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The lifespan of solutions to wave equations with weighted nonlinear terms in one space dimension

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Abstract. In this paper, we consider the initial value problem for nonlinear wave equation with weighted nonlinear terms in one space dimension. Kubo & Osaka & Yazici [4] studied global solvability of the problem under different conditions on the nonlinearity and initial data, together with an upper bound of the lifespan for the problem. The aim of this paper is to improve the upper bound of the lifespan and to derive its lower bound which shows the optimality of our new upper bound.

Key words: nonlinear wave equation, lifespan, one space dimension.

1. Introduction

In this paper we consider the initial value problem for nonlinear wave equations:

$$\begin{cases} u_{tt} - u_{xx} = H(x, u(x,t)), & (x,t) \in \mathbf{R} \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a scalar unknown function of space-time variables, $(f,g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$ and $\varepsilon > 0$ is a "small" parameter. The nonlinear term, H is given by

$$H(x,u) = \frac{F(u(x,t))}{(1+x^2)^{(1+a)/2}}, \tag{1.2}$$

where $a \geq -1$ and $F(u) = |u|^p$ or $|u|^{p-1}u$ with $p > 1$. Let us define the lifespan T_ε of C^2-solution of (1.1) by

$$T_\varepsilon \equiv T_\varepsilon(f,g) := \sup\{T \in (0,\infty) : \text{There exists a unique solution } u \in C^2(\mathbf{R} \times [0,T)) \text{ of (1.1)}\}$$

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with arbitrarily fixed (f,g).

First of all, we recall known results for the case $a=-1$ in general spatial dimensions:

$$\begin{cases} u_{tt}-\Delta u=|u|^p & \text{in} \quad \mathbf{R}^n\times[0,\infty), \\ u(x,0)=\varepsilon f(x),\ u_t(x,0)=\varepsilon g(x), & \text{for } x\in\mathbf{R}^n, \end{cases}$$

where $n\geq 1$. When $n\geq 2$, there exists a critical exponent $p_0(n)$ such that $T_\varepsilon=\infty$ for "small" ε if $p>p_0(n)$, and $T_\varepsilon<\infty$ for "positive" (f,g) if $1<p\leq p_0(n)$. Actually, $p_0(n)$ is a positive root of the quadratic equation $(n-1)p^2-(n+1)p-2=0$. See e.g. Introduction in Takamura & Wakasa [6] for the details.

On the other hand, when $n=1$, and (f,g) has a compact support and satisfies some positivity assumption, Kato [3] showed that $T_\varepsilon<\infty$ for any $p>1$. The difference between the cases $n\geq 2$ and $n=1$ comes from the fact that the solutions to the homogeneous wave equations has a decay estimate, $|u(x,t)|\leq(t+1)^{-(n-1)/2}$. Especially, the solution does not have decay property when $n=1$.

The result due to [3] motivates one to introduce a weight function $(1+x^2)^{-(1+a)/2}$ in the nonlinearity for getting a global solution. Actually, Suzuki [5] showed that $T_\varepsilon=\infty$ with $F(u)=|u|^{p-1}u$ for $p>(1+\sqrt{5})/2$ and $pa>1$ if f and g are odd functions and ε is small enough, and Kubo & Osaka & Yazici [4] have obtained the same conclusion for any $p>1$ satisfying $pa>1$. On the other hand, they showed that $T_\varepsilon<\infty$ for $F(u)=|u|^p$ with $p>1$ and $a\geq -1$ if (f,g) satisfies $f\equiv 0$, $g(x)\geq 0$ for $x\in\mathbf{R}$, and $\int_{\delta/2}^{\delta}g(y)dy>0$ with some $0<\delta<1$. Also, they obtained an upper bound of the lifespan, $T_\varepsilon\leq C\varepsilon^{-p^2}$, where C is a positive constant independent of ε. However, this estimate is not sharp at least in the case of $a=-1$. In fact, Zhou [7] has obtained the following estimate of the lifespan T_ε for any $p>1$,

$$c\varepsilon^{-(p-1)/2}\leq T_\varepsilon\leq C\varepsilon^{-(p-1)/2} \quad \text{if} \quad \int_{\mathbf{R}}g(x)dx\neq 0, \tag{1.3}$$

where c and C are positive constants independent of ε.

Our purpose in this paper is to extend Zhou's result to the case where $a>-1$. To obtain a blow-up result, we require the following assumptions on the data:

$$\begin{aligned}&\text{Let } f \equiv 0 \text{ and } g \in C^1(\mathbf{R}) \text{ does not vanish identically.}\\&\text{Assume } g(x) \geq 0 \text{ for all } x \in \mathbf{R} \text{ and } \int_{-1}^{1} g(y)dy > 0.\end{aligned} \tag{1.4}$$

Then, we have the following blow-up theorem.

Theorem 1.1 *Let $a \geq -1$ and $F(u) = |u|^{p-1}u$ or $|u|^p$ with $p > 1$. Assume (1.4). Then, there exist positive constants $\varepsilon_0 = \varepsilon_0(g, a, p)$ and $C = C(g, a, p)$ such that*

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-(p-1)/(1-a)} & \text{if } -1 \leq a < 0, \\ \phi^{-1}(C\varepsilon^{-(p-1)}) & \text{if } a = 0, \\ C\varepsilon^{-(p-1)} & \text{if } a > 0, \end{cases} \tag{1.5}$$

holds for any ε with $0 < \varepsilon \leq \varepsilon_0$, where $\phi = \phi(s)$ is a function defined by $\phi(s) = s\log(2 + s)$ for $s \geq 0$.

