Hokkaido Mathematical Journal Vol. 33 (2004) p. 443–456

Fredholm and Volterra integral equations with integrable singularities

Ravi P. AGARWAL and Donal O'REGAN

(Received September 24, 2002; Revised April 23, 2003)

Abstract. Using Schauder's fixed point theorem we present a new approach to establishing existence to Fredholm and Volterra integral equations where the nonlinearity may be singular in the dependent variable.

Key words: Integral equations, Fredholm, Volterra, existence, integrable singularities.

1. Introduction

This paper presents a new approach to singular integral equations. In particular we discuss the Fredholm integral equations

$$y(t)=\int_0^1 k(t,\,s)f(s,\,y(s))\,ds \quad \text{for} \quad t\in[0,\,1],$$

$$y(t)=\int_0^\infty k(t,\,s)f(s,\,y(s))\,ds \quad \text{for} \quad t\in[0,\,\infty),$$

and the Volterra equation

$$y(t)=\int_0^t k(t,\,s)f(s,\,y(s))\,ds \quad \text{for} \quad t\in[0,\,T] \text{ (here } T>0 \text{ is fixed).}$$

Our nonlinearity may be singular in the independent variable and may also be singular at $y=0$. There are only a handful of papers in the literature (see [1–9] and the references therein) where integral equations singular in the dependent variable are discussed. These papers involving integral equations with a general kernel only consider the case when $f(t,\,y)=y^{-a}$, $a>0$. For example the problem

$$y(t)=\int_0^1 k(t,\,s)\frac{1}{y(s)}\,ds \quad \text{for} \quad t\in[0,\,1],$$

arises in a problem in communications (for a description see [7]). This type of problem also arises in boundary layer theory in fluid mechanics [9]. In

---

2000 Mathematics Subject Classification : 45B05, 45D05, 45G05.

[5], Karlin and Nirenberg considered the more general problem

$$y(t) = \int_0^1 k(t,\, s)\frac{1}{[y(s)]^a}\, ds \quad \text{for} \quad t \in [0,\, 1],$$

where $a > 0$ is fixed and k is a nonnegative continuous function on $[0,\, 1] \times [0,\, 1]$. In this paper to establish existence we use Schauder's fixed point theorem (and the Schauder–Tychonoff theorem in the last result). Our results are new since new nonlinearities f are considered (i.e. nonlinearities other than $f(t,\, y) = y^{-a}$ are considered). For example in [5] the following restrictive conditions were assumed: (i). $f(t,\, y)$ is bounded as $y \to \infty$, (ii). $k(t,\, s)$ is continuous and bounded, and (iii). $k(t,\, t) > 0$ for all $t > 0$. Condition (i)–(iii) will not be needed in our main theorem.

2. Existence theory

Our first three results concerns the Fredholm integral equation

$$y(t) = \int_0^1 k(t,\, s) f(s,\, y(s))\, ds \quad \text{for} \quad t \in [0,\, 1] \tag{2.1}$$

where f may be singular in the dependent variable. We note that f may be singular also in the independent variable at some set $\Omega \subseteq [0,\, 1]$ with measure zero.

Theorem 2.1 *Let $1 \le p \le \infty$ and let q be the conjugate to p. Suppose the following conditions are satisfied:*

$$(2.2) \quad \begin{cases} \text{for all } t \in [0,\, 1],\ k_t(s) = k(t,\, s) \ge 0 \text{ for a.e. } s \in [0,\, 1] \\ \text{and for a.e. } t \in [0,\, 1],\ k_t(s) > 0 \text{ for a.e. } s \in [0,\, 1] \end{cases}$$

$$(2.3) \quad \begin{cases} k_t(s) \in L^p[0,\, 1] \text{ for each } t \in [0,\, 1] \text{ and the map} \\ t \mapsto k_t \text{ is continuous from } [0,\, 1] \text{ to } L^p[0,\, 1] \end{cases}$$

$$(2.4) \quad \begin{cases} f\colon [0,\, 1] \times (0,\, \infty) \to \mathbf{R} \text{ with } t \mapsto f(t,\, y) \text{ measurable} \\ \text{for every } y \in (0,\, \infty) \text{ and } y \mapsto f(t,\, y) \text{ continuous for} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0,\, 1) \end{cases}$$

$$(2.5) \quad \begin{cases} \text{for any } r > 0,\ \exists \psi_r\colon [0,\, 1] \to \mathbf{R},\ \psi_r > 0 \text{ a.e. on } [0,\, 1], \\ \psi_r \in L^q[0,\, 1] \text{ with } f(t,\, y) \ge \psi_r(t) \text{ a.e. on } [0,\, 1] \text{ for} \\ \text{every } y \in (0,\, r] \end{cases}$$

