/r_1} dy_1 \chi_k \right\|_{L^{p_1(\cdot)}} \\ &\qquad \times \left\| \sum_{l_2=k+2}^{\infty} 2^{-l_2 n} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_{l_2}^j(y_2)|^{r_2} \right)^{1/r_2} dy_1 \chi_k \right\|_{L^{p_2(\cdot)}}.\end{aligned}$$

Then,

$$
\begin{aligned}
I_6 \le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \\
\times \Bigg\| \sum_{l_1=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \Bigg(\sum_{j=1}^{\infty} |f_{l_1}^j(y_1)|^{r_1}\Bigg)^{1/r_1} dy_1 \chi_k \Bigg\|_{L^{p_1(\cdot)}}^{q} \\
\times \Bigg\| \sum_{l_2=k+2}^{\infty} 2^{-l_2 n} \int_{\mathbb{R}^n} \Bigg(\sum_{j=1}^{\infty} |f_{l_2}^j(y_2)|^{r_2}\Bigg)^{1/r_2} dy_2 \chi_k \Bigg\|_{L^{p_2(\cdot)}}^{q} \Bigg\}^{1/q} \\
\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1} \\
\times \Bigg\| \sum_{l_1=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \Bigg(\sum_{j=1}^{\infty} |f_{l_1}^j(y_1)|^{r_1}\Bigg)^{1/r_1} dy_1 \chi_k \Bigg\|_{L^{p_1(\cdot)}}^{q_1} \Bigg\}^{1/q_1} \\
\times 2^{-L\lambda_2}\Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \\
\times \Bigg\| \sum_{l_2=k+2}^{\infty} 2^{-l_2 n} \int_{\mathbb{R}^n} \Bigg(\sum_{j=1}^{\infty} |f_{l_2}^j(y_2)|^{r_2}\Bigg)^{1/r_2} dy_2 \chi_k \Bigg\|_{L^{p_2(\cdot)}}^{q_2} \Bigg\}^{1/q_2} \\
=: C \sup_{L\le 0, L\in\mathbb{Z}} I_{61}(L) I_{62}(L).
\end{aligned}
$$

By the symmetry of f_1^j and f_2^j, we can see that the estimate of $I_{61}(L)$ is similar to that of $I_{22}(L)$ and $I_{62}(L) = I_{32}(L)$.

Step 6: In the last step, we estimate I_9. Note that $l_2 \ge k+2$ and $|x-y_i| > 2^{l_i-2}$ for $x \in D_k,\ y_i \in D_{l_i},\ i = 1,\ 2,$

$$
|K(x,y_1,y_2)| \le C(|x-y_1| + |x-y_2|)^{-2n} \le C2^{-l_1 n}2^{-l_2 n}.
$$

$$
\begin{aligned}
\left|T(f_{l_1}^j, f_{l_2}^j)(x)\right| &\le \int_{\mathbb{R}^n} |K(x,y_1,y_2)| \prod_{i=1}^{2} |f_{l_i}^j(y_i)| dy_i \\
&\le C2^{-l_1 n}2^{-l_2 n} \prod_{i=1}^{2} \int_{\mathbb{R}^n} |f_{l_i}^j(y_i)| dy_i.
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}T(f_{l_1}^j,f_{l_2}^j)\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}\\
&\quad\le C\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}2^{-l_1n}\int_{\mathbb{R}^n}f_{l_1}^j(y_1)dy_1 2^{-l_2n}\int_{\mathbb{R}^n}f_{l_2}^j(y_2)dy_2\right|^r\right)^{1/r}\chi_k\right\|_{L^{p(\cdot)}}\\
&\quad\le C\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k+2}^{\infty}2^{-l_1n}\int_{\mathbb{R}^n}f_{l_1}^j(y_1)dy_1\right|^{r_1}\right)^{1/r_1}\right.\\
&\qquad\times\left.\left(\sum_{j=1}^{\infty}\left|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}f_{l_2}^j(y_2)dy_2\right|^{r_2}\right)^{1/r_2}\chi_k\right\|_{L^{p(\cdot)}}\\
&\quad\le C\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_1=k+2}^{\infty}2^{-l_1n}\int_{\mathbb{R}^n}f_{l_1}^j(y_1)dy_1\right|^{r_1}\right)^{1/r_1}\chi_k\right\|_{L^{p_1(\cdot)}}\\
&\qquad\times\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}f_{l_2}^j(y_2)dy_2\right|^{r_2}\right)^{1/r_2}\chi_k\right\|_{L^{p_2(\cdot)}}\\
&\quad\le C\left\|\sum_{l_1=k+2}^{\infty}2^{-l_1n}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\right)^{1/r_1}dy_1\chi_k\right\|_{L^{p_1(\cdot)}}\\
&\qquad\times\left\|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\right)^{1/r_2}dy_1\chi_k\right\|_{L^{p_2(\cdot)}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_9\le C\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda}\Bigg\{\sum_{k=-\infty}^{L}&2^{k\alpha(0)q}\\
&\times\left\|\sum_{l_1=k+2}^{\infty}2^{-l_1n}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\right)^{1/r_1}dy_1\chi_k\right\|_{L^{p_1(\cdot)}}^q\\
&\times\left\|\sum_{l_2=k+2}^{\infty}2^{-l_2n}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\right)^{1/r_2}dy_2\chi_k\right\|_{L^{p_2(\cdot)}}^q\Bigg\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\Bigg\{\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1} \\
&\qquad \times \Bigg\|\sum_{l_1=k+2}^{\infty} 2^{-l_1 n}\int_{\mathbb{R}^n}\Bigg(\sum_{j=1}^{\infty}|f_{l_1}^j(y_1)|^{r_1}\Bigg)^{1/r_1} dy_1\chi_k\Bigg\|_{L^{p_1(\cdot)}}^{q_1}\Bigg\}^{1/q_1} \\
&\quad \times 2^{-L\lambda_2}\Bigg\{\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \\
&\qquad \times \Bigg\|\sum_{l_2=k+2}^{\infty} 2^{-l_2 n}\int_{\mathbb{R}^n}\Bigg(\sum_{j=1}^{\infty}|f_{l_2}^j(y_2)|^{r_2}\Bigg)^{1/r_2} dy_2\chi_k\Bigg\|_{L^{p_2(\cdot)}}^{q_2}\Bigg\}^{1/q_2} \\
&=: C \sup_{L\le 0, L\in\mathbb{Z}} I_{91}(L) I_{92}(L).
\end{aligned}
$$

Obviously, the estimate of $I_{9i}(L), i = 1, 2$ is similar to that of $I_{32}(L)$.

Combing all estimates for $I_i, i = 1, 2, \ldots, 9$, together, we get

$$
E \le C\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_1^j|^{r_1}\Bigg)^{1/r_1}\Bigg\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha_1(\cdot),\lambda_1}(\mathbb{R}^n)}\Bigg\|\Bigg(\sum_{j=1}^{\infty}|f_2^j|^{r_2}\Bigg)^{1/r_2}\Bigg\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha_2(\cdot),\lambda_2}(\mathbb{R}^n)}.
$$

Thus we complete the proof. □

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