The proof of this theorem is done by an iteration argument concerning point-wise estimates. Such kind of framework was introduced by John [2] in three space dimensions. The first step of the iteration argument comes from the linear estimate of the solution to the homogeneous wave equation from below. Kubo & Osaka & Yazici [4] obtained such an estimate only in a strip domain, $\{0 \leq x - t \leq \delta/2\}$, where $0 < \delta < 1$ is a constant. On the other hand, we are able to show a similar estimate in unbounded domain, $\{t - x \geq 1\}$. This improvement enable us to establish sharp upper bound of T_ε. See Lemma 3.2 and Remark 3.2 for details.

To show the optimality of the upper bounds in Theorem 1.1, we require the following assumptions on (f, g)

$$\begin{aligned}&f \in C^2(\mathbf{R}) \text{ and } g \in C^1(\mathbf{R}) \text{ satisfy } \|f\|_{L^\infty(\mathbf{R})} < \infty\\&\text{and } \|g\|_{L^1(\mathbf{R})} < \infty.\end{aligned} \tag{1.6}$$

Then, we have the following theorem.

Theorem 1.2 *Let $a \geq -1$ and $F(u) = |u|^{p-1}u$ or $|u|^p$ with $p > 1$. Assume (1.6). Then, there exists a positive constant $c = c(f, g, a, p)$ such that*

$$T_\varepsilon \geq \begin{cases} c\varepsilon^{-(p-1)/(1-a)} & \text{if } -1 \leq a < 0, \\ \phi^{-1}(c\varepsilon^{-(p-1)}) & \text{if } a = 0, \\ c\varepsilon^{-(p-1)} & \text{if } a > 0, \end{cases} \tag{1.7}$$

holds for $\varepsilon > 0$, *where* ϕ *is the function in Theorem* 1.1.

Remark 1.1 One can easily generalize the assumption on F in Theorem 1.2 as follows:

$$\begin{aligned} &F \in C^1(\mathbf{R}) \text{ satisfies } F(0) = F'(0) = 0 \text{ and} \\ &|F'(s)| \leq pA|s|^{p-1} \text{ for } s \in \mathbf{R}, \text{ where } p > 1 \text{ and } A > 0. \end{aligned} \tag{1.8}$$

This paper is organized as follows. In the next section, we prepare some notations. The upper bounds of the lifespan and lower bounds of the lifespan are obtained in Section 3 and Section 4, respectively.

2. Notations

In this section, we give some notations and definitions.

We define

$$u^0(x,t) = \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy \tag{2.1}$$

and

$$L(V)(x,t) = \frac{1}{2}\iint_{D(x,t)} V(y,s)dyds \tag{2.2}$$

for $V \in C(\mathbf{R} \times [0,\infty))$, where

$$D(x,t) = \{(y,s) \in \mathbf{R} \times [0,\infty) : 0 \leq s \leq t, x - t + s \leq y \leq x + t - s\}.$$

For $(f,g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$, if $u \in C(\mathbf{R} \times [0,\infty))$ is a solution of

$$u(x,t) = \varepsilon u^0(x,t) + L(H(\cdot,u))(x,t), \quad (x,t) \in \mathbf{R} \times [0,\infty), \tag{2.3}$$

then $u \in C^2(\mathbf{R} \times [0,\infty))$ is the solution to the initial value problem (1.1).

For $T > 0$, we define the following domains:

$$\begin{aligned} \Gamma_1 &= \{(x,t) \in [0,\infty) \times [0,T] : t - x \geq 1\}, \\ \Gamma_2 &= \{(x,t) \in [0,\infty) \times [0,T] : x \geq t - x \geq 1\}, \\ \Sigma_j &= \{(x,t) \in [0,\infty) \times [0,T] : t - x \geq l_j\}, \end{aligned} \tag{2.4}$$

where

$$\begin{cases} l_1 = 3 \\ l_j = l_1 + \displaystyle\sum_{k=1}^{j-1} 2^{-(k-1)} = l_1 + 2\left(1 - \frac{1}{2^{j-1}}\right) \quad \text{for } j \geq 2. \end{cases} \tag{2.5}$$

3. Upper bound of the lifespan

In this section, we prove Theorem 1.1. It is sufficient to show that the solution to the integral equation,

$$u(x,t) = \varepsilon u^0(x,t) + \frac{1}{2} \iint_{D(x,t)} \frac{|u(y,s)|^p dyds}{(1+y^2)^{(1+a)/2}}, \quad (x,t) \in \mathbf{R} \times [0,\infty), \tag{3.1}$$

blows up in finite time. Because, if $u \in C(\mathbf{R} \times [0,\infty))$ is a solution of (3.1), then u satisfies $u(x,t) \geq 0$ for $(x,t) \in \mathbf{R} \times [0,\infty)$ by the assumptions in (1.4). Therefore, this u must solve the equation (2.3) with $F(u) = |u|^{p-1}u$ by the uniqueness of solutions to (1.1).