$$
(2.6) \qquad \begin{cases} \textit{for any } r > 0 \textit{ with } \int_0^1 k(t,\, s)\psi_r(s)\, ds \le r \textit{ for } t \in [0,\, 1], \\ \exists h_r \colon [0,\, 1] \to \mathbf{R},\ h_r \ge 0 \textit{ a.e. on } [0,\, 1],\ h_r \in L^q[0,\, 1] \\ \textit{with } f(t,\, y) \le h_r(t) \textit{ for a.e. } t \in [0,\, 1] \textit{ and} \\ y \in \left[\int_0^1 k(t,\, s)\psi_r(s)\, ds,\, r\right] \end{cases}
$$

and

$$
(2.7) \qquad \begin{cases} \exists M > 0 \textit{ with } M \ge \int_0^1 k(t,\, s) h_M(s)\, ds \ge \int_0^1 k(t,\, s)\psi_M(s)\, ds \\ \textit{for } t \in [0,\, 1]. \end{cases}
$$

Then (2.1) *has a solution* $y \in C[0,\, 1]$ *with* $y(t) > 0$ *for a.e.* $t \in [0,\, 1]$.

Remark 2.1 In Theorem 2.1 it is possible to replace (2.6) with

$$
\begin{cases} \text{for any } r > 0 \text{ with } \int_0^1 k(t,\, s)\psi_r(s)\, ds \le r \\ \text{for } t \in [0,\, 1], \text{ assume } h_r \in L^q[0,\, 1] \text{ where} \\ h_r(t) = \sup\left\{ f(t,\, y) \colon y \in \left[\int_0^1 k(t,\, s)\psi_r(s)\, ds,\, r\right]\right\}. \end{cases}
$$

Proof. Choose $M > 0$ so that (2.5), (2.6) and (2.7) hold. Let

$$
Q = \left\{ u \in C[0,\, 1] \colon \int_0^1 k(t,\, s)\psi_M(s)\, ds \le u(t) \le \int_0^1 k(t,\, s) h_M(s)\, ds,\ t \in [0,\, 1] \right\}.
$$

Clearly Q is a closed, convex subset of $C[0,\, 1]$. To establish our result we will apply Schauder's fixed point theorem to the operator N; here N is given by

$$
Ny(t) = \int_0^1 k(t,\, s) f(s,\, y(s))\, ds.
$$

First we show $N \colon Q \to Q$. To see this let $u \in Q$ so for $t \in [0,\, 1]$ we have (use (2.7))

$$
\int_0^1 k(t,\, s)\psi_M(s)\, ds \le u(t) \le \int_0^1 k(t,\, s) h_M(s)\, ds \le M. \tag{2.8}
$$

As a result (2.5) implies

$$
f(s,\, u(s)) \ge \psi_M(s) \quad \text{a.e. on} \quad [0,\, 1],
$$

so

$$Nu(t) = \int_0^1 k(t,\, s) f(s,\, u(s))\, ds \geq \int_0^1 k(t,\, s)\psi_M(s)\, ds \qquad \text{for} \quad t \in [0,\, 1].$$

Also (2.6) and (2.8) imply

$$f(s,\, u(s)) \leq h_M(s) \quad \text{a.e. on} \quad [0,\, 1],$$

so

$$Nu(t) = \int_0^1 k(t,\, s) f(s,\, u(s))\, ds \leq \int_0^1 k(t,\, s) h_M(s)\, ds \qquad \text{for} \quad t \in [0,\, 1].$$

Thus $N\colon Q \to Q$. Next we show N is continuous. To see this notice if $y_n \in Q$ and $y_n \to y$ in $C[0,\, 1]$ then for $t \in [0,\, 1]$ we have

$$|Ny_n(t) - Ny(t)| \leq \int_0^1 k(t,\, s)|f(s,\, y_n(s)) - f(s,\, y(s))|\, ds$$

and so the Lebesgue dominated convergence theorem implies

$$\begin{aligned} \sup_{t\in[0,\,1]} |Ny_n(t) - Ny(t)| \leq & \left(\sup_{t\in[0,\,1]} \int_0^1 [k(t,\, s)]^p\, ds \right)^{1/p} \\ & \times \left(\int_0^1 |f(s,\, y_n(s)) - f(s,\, y(s))|^q\, ds \right)^{1/q} \\ & \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

since

$$\int_0^1 |f(s,\, y_n(s)) - f(s,\, y(s))|^q\, ds \leq 2^q \int_0^1 [h_M(s)]^q\, ds.$$