Before proving Theorem 1.1, we prepare the following lemmas:

Lemma 3.1 *Let $p > 1$, $a \geq -1$ and let us define a sequence*

$$\begin{cases} C_{a,j} = \exp\{p^{j-1}(\log(C_{a,1} F_{p,a}^{-S_j} E_{p,a}^{1/(p-1)})) - \log E_{p,a}^{1/(p-1)}\} \ (j \geq 2), \\ C_{a,1} = c_0^p k_a \varepsilon^p, \end{cases} \tag{3.2}$$

where

$$E_{p,a} = \begin{cases} (p-1)^2/(2^{a+5}p^2), & \text{if } -1 \leq a < 0, \\ (p-1)^2/(2p^2), & \text{if } a = 0, \\ (p-1)/(2^{a+2}p). & \text{if } a > 0, \end{cases} \tag{3.3}$$

$$F_{p,a} = \begin{cases} p^2, & \textit{if } -1 \le a \le 0, \\ 2p & \textit{if } a > 0, \end{cases} \tag{3.4}$$

$$k_a = \begin{cases} 2^{-(a+4)}, & \textit{if } -1 \le a < 0, \\ 2^{-1}, & \textit{if } a = 0, \\ 2^{-(a+2)}. & \textit{if } a > 0, \end{cases} \tag{3.5}$$

and

$$S_j = \sum_{i=1}^{j-1} \frac{i}{p^i}. \tag{3.6}$$

Then, we have the following relation:

$$C_{a,j+1} = \frac{C_{a,j}^p E_{p,a}}{F_{p,a}^j} \quad (j \in \mathbf{N}). \tag{3.7}$$

Proof. First, we shall show (3.7) for $j = 1$. One can easily get

$$\begin{aligned} \log\left(\frac{C_{a,1}^p E_{p,a}}{F_{p,a}}\right) &= p\log(C_{a,1}F_{p,a}^{-1/p}) + \log E_{p,a} \\ &= p\log(C_{a,1}F_{p,a}^{-1/p}E_{p,a}^{1/(p-1)}) - \log E_{p,a}^{1/(p-1)} = \log C_{a,2}. \end{aligned}$$

Hence (3.7) holds for $j = 1$. Next, we shall show (3.7) for $j \ge 2$. Note that (3.7) is equivalent to

$$\log C_{a,j+1} = p\log C_{a,j} - j\log F_{p,a} + \log E_{p,a}.$$

By (3.2) and the expression of S_j in (3.6), the right-hand side of this identity is equal to

$$\begin{aligned} &p^j\big\{\log(C_{a,1}F_{p,a}^{-S_j}E_{p,a}^{1/(p-1)})\big\} - p\log E_{p,a}^{1/(p-1)} - j\log F_{p,a} + \log E_{p,a} \\ &\quad = p^j\big\{\log(C_{a,1}F_{p,a}^{-S_{j+1}}E_{p,a}^{1/(p-1)})\big\} \\ &\qquad + p^j\log F_{p,a}^{j/p^j} - j\log F_{p,a} - \log E_{p,a}^{1/(p-1)} \\ &\quad = p^j\big\{\log(C_{a,1}F_{p,a}^{-S_{j+1}}E_{p,a}^{1/(p-1)})\big\} - \log E_{p,a}^{1/(p-1)}. \end{aligned}$$

Hence, we obtain (3.7) by (3.2) with j replaced by $j+1$. This completes the proof. □

Remark 3.1 For the proof of Lemma 3.1 itself, the definitions of $E_{p,a}$ (3.3), $F_{p,a}$ (3.4) and k_a (3.5) are not necessary, but only the positivity of them is enough.

Next, we derive a lower bound of the solution to (3.1) which is a starting point of our iteration argument.

Lemma 3.2 *Suppose that the assumptions in Theorem* 1.1 *are fulfilled. Let* $u \in C(\mathbf{R} \times [0,T])$ *be the solution of* (3.1)*. Then, u satisfies*

$$u(x,t) \geq \varepsilon c_0 \quad \textit{for} \quad (x,t) \in \Gamma_1, \tag{3.8}$$

where $c_0 = (1/2)\int_{-1}^{1} g(y)dy > 0$ *and* $\Gamma_1 (= \{(x,t) \in [0,\infty) \times [0,T] : t-x \geq 1\})$ *is the one in* (2.4)*.*

Proof. By (1.4) and (2.1), we get

$$\varepsilon u^0(x,t) = \frac{\varepsilon}{2}\int_{x-t}^{x+t} g(y)dy \geq \varepsilon c_0 \quad \text{for} \quad (x,t) \in \Gamma_1.$$

Making use of the positivity of the second term of right-hand side in (3.1), we have (3.8). This completes the proof. □

Remark 3.2 In three space dimensions, the following estimate which is necessary to get the first step of the iteration argument was obtained by John [2] in a strip domain: For $(x,t) \in S$, we have

$$u^0(x,t) \geq Cr^{-1},$$

where $r = |x|$, C is a positive constant and $S = \{(r,t) \in (0,\infty) \times [0,\infty) : \delta \leq t - r \leq \delta'\}$, with some δ', δ ($\delta' > \delta > 0$).

On the contrary, our estimate holds in some domain without any restriction of upper bound for $t-x$. This is the key point to obtain sharp upper bound of T_ε.

Our iteration argument will be done by using the following estimates.