As a result $N\colon Q \to Q$ is continuous. It remains to show $N\colon Q \to Q$ is compact. This follows from the Arzela–Ascoli theorem and the following (here $y \in Q$ and $t,\, t' \in [0,\, 1]$),

$$\sup_{t\in[0,\,1]} |Ny(t)| \leq \sup_{t\in[0,\,1]} \int_0^1 k(t,\, s) h_M(s)\, ds \leq M$$

and

$$|Ny(t) - Ny(t')| \leq \int_0^1 |k(t,\, s) - k(t',\, s)| h_M(s)\, ds$$

$$\leq \left(\int_0^1 |k_t(s) - k_{t'}(s)|^p \, ds \right)^{1/p} \left(\int_0^1 [h_M(s)]^q \, ds \right)^{1/q}$$
$$\to 0 \quad \text{as} \quad t \to t' \qquad \text{(from (2.3).)}$$

Now Schauder's fixed point theorem guarantees that N has a fixed point in Q. □

Our next result is a more "applicable" version of Theorem 2.1.

Theorem 2.2 *Let* $1 \leq p \leq \infty$, *q the conjugate to p and suppose* (2.2), (2.3), (2.4) *and* (2.5) *hold. In addition assume the following conditions are satisfied:*

$$(2.9) \qquad \begin{cases} f(t, y) \leq \phi(t)[g(y) + \tau(y)] \text{ on } [0, 1] \times (0, \infty) \text{ with } g > 0 \\ \text{continuous and nonincreasing on } (0, \infty), \ \tau \geq 0 \\ \text{continuous and nondecreasing on } (0, \infty) \text{ and} \\ \phi\colon [0, 1] \to \mathbf{R} \text{ with } \phi > 0 \text{ a.e. on } [0, 1] \end{cases}$$

$$(2.10) \qquad \begin{cases} \phi \in L^q[0, 1] \text{ and } \phi(t) g\big(\int_0^1 k(t, s)\psi_r(s)\, ds\big) \in L^q[0, 1] \\ \text{for any } r > 0 \end{cases}$$

and there exists $M > 0$ with

$$(2.11) \qquad \begin{cases} M \geq \int_0^1 k(t, s)\phi(s)\big[\tau(M) + g\big(\int_0^1 k(s, x)\psi_M(x)\, dx\big)\big]\, ds \\ \quad \geq \int_0^1 k(t, s)\psi_M(s)\, ds \quad \text{for} \quad t \in [0, 1]. \end{cases}$$

Then (2.1) *has a solution $y \in C[0, 1]$ with $y(t) > 0$ for a.e. $t \in [0, 1]$.*

Proof. The result follows from Theorem 2.1 once we show (2.6) and (2.7) hold. Notice for a.e. $t \in [0, 1]$ and $y \in \big[\int_0^1 k(t, s)\psi_r(s)\, ds, r\big]$ that (2.9) yields

$$f(t, y) \leq \phi(t) \left[\tau(r) + g\left(\int_0^1 k(t, s)\psi_r(s)\, ds \right) \right].$$

If we take

$$h_r(t) = \phi(t) \left[\tau(r) + g\left(\int_0^1 k(t, s)\psi_r(s)\, ds \right) \right],$$

then (2.6) is immediate since (2.10) guarantees that $h_r \in L^q[0, 1]$. Also (2.11) guarantees (2.7). □

To show how Theorem 2.2 can be applied in practice consider

$$y(t) = \int_0^1 k(t,\, s)\phi(s)[g(y(s)) + \tau(y(s))]\, ds \quad \text{for} \quad t \in [0,\, 1]. \tag{2.12}$$

Theorem 2.3 *Let $1 \le p \le \infty$, q the conjugate to p and suppose* (2.2) *and* (2.3) *hold. In addition assume the following conditions are satisfied:*

$$(2.13) \quad \begin{cases} g > 0 \text{ is continuous and nonincreasing on } (0,\, \infty),\ \tau \ge 0 \\ \text{is continuous and nondecreasing on } (0,\, \infty) \text{ and} \\ \phi\colon [0,\, 1] \to \mathbf{R} \text{ is measurable with } \phi > 0 \text{ a.e. on } [0,\, 1] \end{cases}$$

$$(2.14) \quad \begin{cases} \phi \in L^q[0,\, 1] \text{ and } \phi(t) g\big(g(r) \int_0^1 k(t,\, s)\phi(s)\, ds\big) \in L^q[0,\, 1] \\ \text{for any } r > 0 \end{cases}$$

and there exists $M > 0$ with

$$(2.15) \quad \begin{cases} M \ge \int_0^1 k(t,\, s)\phi(s)\big[\tau(M) + g\big(g(M) \int_0^1 k(s,\, x)\phi(x)\, dx\big)\big]\, ds \\ \qquad \ge g(M) \int_0^1 k(t,\, s)\phi(s)\, ds \quad \text{for} \quad t \in [0,\, 1]. \end{cases}$$

Then (2.12) *has a solution $y \in C[0,\, 1]$ with $y(t) > 0$ for a.e. $t \in [0,\, 1]$.*

Proof. The result follows from Theorem 2.2 once we notice that we can take $\psi_r(t) = \phi(t)g(r)$. □

Remark 2.2 If $g(y) = y^{-\alpha}$, $\alpha > 0$ and for $y \ge 0$ we have $\tau(y) = Ay^{\beta} + B$, $A \ge 0$, $B \ge 0$, $\beta \ge 0$ then (2.15) reduces to

$$\begin{cases} M \ge \int_0^1 k(t,\, s)\phi(s)\big[AM^{\beta} + B + M^{\alpha^2}\big(\int_0^1 k(s,\, x)\phi(x)\, dx\big)^{-\alpha}\big]\, ds \\ \qquad \ge M^{-\alpha} \int_0^1 k(t,\, s)\phi(s)\, ds \quad \text{for} \quad t \in [0,\, 1]. \end{cases}$$

Of course if α and $\beta < 1$ then this inequality is satisfied for M large.

Remark 2.3 It is also possible to obtain an analogue of Theorem 2.1 in the nonsingular case i.e. when $f\colon [0,\, 1] \times [0,\, \infty) \to \mathbf{R}$. In this case (2.5) and (2.6) are replaced by

$$(2.16\text{a}) \quad \begin{cases} \text{for any } r > 0,\ \exists h_r\colon [0,\, 1] \to \mathbf{R},\ h_r \ge 0 \text{ a.e. on } [0,\, 1], \\ h_r \in L^q[0,\, 1] \text{ with } 0 \le f(t,\, y) \le h_r(t) \text{ a.e. on } [0,\, 1] \text{ for} \\ \text{every } y \in [0,\, r] \end{cases}$$

and (2.7) is replaced by

$$\text{(2.16b)} \quad \begin{cases} \exists M > 0 \text{ with } M \geq \int_0^1 k(t,\, s) h_M(s)\, ds \\ \text{for } t \in [0,\, 1]. \end{cases}$$

The details are left to the reader (also in this case the solution y constructed satisfies $y(t) \geq 0$ for $t \in [0,\, 1]$).

Next we discuss the Volterra integral equation

$$y(t) = \int_0^t k(t,\, s) f(s,\, y(s))\, ds \quad \text{for} \quad t \in [0,\, T] \tag{2.17}$$

(here $T > 0$ is fixed).

Theorem 2.4 *Let $1 \leq p \leq \infty$ and let q be the conjugate to p. Suppose the following conditions are satisfied:*

$$\text{(2.18)} \quad \begin{cases} \textit{for all } t \in [0,\, T],\ k_t(s) = k(t,\, s) \geq 0 \textit{ for a.e. } s \in [0,\, t] \\ \textit{and for a.e. } t \in [0,\, T],\ k_t(s) > 0 \textit{ for a.e. } s \in [0,\, t] \end{cases}$$

$$\text{(2.19)} \quad k_t(s) \in L^p[0,\, t] \quad \textit{for each} \quad t \in [0,\, T] \quad \textit{and} \quad \sup_{t \in [0,T]} \int_0^t [k_t(s)]^p\, ds < \infty$$

$$\text{(2.20)} \quad \begin{cases} \textit{for any } t,\, t' \in [0,\, T],\ \int_0^{t^\star} |k_t(s) - k_{t'}(s)|^p\, ds \to 0 \\ \textit{as } t \to t' \textit{ where } t^\star = \min\{t,\, t'\} \end{cases}$$

$$\text{(2.21)} \quad \begin{cases} f\colon [0,\, T] \times (0,\, \infty) \to \mathbf{R} \textit{ with } t \mapsto f(t,\, y) \textit{ measurable} \\ \textit{for every } y \in (0,\, \infty) \textit{ and } y \mapsto f(t,\, y) \textit{ continuous for} \\ \textit{a.e. } t \in (0,\, T) \end{cases}$$