Proposition 3.1 *Suppose that the assumptions in Theorem* 1.1 *are fulfilled. Let $j\in\mathbf{N}$ and let $u\in C(\mathbf{R}\times[0,T])$ be the solution of* (3.1). *Then, u satisfies*

$$u(x,t)\geq C_{a,j}\{(t-x)^{-(a+1)}(t-x-1)^2\}^{a_j}\quad \text{if } -1\leq a<0, \tag{3.9}$$

for $(x,t)\in\Gamma_2$, and

$$u(x,t)\geq C_{0,j}\{(t-x-1)\log(1+x)\}^{a_j}\quad \text{if } a=0, \tag{3.10}$$

for $(x,t)\in\Gamma_1$, and

$$u(x,t)\geq C_{a,j}(t-x-l_j)^{a_j}\quad \text{if } a>0, \tag{3.11}$$

for $(x,t)\in\Sigma_j$, where Γ_1, Γ_2 and Σ_j are defined in (2.4). *Here $C_{a,j}$ is the one in* (3.2) *and a_j is defined by*

$$a_j=\frac{p^j-1}{p-1}\quad (j\in\mathbf{N}). \tag{3.12}$$

Proof. We shall show (3.9), (3.10) and (3.11) by induction. Noticing that $u^0(x,t)\geq 0$ for $(x,t)\in\mathbf{R}\times[0,\infty)$ and $(1+y^2)^{1/2}\leq 1+|y|$, we get

$$u(x,t)\geq\frac{1}{2}\iint_{D(x,t)}\frac{|u(y,s)|^p}{(1+|y|)^{1+a}}dyds\quad \text{in } \mathbf{R}\times[0,\infty). \tag{3.13}$$

(i) Estimate in the case of $-1\leq a<0$.

Let $(x,t)\in\Gamma_2$. Define

$$T_1(x,t):=\{(y,s)\in D(x,t):1\leq s-y\leq t-x\leq y, s+y\leq t+x\}.$$

Changing the variables in the integral of (3.13) by

$$\alpha=s+y,\ \beta=s-y \tag{3.14}$$

and replacing the domain of integration by $T_1(x,t)$, we get

$$u(x,t)\geq\frac{1}{4}\int_1^{t-x}d\beta\int_{2(t-x)+\beta}^{t+x}\frac{|u(y,s)|^p}{\{1+(\alpha-\beta)/2\}^{1+a}}d\alpha\quad \text{in } \Gamma_2. \tag{3.15}$$

Making use of (3.8) and $T_1(x,t) \subset \Gamma_1$ for $(x,t) \in \Gamma_2$, we have

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

Note that $x \geq t-x$ is equivalent to $t+x \geq 3(t-x)$, we get

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{3(t-x)} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

It follows from

$$1+\frac{\alpha-\beta}{2} \leq 1+\frac{3(t-x)-1}{2} \leq 2(t-x)$$

for $\alpha \leq 3(t-x)$, $\beta \geq 1$ and $t-x \geq 1$ that

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{2^{a+3}(t-x)^{1+a}} \int_1^{t-x} (t-x-\beta) d\beta = C_{a,1} \frac{(t-x-1)^2}{(t-x)^{1+a}} \quad \text{in } \Gamma_2.$$

Therefore, (3.9) holds for $j=1$.

Assume that (3.9) holds. Noticing that $T_1(x,t) \subset \Gamma_2$ for $(x,t) \in \Gamma_2$ and putting (3.9) into (3.15), we have

$$u(x,t) \geq \frac{C_{a,j}^p}{4} \int_1^{t-x} \frac{(\beta-1)^{2pa_j}}{\beta^{p(a+1)a_j}} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

Analogously to the case of $j=1$, we get

$$\begin{aligned} u(x,t) &\geq \frac{C_{a,j}^p}{2^{a+3}(t-x)^{(a+1)(pa_j+1)}} \int_1^{t-x} (\beta-1)^{2pa_j} d\beta \int_{2(t-x)+\beta}^{3(t-x)} d\alpha \\ &= \frac{C_{a,j}^p}{2^{a+3}(t-x)^{(a+1)(pa_j+1)}} \int_1^{t-x} (\beta-1)^{2pa_j} (t-x-\beta) d\beta \end{aligned}$$

in Γ_2. Making use of integration by parts to the integral above, we have

$$u(x,t) \geq \frac{C_{a,j}^p (t-x-1)^{2(pa_j+1)}}{2^{a+5}(pa_j+1)^2 (t-x)^{(a+1)(pa_j+1)}} \quad \text{in } \Gamma_2.$$

Recalling the definition of a_j, we have

$$a_{j+1} = pa_j + 1 \leq \frac{p^{j+1}}{p-1}. \tag{3.16}$$

Making use of (3.7), we get

$$u(x,t) \geq \frac{C_{a,j}^p (p-1)^2}{2^{a+5} p^{2(j+1)}} \cdot \frac{(t-x-1)^{2a_{j+1}}}{(t-x)^{(a+1)a_{j+1}}} = C_{a,j+1} \frac{(t-x-1)^{2a_{j+1}}}{(t-x)^{(a+1)a_{j+1}}}$$

in Γ_2. Therefore, (3.9) holds for all $j \in \mathbf{N}$.

(ii) Estimate in the case of $a = 0$.