$$\text{(2.22)} \quad \begin{cases} \textit{for any } r > 0,\ \exists \psi_r\colon [0,\, T] \to \mathbf{R},\ \psi_r > 0 \textit{ a.e. on } [0,\, T], \\ \psi_r \in L^q[0,\, T] \textit{ with } f(t,\, y) \geq \psi_r(t) \textit{ a.e. on } [0,\, T] \textit{ for} \\ \textit{every } y \in (0,\, r] \end{cases}$$

$$\text{(2.23)} \quad \begin{cases} \textit{for any } r > 0 \textit{ with } \int_0^t k(t,\, s)\psi_r(s)\, ds \leq r \textit{ for } t \in [0,\, T], \\ \exists h_r\colon [0,\, T] \to \mathbf{R},\ h_r \geq 0 \textit{ a.e. on } [0,\, T],\ h_r \in L^q[0,\, T] \\ \textit{with } f(t,\, y) \leq h_r(t) \textit{ for a.e. } t \in [0,\, T] \textit{ and} \\ y \in \left[\int_0^t k(t,\, s)\psi_r(s)\, ds,\, r\right] \end{cases}$$

and

$$\text{(2.24)} \quad \begin{cases} \exists M > 0 \textit{ with } M \geq \int_0^t k(t,\, s) h_M(s)\, ds \geq \int_0^t k(t,\, s)\psi_M(s)\, ds \\ \textit{for } t \in [0,\, T]. \end{cases}$$

Then (2.17) has a solution $y \in C[0,\, T]$ with $y(t) > 0$ for a.e. $t \in [0,\, T]$.

Proof. Choose $M > 0$ so that (2.22), (2.23) and (2.24) hold. Let

$$Q = \left\{u \in C[0, T]\colon \int_0^t k(t, s)\psi_M(s)ds \le u(t) \le \int_0^t k(t, s)h_M(s)ds,\ t \in [0, T]\right\}$$

and

$$Ny(t) = \int_0^t k(t, s)f(s, y(s))\, ds.$$

Essentially the same reasoning as in Theorem 2.1 guarantees that $N\colon Q \to Q$. Also $N\colon Q \to Q$ is continuous. To see this notice if $y_n \in Q$ and $y_n \to y$ in $C[0, T]$ then the Lebesgue dominated convergence theorem implies

$$\begin{aligned}\sup_{t\in[0,T]} |Ny_n(t) - Ny(t)| &\le \left(\sup_{t\in[0,T]} \int_0^t [k(t, s)]^p\, ds\right)^{1/p} \\ &\quad \times \left(\int_0^T |f(s, y_n(s)) - f(s, y(s))|^q\, ds\right)^{1/q} \\ &\quad \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

In addition $N\colon Q \to Q$ is compact from the Arzela–Ascoli theorem and the following (here $y \in Q$ and $t, t' \in [0, T]$ with $t' < t$),

$$\sup_{t\in[0,T]} |Ny(t)| \le \sup_{t\in[0,T]} \int_0^t k(t, s)h_M(s)ds \le M$$

and

$$\begin{aligned}|Ny(t) - Ny(t')| &\le \int_0^{t'} |k_t(s) - k_{t'}(s)| f(s, y(s))\, ds \\ &\quad + \int_{t'}^t k_t(s) f(s, y(s))\, ds \\ &\le \left(\int_0^{t'} |k_t(s) - k_{t'}(s)|^p\, ds\right)^{1/p} \left(\int_0^T [h_M(s)]^q\, ds\right)^{1/q} \\ &\quad + \left(\sup_{t\in[0,T]} \int_0^t [k_t(s)]^p\, ds\right)^{1/p} \left(\int_{t'}^t [h_M(s)]^q\, ds\right)^{1/q} \\ &\quad \to 0 \quad \text{as} \quad t \to t' \quad \text{(from (2.19) and (2.20))}.\end{aligned}$$

A similar calculation holds for $t' > t$. Now apply Schauder's fixed point theorem. □

Remark 2.4 In fact if (2.20) is replaced by

$$\begin{cases} \text{for any } t,\, t' \in [0,T],\ \int_0^{t^\star} |k_t(s) - k_{t'}(s)|^p\,ds + \int_{t^\star}^{t^{\star\star}} [k_{t^{\star\star}}(s)]^p\,ds \to 0 \\ \text{as } t \to t' \text{ where } t^\star = \min\{t,\, t'\} \text{ and } t^{\star\star} = \max\{t,\, t'\} \end{cases}$$

then automatically $\sup_{t\in[0,T]} \int_0^t [k_t(s)]^p\,ds < \infty$ in (2.19).

Theorem 2.5 *Let* $1 \le p \le \infty$, *q the conjugate to p and suppose* (2.18), (2.19), (2.20), (2.21) *and* (2.22) *hold. In addition assume the following conditions are satisfied*:

$$(2.25)\qquad \begin{cases} f(t,\, y) \le \phi(t)[g(y) + \tau(y)] \text{ on } [0,T] \times (0,\infty) \text{ with } g > 0 \\ \text{continuous and nonincreasing on } (0,\infty),\ \tau \ge 0 \\ \text{continuous and nondecreasing on } (0,\infty) \text{ and} \\ \phi\colon [0,T] \to \mathbf{R} \text{ with } \phi > 0 \text{ a.e. on } [0,T] \end{cases}$$

$$(2.26)\qquad \begin{cases} \phi \in L^q[0,T] \text{ and } \phi(t)g\big(\int_0^t k(t,\,s)\psi_r(s)\,ds\big) \in L^q[0,T] \\ \text{for any } r > 0 \end{cases}$$

and there exists $M > 0$ *with*

$$(2.27)\qquad \begin{cases} M \ge \int_0^t k(t,\,s)\phi(s)\big[\tau(M) + g\big(\int_0^s k(s,\,x)\psi_M(x)\,dx\big)\big]\,ds \\ \quad\ \ge \int_0^t k(t,\,s)\psi_M(s)\,ds \quad \text{for} \quad t \in [0,T]. \end{cases}$$

Then (2.17) *has a solution $y \in C[0,T]$ with $y(t) > 0$ for a.e. $t \in [0,T]$.*

Proof. The result follows from Theorem 2.4 with

$$h_r(t) = \phi(t)\left[\tau(r) + g\left(\int_0^t k(t,\,s)\psi_r(s)\,ds\right)\right].$$

□

Remark 2.5 There is also an analogue of Theorem 2.3 for the Volterra equation

$$y(t) = \int_0^t k(t,\,s)\phi(s)[g(y(s)) + \tau(y(s))]\,ds \quad \text{for} \quad t \in [0,T].$$

We leave the details to the reader.

Next we examine the integral equation

$$y(t) = \int_0^{\infty} k(t,\, s) f(s,\, y(s)) ds \quad \text{for} \quad t \in [0,\, \infty). \tag{2.28}$$

We begin by looking for solutions to (2.28) in $C_l[0,\, \infty)$. Recall $C_l[0,\, \infty)$ is the subset of $BC[0,\, \infty)$ (the space of bounded continuous functions on $[0,\, \infty)$) which consists of all $y \in BC[0,\, \infty)$ such that $\lim_{t\to\infty} y(t)$ exist.

Theorem 2.6 *Let $1 \le p \le \infty$ and let q be the conjugate to p. Suppose the following conditions are satisfied:*

$$(2.29) \quad \begin{cases} \text{for all } t \in [0,\, \infty),\ k_t(s) = k(t,\, s) \ge 0 \text{ for a.e. } s \in [0,\, \infty) \\ \text{and for a.e. } t \in [0,\, \infty),\ k_t(s) > 0 \text{ for a.e. } s \in [0,\, \infty) \end{cases}$$

$$(2.30) \quad \begin{cases} k_t(s) \in L^p[0,\, \infty) \text{ for each } t \in [0,\, \infty) \text{ and the map} \\ t \mapsto k_t \text{ is continuous from } [0,\, \infty) \text{ to } L^p[0,\, \infty) \end{cases}$$

$$(2.31) \quad \begin{cases} \text{there exists } \tilde{k} \in L^p[0,\, \infty) \text{ such that} \\ k_t \to \tilde{k} \text{ in } L^p[0,\, \infty) \text{ as } t \to \infty \end{cases}$$

$$(2.32) \quad \begin{cases} f\colon [0,\, \infty) \times (0,\, \infty) \to \mathbf{R} \text{ with } t \mapsto f(t,\, y) \text{ measurable} \\ \text{for every } y \in (0,\, \infty) \text{ and } y \mapsto f(t,\, y) \text{ continuous for} \\ \text{a.e. } t \in [0,\, \infty) \end{cases}$$

$$(2.33) \quad \begin{cases} \text{for any } r > 0,\ \exists \psi_r\colon [0,\, \infty) \to \mathbf{R},\ \psi_r > 0 \text{ a.e. on } [0,\, \infty), \\ \psi_r \in L^q[0,\, \infty) \text{ with } f(t,\, y) \ge \psi_r(t) \text{ a.e. on } [0,\, \infty) \text{ for} \\ \text{every } y \in (0,\, r] \end{cases}$$