Let $(x,t) \in \Gamma_1$. Define

$$T_2(x,t) := \{(y,s) \in D(x,t) : 1 \leq s-y \leq t-x, s+y \leq t+x, y \geq 0\}.$$

Changing the variables by (3.14) in the integral of (3.13) and replacing the domain of integration by $T_2(x,t)$, we get

$$u(x,t) \geq \frac{1}{4} \int_1^{t-x} d\beta \int_\beta^{t+x} \frac{|u(y,s)|^p}{1+(\alpha-\beta)/2} d\alpha \quad \text{in } \Gamma_1. \tag{3.17}$$

By making use of (3.8) and $T_2(x,t) \subset \Gamma_1$ for $(x,t) \in \Gamma_1$, we get

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_\beta^{t+x} \frac{d\alpha}{1+(\alpha-\beta)/2} \quad \text{in } \Gamma_1.$$

Noticing that

$$\int_\beta^{t+x} \frac{d\alpha}{1+(\alpha-\beta)/2} = 2\log\left(1 + \frac{t+x-\beta}{2}\right) \geq 2\log(1+x),$$

for $\beta \leq t-x$, we obtain

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{2} \log(1+x) \int_1^{t-x} d\beta = C_{0,1}(t-x-1)\log(1+x) \quad \text{in } \Gamma_1.$$

Therefore, (3.10) holds for $j = 1$.

Assume that (3.10) holds. Noticing that $T_2(x,t) \subset \Gamma_1$ for $(x,t) \in \Gamma_1$ and putting (3.10) into (3.17), we have

$$u(x,t) \geq \frac{C_{0,j}^p}{4} \int_1^{t-x} (\beta-1)^{pa_j} d\beta \int_\beta^{t+x} \frac{\{\log(1+(\alpha-\beta)/2)\}^{pa_j} d\alpha}{1+(\alpha-\beta)/2} \quad \text{in } \Gamma_1.$$

Analogously to the case of $j=1$, we get

$$\begin{aligned} u(x,t) &\geq \frac{C_{0,j}^p}{2(pa_j+1)} \int_1^{t-x} (\beta-1)^{pa_j} \left\{ \log\left(1+\frac{t+x-\beta}{2}\right)\right\}^{pa_j+1} d\beta \\ &\geq \frac{C_{0,j}^p \{\log(1+x)\}^{pa_j+1}}{2(pa_j+1)} \int_1^{t-x} (\beta-1)^{pa_j} d\beta \end{aligned}$$

in Γ_1. It follows from (3.16) and (3.7) that

$$\begin{aligned} u(x,t) &\geq \frac{C_{0,j}^p (p-1)^2}{2p^{2(j+1)}} \cdot \{(t-x-1)\log(1+x)\}^{a_{j+1}} \\ &= C_{0,j+1}\{(t-x-1)\log(1+x)\}^{a_{j+1}} \end{aligned}$$

in Γ_1. Therefore, (3.10) holds for all $j \in \mathbf{N}$.

(iii) Estimate in the case of $a > 0$.

In this case, we use the "slicing" method which is introduced by Agemi & Kurokawa & Takamura [1]. Let $(x,t) \in \Sigma_1$. Define

$$L_1(x,t) := \{(y,s) \in D(x,t) : 1 \leq s-y \leq t-x-2, 0 \leq y \leq 1\}.$$

Changing the variables by (3.14) in the integral of (3.13) and replacing the domain of integration by $L_1(x,t)$, we get

$$u(x,t) \geq \frac{1}{4} \int_1^{t-x-2} d\beta \int_\beta^{2+\beta} \frac{|u(y,s)|^p d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Sigma_1.$$

By making use of (3.8) and $L_1(x,t) \subset \Gamma_1$ for $(x,t) \in \Sigma_1$, we have

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x-2} d\beta \int_\beta^{2+\beta} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Sigma_1.$$

It follows from $1+(\alpha-\beta)/2 \le 2$ for $\alpha \le 2+\beta$ that

$$u(x,t) \ge \frac{c_0^p \varepsilon^p}{2^{a+2}} \int_1^{t-x-2} d\beta = C_{a,1}(t-x-3) \quad \text{in } \Sigma_1.$$

Therefore, (3.11) holds for $j=1$.

Assume that (3.11) holds. Let $(x,t) \in \Sigma_{j+1}$. Define

$$L_j(x,t) := \{(y,s) \in D(x,t) : l_j \le s-y \le t-x-2^{-(j-1)}, 0 \le y \le 2^{-j}\}$$

for $j \ge 1$, where l_j is defined in (2.5). Making use of (3.14) and replacing the domain of integration in (3.13) by $L_j(x,t)$, we have

$$u(x,t) \ge \frac{1}{4} \int_{l_j}^{t-x-2^{-(j-1)}} d\beta \int_\beta^{2^{-(j-1)}+\beta} \frac{|u(y,s)|^p d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Sigma_{j+1}.$$

Noticing that $L_j(x,t) \subset \Sigma_j$ for $(x,t) \in \Sigma_{j+1}$ and putting (3.11) into the integral above, we have

$$u(x,t) \ge \frac{C_{a,j}^p}{4} \int_{l_j}^{t-x-2^{-(j-1)}} (\beta - l_j)^{pa_j} d\beta \int_\beta^{2^{-(j-1)}+\beta} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}}$$

in Σ_{j+1}. Note that

$$1+\frac{\alpha-\beta}{2} \le 1+\frac{1}{2^j} \le 2$$

for $\alpha \le 2^{-(j-1)}+\beta$, we get

$$u(x,t) \ge \frac{C_{a,j}^p}{2^{a+2+j}} \int_{l_j}^{t-x-2^{-(j-1)}} (\beta - l_j)^{pa_j} d\beta \quad \text{in } \Sigma_{j+1}.$$