$$(2.34) \quad \begin{cases} \text{for any } r > 0 \text{ with } \int_0^\infty k(t,\, s)\psi_r(s)\, ds \le r \text{ for } t \in [0,\, \infty), \\ \exists h_r\colon [0,\, \infty) \to \mathbf{R},\ h_r \ge 0 \text{ a.e. on } [0,\, \infty),\ h_r \in L^q[0,\, \infty) \\ \text{with } f(t,\, y) \le h_r(t) \text{ for a.e. } t \in [0,\, \infty) \text{ and} \\ y \in \left[\int_0^\infty k(t,\, s)\psi_r(s)\, ds,\, r\right] \end{cases}$$

and

$$(2.35) \quad \begin{cases} \exists M > 0 \text{ with } M \ge \int_0^\infty k(t,\, s)\, h_M(s)\, ds \ge \int_0^\infty k(t,\, s)\psi_M(s)\, ds \\ \text{for } t \in [0,\, \infty). \end{cases}$$

Then (2.28) *has a solution $y \in C_l[0,\, \infty)$ with $y(t) > 0$ for a.e. $t \in [0,\, \infty)$.*

Proof. Choose $M > 0$ so that (2.33), (2.34) and (2.35) hold. Let N be given by

$$Ny(t) = \int_0^\infty k(t,\, s) f(s,\, y(s))\, ds$$

and

$$Q = \left\{ u \in C_l[0, \infty) \colon \begin{array}{l} \displaystyle\int_0^\infty k(t, s)\psi_M(s)ds \le u(t) \\ \displaystyle\le \int_0^\infty k(t, s)h_M(s)ds \quad \text{for} \quad t \in [0, \infty) \end{array} \right\}.$$

Now since $C_l[0, \infty)$ is a closed subspace of $BC[0, \infty)$ it is clear that q is a closed subset of $C_l[0, \infty)$. First we show $N \colon Q \to Q$. To see this let $u \in Q$ so as in Theorem 2.1 it is easy to see from (2.33) that

$$f(s, u(s)) \ge \psi_M(s) \quad \text{a.e. on} \quad [0, \infty),$$

and from (2.34) that

$$f(s, u(s)) \le h_M(s) \quad \text{a.e. on} \quad [0, \infty).$$

Thus

$$\int_0^\infty k(t, s)\psi_M(s)\, ds \le Nu(t) \le \int_0^\infty k(t, s)h_M(s)\, ds \tag{2.36}$$

for $t \in [0, \infty)$. It remains to show $Nu \in C_l[0, \infty)$. To see that $Nu \in C_l[0, \infty)$ notice

$$\lim_{t \to \infty} \int_0^\infty k(t, s)f(s, u(s))\, ds = \int_0^\infty \tilde{k}(s)f(s, u(s))\, ds$$

since (2.31) implies

$$\begin{aligned} \int_0^\infty \left|[k(t, s) - \tilde{k}(s)]\, f(s, u(s))\right| ds &\le \int_0^\infty |k(t, s) - \tilde{k}(s)|\, h_M(s)\, ds \\ &\le \left(\int_0^\infty |k_t(s) - \tilde{k}(s)|^p\, ds \right)^{1/p} \\ &\times \left(\int_0^\infty |h_M(s)|^q\, ds \right)^{1/q} \\ &\to 0 \text{ as } t \to \infty. \end{aligned}$$

Thus $N \colon Q \to Q$. Essentially the same reasoning as in Theorem 5.2.3 in [8] guarantees that $N \colon Q \to Q$ is continuous and compact. Now apply Schauder's fixed point theorem. □

Next rather than require solutions of (2.28) to lie in $C_l[0, \infty)$ we now seek solutions to (2.28) in $C[0, \infty)$ (the space of continuous functions on

$[0, \infty)$). Recall $C[0, \infty)$ is a Fréchet space so we will apply the Schauder–Tychonoff theorem instead of Schauder's theorem in this case.

Theorem 2.7 *Let $1 \le p \le \infty$, q the conjugate to p and suppose* (2.29), (2.30), (2.32), (2.33), (2.34) *and* (2.35) *hold. Then* (2.28) *has a solution $y \in C[0, \infty)$ (in fact $y \in BC[0, \infty)$) with $y(t) > 0$ for a.e. $t \in [0, \infty)$.*

Proof. Let M, ψ_M, h_M and N be as in Theorem 2.6. Now let

$$Q = \left\{ u \in C[0, \infty) \colon \begin{array}{c} u \in BC[0, \infty) \quad \text{and} \\ \int_0^\infty k(t, s)\psi_M(s)ds \le u(t) \\ \le \int_0^\infty k(t, s)h_M(s)\, ds \quad \text{for} \quad t \in [0, \infty) \end{array} \right\}.$$

Clearly, Q is a closed (note (2.35)) convex subset of the Fréchet space $C[0, \infty)$.

First we show $N\colon Q \to Q$. To see this take $u \in Q$. Then as in Theorem 2.6 we have (2.36). Also $Nu \in C[0, \infty)$ since if $t, t' \in [0, \infty)$ then (2.30) implies

$$\begin{aligned} |Nu(t) - Nu(t')| &\le \int_0^\infty |k_t(s) - k_{t'}(s)| h_M(s)\, ds \\ &\le \left(\int_0^\infty |k_t(s) - k_{t'}(s)|^p\, ds\right)^{1/p} \left(\int_0^\infty [h_M(s)]^q\, ds\right)^{1/q} \\ &\to 0 \quad \text{as} \quad t \to t'. \end{aligned}$$

Thus $N\colon Q \to Q$. Next we show $N\colon Q \to Q$ is compact. To see this we just need to notice that $N(Q)$ is uniformly bounded and equicontinuous on each compact subinterval on $[0, \infty)$ and this is immediate since (here $y \in Q$ and $t, t' \ge 0$),

$$|Ny(t)| \le \sup_{t\in[0,\infty)} \int_0^\infty k(t, s)h_M(s) \le M$$

and

$$|Ny(t) - Ny(t')| \le \left(\int_0^\infty |k_t(s) - k_{t'}(s)|^p\, ds\right)^{1/p} \left(\int_0^\infty [h_M(s)]^q\, ds\right)^{1/q}.$$

As a result $N\colon Q \to Q$ is a compact map. Finally we show $N\colon Q \to Q$ is continuous. Suppose $y_n \in Q$ with $y_n \to y$ in $C[0, \infty)$. Then $y_n \to y$ in

$C[0, m]$ for each $m \in \{1, 2, \ldots\} = N_0$ and also y_n converges pointwise to y on $[0, \infty)$. Fix $m \in N_0$. Essentially the same argument as in Theorem 2.1 guarantees that $Ny_n(t) \to Ny(t)$ for each $t \in [0, \infty)$ as $n \to \infty$ and $Ny_n \to Ny$ in $C[0, m]$ as $n \to \infty$. This is true for each $m \in N_0$ so $Ny_n \to Ny$ in $C[0, \infty)$. Now apply the Schauder–Tychonoff fixed point theorem. □

Acknowledgment The authors would like to thank the referee for many suggestions which improved considerably the original manuscript.

References

[1] Agarwal R.P. and O'Regan D., *Volterra integral equations: the singular case.* Hokkaido Math. J. **32** (2003), 371–381.

[2] Agarwal R.P. and O'Regan D., *Singular integral equations arising in Homann flow.* Dynamics of Continuous, Discrete and Impulsive Systems (Applications and Algorithms) **9** (2002), 481–488.

[3] Bonanno G., *An existence theorem of positive solutions to a singular nonlinear boundary value problem.* Comment. Math. Univ. Carolinae **36** (1995), 609–614.

[4] Corduneanu C., *Integral equations and applications.* Cambridge University Press, New York, 1990.

[5] Karlin S. and Nirenberg L., *On a theorem of P. Nowosad.* J. Math. Anal. Appl. **17** (1967), 61–67.

[6] Meehan M. and O'Regan D., *Positive solutions of singular integral equations.* J. Integral Eqns. and Appl. **12** (2000), 271–280.

[7] Nowosad P., *On the integral equation $\kappa f = 1/f$ arising in a problem in communications.* J. Math. Anal. Appl. **14** (1966), 484–492.

[8] O'Regan D. and Meehan M., *Existence theory for nonlinear integral and integrodifferential equations.* Kluwer Acad. Press, Dordrecht, 1998.

[9] Wang J., Gao W. and Zhang Z., *Singular nonlinear boundary value problems arising in boundary layer theory.* J. Math. Anal. Appl. **233** (1999), 246–256.

R.P. Agarwal
Department of Mathematical Science
Florida Institute of Technology
Melbourne, Florida 32901
USA
E-mail: agarwal@fit.edu

D. O'Regan
Department of Mathematics
National University of Ireland
Galway
Ireland