It follows from $l_j + 2^{-(j-1)} = l_{j+1}$, (3.16) and (3.7) that

$$u(x,t) \ge \frac{(p-1)C_{a,j}^p}{2^{a+2+j} p^{j+1}} \cdot (t-x-l_{j+1})^{a_{j+1}} = C_{a,j+1}(t-x-l_{j+1})^{a_{j+1}}$$

in Σ_{j+1}. Therefore, (3.11) holds for all $j \in \mathbf{N}$. The proof of Proposition 3.1

is now completed. □

End of the proof of Theorem 1.1. Let $u \in C(\mathbf{R} \times [0,T])$ be the solution of the integral equation, (3.1). Setting $S = \lim_{j\to\infty} S_j$, we see from (3.6) that $S_j \le S$ for all $j \in \mathbf{N}$. Therefore, (3.2) yields

$$\begin{aligned} C_{a,j} &\ge \exp\left\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\} - \log E_{p,a}^{1/(p-1)}\right\} \\ &= E_{p,a}^{-1/(p-1)} \exp\left\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\right\}. \end{aligned} \tag{3.18}$$

(i) The lifespan in the case of $-1 \le a < 0$.

We take $\varepsilon_0 = \varepsilon_0(g,a,p) > 0$ so small that

$$B_1 \varepsilon_0^{-(p-1)/(1-a)} \ge 4,$$

where we set

$$B_1 = (c_0^p 2^{-(a+4)+p(a-3)/(p-1)} p^{-2S} E_{p,a}^{1/(p-1)})^{-(p-1)/p(1-a)} > 0.$$

Next, for a fixed $\varepsilon \in (0, \varepsilon_0]$, we suppose that T satisfies

$$T > B_1 \varepsilon^{-(p-1)/(1-a)} \ (\ge 4). \tag{3.19}$$

Combining (3.18) with (3.9), we have

$$\begin{aligned} u(x,t) \ge\ & E_{p,a}^{-1/(p-1)} \exp\left\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\right\} \\ & \times \left\{\frac{(t-x-1)^2}{(t-x)^{(1+a)}}\right\}^{(p^j-1)/(p-1)} \end{aligned}$$

in Γ_2. Note that $t-x-1 \ge (t-x)/2$ is equivalent to $t-x \ge 2$. Furthermore, we have $(t/2,t) \in \Gamma_2$ for $t \in [4,T]$. Hence we get

$$\begin{aligned} u(t/2,t) &\ge (2^{a-3}E_{p,a})^{-1/(p-1)} \\ &\quad \times \exp\left\{p^{j-1}\{\log(2^{p(a-3)/(p-1)}C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\right\} \\ &\quad \times t^{(1-a)(p^j-1)/(p-1)} \\ &= (2^{a-3}E_{p,a})^{-1/(p-1)} \exp\{p^{j-1}K_1(t)\} t^{-(1-a)/(p-1)} \end{aligned}$$

for $t \in [4, T]$, where we set

$$K_1(t) = \log\left(\varepsilon^p c_0^p 2^{-(a+4)+p(a-3)/(p-1)} p^{-2S} E_{p,a}^{1/(p-1)} t^{p(1-a)/(p-1)}\right)$$

(recall (3.4) and (3.5)).

By (3.19) and the definition of B_1, we have $K_1(T) > 0$. Therefore we get $u(T/2, T) \to \infty$ as $j \to \infty$. Hence, (3.19) implies that $T_\varepsilon \le B_1 \varepsilon^{-(p-1)/(1-a)}$ for $0 < \varepsilon \le \varepsilon_0$.

(ii) The lifespan in the case of $a = 0$.

We take $\varepsilon_1 = \varepsilon_1(g, p) > 0$ so small that

$$\phi^{-1}(B_2 \varepsilon_1^{-(p-1)}) \ge 4,$$

where ϕ is the one in Theorem 1.1 and

$$B_2 = \left(c_0^p 2^{-1-3p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)}\right)^{-(p-1)/p} > 0.$$

Next, for a fixed $\varepsilon \in (0, \varepsilon_1]$, we suppose that T satisfies

$$T > \phi^{-1}(B_2 \varepsilon^{-(p-1)}) \ (\ge 4). \tag{3.20}$$

Combining the estimates (3.18) and (3.10), we have

$$\begin{aligned} u(t/2, t) \ge {} & (2^{-2} E_{p,0})^{-1/(p-1)} \\ & \times \exp\left\{p^{j-1}\{\log(\varepsilon^p c_0^p 2^{-1-2p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)})\}\right\} \\ & \times \{t \log(1 + t/2)\}^{(p^j-1)/(p-1)} \end{aligned}$$

for $4 \le t \le T$. Noticing that

$$\log\left(1 + \frac{t}{2}\right) = \log(2 + t) - \log 2 \ge \frac{\log(2 + t)}{2} \quad \text{for } t \ge 2,$$

we get

$$u(t/2, t) \ge (2^{-3} E_{p,0})^{-1/(p-1)} \exp\{p^{j-1} K_2(t)\} \phi(t)^{-1/(p-1)}$$

for $4 \le t \le T$, where we set

$$K_2(t) = \log\left(\varepsilon^p c_0^p 2^{-1-3p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)} \{\phi(t)\}^{p/(p-1)}\right).$$

Analogously to the case of $-1 \le a < 0$, we have $K_2(T) > 0$ by (3.20) and the definition of B_2. Therefore we get $u(T/2,T) \to \infty$ as $j \to \infty$. Hence, (3.20) implies that $T_\varepsilon \le \phi^{-1}(B_2\varepsilon^{-(p-1)})$ for $0 < \varepsilon \le \varepsilon_1$.

(iii) The lifespan in the case of $a > 0$.
We take $\varepsilon_2 = \varepsilon_2(g,a,p) > 0$ so small that

$$B_3\varepsilon_2^{-(p-1)} \ge 20,$$

where we set

$$B_3 = \left(c_0^p 2^{-(a+2)-2p/(p-1)} (2p)^{-S} E_{p,a}^{1/(p-1)}\right)^{-(p-1)/p} > 0.$$

Next, for a fixed $\varepsilon \in (0, \varepsilon_2]$, we suppose that T satisfies

$$T > B_3\varepsilon^{-(p-1)} \ (\ge 20). \tag{3.21}$$

Combining the estimates (3.18) with (3.11), we have

$$u(t/2,t) \ge (2^{-2}E_{p,a})^{-1/(p-1)} \exp\{p^{j-1}K_3(t)\} t^{-1/(p-1)}$$

for $20 \le t \le T$, where we set

$$K_3(t) = \log\left(\varepsilon^p c_0^p 2^{-(a+2)-2p/(p-1)} (2p)^{-S} E_{p,a}^{1/(p-1)} t^{p/(p-1)}\right).$$

Since $K_3(T) > 0$, by (3.21) and the definition of B_3, we get $u(T/2,T) \to \infty$ as $j \to \infty$. Hence, (3.21) implies that $T_\varepsilon \le B_3\varepsilon^{-(p-1)}$ for $0 < \varepsilon \le \varepsilon_2$. Therefore, the proof of Theorem 1.1 is now completed. □

4. Lower bound of the lifespan

In this section, we prove Theorem 1.2. First of all, we introduce a Banach space

$$X = \{u \in C(\mathbf{R} \times [0,T]) : \|u\|_{L^\infty(\mathbf{R}\times[0,T])} < \infty\}, \tag{4.1}$$

which is equipped with a norm

$$\|u\|_{L^\infty(\mathbf{R}\times[0,T])} = \sup_{(x,t)\in\mathbf{R}\times[0,T]} |u(x,t)|. \tag{4.2}$$

We also define a closed subspace Y in X by

$$Y = \{u \in X : \|u\|_{L^\infty(\mathbf{R}\times[0,T])} \le 2M\varepsilon\},$$

where we set

$$M = \|f\|_{L^\infty(\mathbf{R})} + \|g\|_{L^1(\mathbf{R})}.$$

We shall construct a solution of the integral equation (2.3) in Y under suitable assumption on T such as (4.7) below. Define a sequence of functions $\{u_n\}_{n\in\mathbf{N}}$ by

$$u_n = u_0 + L(H(\cdot, u_{n-1})), \quad u_0 = \varepsilon u^0, \tag{4.3}$$

where L, H and u^0 are given by (2.2), (1.2) and (2.1), respectively. Since $\|u_0\|_{L^\infty(\mathbf{R}\times[0,T])} \le M\varepsilon$ by (2.1), we have $u_0 \in Y$.

The following *a priori* estimate plays a key role in the proof of Theorem 1.2.

Lemma 4.1 *Let $V \in X$, $a \ge -1$, and let $D = D(\tau)$ is a function defined by*

$$D(\tau) = \begin{cases} (1+\tau)^{1-a} & \text{if } -1 \le a < 0, \\ \phi(\tau) & \text{if } a = 0, \\ 1+\tau & \text{if } a > 0, \end{cases} \tag{4.4}$$

for $\tau \ge 0$, where ϕ is the one in Theorem 1.1*. Then, there exists a positive constant C_a such that*

$$\left\| L\left(\frac{V}{(1+|\cdot|^2)^{(1+a)/2}} \right) \right\|_{L^\infty(\mathbf{R}\times[0,T])} \le C_a D(T) \|V\|_{L^\infty(\mathbf{R}\times[0,T])}. \tag{4.5}$$

Proof. Noticing that $(1+y^2) \ge (1+|y|)^2/2$, the left-hand side in (4.5) is dominated by

$$C_a\|V\|_{L^\infty(\mathbf{R}\times[0,T])}\iint_{D(x,t)}\frac{dyds}{\langle y\rangle^{1+a}},$$

where we set $\langle y\rangle = 1+|y|$. Thus, it is enough to show the inequality,

$$I(x,t)\le C_a D(T) \quad \text{for } (x,t)\in\mathbf{R}\times[0,T], \tag{4.6}$$

where we set

$$I(x,t)=\iint_{D(x,t)}\frac{dyds}{\langle y\rangle^{1+a}}.$$

We may assume $x\ge 0$. Because $I(x,t)$ is an even function with respect to x. When $t\ge x\ge 0$, we divide the integral domain $D(x,t)$ into two parts $D_j(x,t)$ $(j=1,2)$, where

$$D_1(x,t)=\{(y,s)\in\mathbf{R}\times[0,\infty): 0\le s\le t-x, x-t+s\le y\le t-x-s\},$$
$$D_2(x,t)=\{(y,s)\in[0,\infty)^2: 0\le s\le t, |x-t+s|\le y\le x+t-s\}.$$

Namely, we set

$$I_j(x,t)=\iint_{D_j(x,t)}\frac{1}{\langle y\rangle^{1+a}}dyds \quad (j=1,2),$$

so that $I(x,t)=I_1(x,t)+I_2(x,t)$. We shall estimate I_1. Since $\langle y\rangle$ is an even function, we obtain

$$I_1(x,t)=2\int_0^{t-x}ds\int_0^{t-x-s}\frac{dy}{(1+y)^{1+a}} \quad \text{for } t\ge x\ge 0.$$

Then, the y-integral is dominated by

$$\begin{cases} -a^{-1}(1+t-x)^{-a} & \text{if } a<0,\\ \log(1+t-x) & \text{if } a=0,\\ a^{-1} & \text{if } a>0.\end{cases}$$

Hence, we get

$$I_1(x,t) \le C_a D(t-x) \le C_a D(T) \quad \text{for } 0 \le x \le t \le T.$$

Next, we shall estimate I_2. It follows that

$$I_2(x,t) = \int_0^t ds \int_{|x-t+s|}^{t-x-s} \frac{dy}{(1+y)^{1+a}} \le \int_0^t ds \int_0^{t+x-s} \frac{dy}{(1+y)^{1+a}}$$

for $t \ge x \ge 0$, and that the y-integral is dominated by

$$\begin{cases} -a^{-1}(1+t+x)^{-a} & \text{if } a<0, \\ \log(1+t+x) & \text{if } a=0, \\ a^{-1} & \text{if } a>0. \end{cases}$$

Noticing that

$$\log(1+2t) \le \log 2 + \log(2+t) \le 2\log(2+t) \quad \text{for } t \ge 0,$$

we get

$$I_2(x,t) \le C_a D(t+x) \le C_a D(T) \quad \text{for } 0 \le x \le t \le T.$$

When $x \ge t$, we have

$$I(x,t) \le \int_0^t \frac{ds}{(1+s)^{1+a}} \int_{x-t+s}^{x+t-s} dy \le 2t \int_0^t \frac{ds}{(1+s)^{1+a}} \le C_a D(T).$$

Therefore, the proof of Lemma 4.1 is ended. □

Now, we move on to the proof of Theorem 1.2. First of all, we take $T>0$ such that

$$2^{p+1} p C_a D(T) M^{p-1} \varepsilon^{p-1} \le 1, \tag{4.7}$$

where C_a is the one in Lemma 4.1. We shall show

$$\|u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le 2M\varepsilon \quad (n \in \mathbf{N}), \tag{4.8}$$

by induction. Assume that $\|u_{n-1}\|_{L^\infty(\mathbf{R}\times[0,T])} \le 2M\varepsilon$ $(n \ge 2)$. It follows from (4.3) and Lemma 4.1 that

$$\begin{aligned}\|u_n\|_{L^\infty(\mathbf{R}\times[0,T])} &\le \|u_0\|_{L^\infty(\mathbf{R}\times[0,T])} + \|L(H(\cdot,u_{n-1}))\|_{L^\infty(\mathbf{R}\times[0,T])} \\ &\le M\varepsilon + C_a D(T)\|u_{n-1}\|^p_{L^\infty(\mathbf{R}\times[0,T])}.\end{aligned}$$

The assumption of the induction yields that

$$\|u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le M\varepsilon + C_a(2M\varepsilon)^p D(T).$$

This inequality shows (4.8), provided (4.7) holds.

Next we shall estimate the differences of $\{u_n\}_{n\in\mathbf{N}}$. Since

$$\begin{aligned}|H(y,u_n) - H(y,u_{n-1})| \le \frac{p}{(1+y^2)^{(1+a)/2}} &(|u_{n-1}(y,s)|^{p-1} + |u_n(y,s)|^{p-1}) \\ &\times |u_n(y,s) - u_{n-1}(y,s)|\end{aligned}$$

for $(y,s)\in\mathbf{R}\times[0,\infty)$, we see from Lemma 4.1 that

$$\begin{aligned}&\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \\ &\quad\le pC_a D(T)\big(\|u_n\|^{p-1}_{L^\infty(\mathbf{R}\times[0,T])} + \|u_{n-1}\|^{p-1}_{L^\infty(\mathbf{R}\times[0,T])}\big) \\ &\qquad\times \|u_n - u_{n-1}\|_{L^\infty(\mathbf{R}\times[0,T])}.\end{aligned}$$

Making use of (4.8), we have

$$\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le \frac{1}{2}\|u_n - u_{n-1}\|_{L^\infty(\mathbf{R}\times[0,T])} \quad \text{for } n\in\mathbf{N}$$

provided (4.7) holds. Hence, we obtain

$$\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le \frac{1}{2^n}\|u_1 - u_0\|_{L^\infty(\mathbf{R}\times[0,T])} \quad \text{for } n\in\mathbf{N}.$$

Therefore, $\{u_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in Y provided (4.7) holds. Since Y is complete, there exists $u\in Y$ such that u_n converges uniformly to u in Y. Therefore, by taking limits under the integral sign, u satisfies the integral equation (2.3), so that u is the C^2-solution of (1.1). Hence, the proof of Theorem 1.2 is completed. □

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