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Transitive Lie algebras admitting differential systems

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Introduction.

In this paper we define the transitive filtered Lie algebras of depth μ and prove the structure theorems on these Lie algebras.

According to Guillemin-Sternberg [2], a Lie algebra L is called a transitive Lie algebra if it possesses a filtration $\{L^p\}_{p\in Z}$ satisfying: 0) $L=L^{-1}$, i) $L^p\supset L^{p+1}$, ii) $[L^p, L^q]\subset L^{p+q}$, iii) $\dim L^p/L^{p+1}<\infty$, iv) $\bigcap_{p\in Z} L^p=0$, v) $L^{p+1}=\{x\in L^p \mid [x, L^a]\subset L^{p+a+1}$ for all $a<0\}$ for $p\geq 0$. As well-known, to a transitive Lie pseudo-group corresponds a transitive Lie algebra as its formal algebra and algebraic theories of transitive Lie algebras have been adequately developed by many authors, in particular by Guillemin-Sternberg [2] and Singer-Sternberg [11].

On the other hand if a transitive Lie pseudo-group acting on a manifold M admits (i.e., leaves invariant) a sequence $\{D^p\}_{p<0}$ of differential systems (i.e., subbundles of the tangent bundle TM of M) such that 0) $TM=D^{-\mu}$ for an integer $\mu\geq 1$, i) $D^p\supset D^{p+1}$, ii) $[\mathscr{D}^p, \mathscr{D}^q]\subset \mathscr{D}^{p+q}$, where $\mathscr{D}^p$ denotes the sheaf of the local sections of D^p, then the transitive Lie algebra L corresponding to this pseudo-group admits in a natural way another filtration $\{L^p\}$ more refined than the usual one (See § 1). This new filtration starts with $L^{-\mu}$ instead of L^{-1}:

$$L=L^{-\mu}\supset\cdots\supset L^{-1}\supset L^0\supset\cdots$$

and satisfies the same conditions i), ii)$\cdots$, v) as mentioned above. A Lie algebra endowed with such a filtration will be called a transitive filtered Lie algebra of depth μ. If $\mu=1$ the filtration reduces to a usual one. The contact Lie algebra $C(n)$ (See § 5) is a typical example of transitive Lie algebras possessing transitive filtrations of depth 2. This filtration has already played an important rôle in the classification of the infinite primitive Lie algebras ([6]). Moreover there appear many examples of transitive filtered Lie algebras of depth greater than 1 in geometry of differential systems (cf. Tanaka [12], [14]) and in higher order contact geometry (cf. Yamaguchi [15], [16]).

This leads us to the general study of transitive filtered Lie algebras of depth μ. Our results will show that the structure theorems on transitive filtered Lie algebras established in the case $\mu=1$ (such as in Guillemin-Sternberg [2], Singer-Sternberg [11], Rim [10], Hayashi [3]) can be generalized quite naturally to the case of arbitrary $\mu\geq 1$.

Recall that in the case $\mu=1$ the algebraic theory of Guillemin-Sternberg and others is buit on three pillars: First the notion of prolongation concerning the associated graded Lie algebras, second the Spencer cohomology group attached to the graded Lie algebra, especially its finiteness (i. e., finite dimensionality), and third the notion of truncated structures of filtered Lie algebras.

Now in the case $\mu\geq 1$, the theory can be developed in the parallel way. Let $\boldsymbol{L}=(L, \{L^p\})$ be a transitive filtered Lie algebra of depth μ. Then the associated graded Lie algebra $\mathrm{gr}\,\boldsymbol{L}=\bigoplus_{p\in\boldsymbol{Z}}\mathrm{gr}_p\,\boldsymbol{L}$ belongs to those Lie algebras that are called graded Lie algebras of the μ-th kind and studied by Tanaka in geometry of differential systems. Thus the notion of prolongation to these Lie algebras is already familiar (See [12], [13]). Moreover the Spencer cohomology group is generalized as the cohomology group $H(\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})$ associated with the adjoint representation on $\mathrm{gr}\,\boldsymbol{L}$ of the nilpotent Lie algebra $\mathrm{gr}_-\boldsymbol{L}=\bigoplus_{p<0}\mathrm{gr}_p\,\boldsymbol{L}$, which is, by means of the gradation of $\mathrm{gr}\,\boldsymbol{L}$, endowed with a natural bigradation: $H(\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})=\bigoplus_{p,r\in\boldsymbol{Z}}H_r^p$, where H_r^p is the cohomology group at $\mathrm{Hom}(\Lambda^p\,\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})_r$ (the subspace of $\mathrm{Hom}(\Lambda^p\,\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})$ consisting of all elements of degree r) (See § 2). This cohomology group was first introduced by Tanaka [14] with the following bigradation: $H=\bigoplus H^{s,p}$ where $H^{s,p}=H^p_{s+p-1}$. Note that if $\mu=1$ then $\mathrm{gr}_-\boldsymbol{L}$ is abelian and $H^{s,p}$ is just the Spencer cohomology group at (s, p).

Our starting-point is the finiteness of this generalized Spencer cohomology group:

THEOREM I. *Let* $\boldsymbol{L}$ *be a transitive filtered Lie algebra of depth* μ, *then* $H_r(\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})=0$ *for sufficiently large* r. *In other words, there exists an integer* k *such that* $H^{s,p}(\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})=0$ *for all* p *if* $s\geq k$.

The proof is based on the fact that the universal enveloping algebra of a finite dimensional Lie algebra is noetherian and will be given in section 2.

Now our main results may be stated as follows: For a transitive filtered Lie algebra $\boldsymbol{L}$, let k_0 be the smallest non-negative integer such that

$$H_r^1(\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})=H_r^2(\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})=H_s^3(\mathrm{gr}_-\boldsymbol{L}, \mathrm{gr}\,\boldsymbol{L})=0$$

for $r\geq k_0+1$, $s\geq \mathrm{Max}\{k_0, 1\}+1$ and call the integer k_0 the structural order of

$\boldsymbol{L}$. Then we have

THEOREM II. *A transitive filtered Lie algebra L of depth μ can be completely determined, up to isomorphism, by its truncated Lie algebra* $\mathrm{Trun}_k\, \boldsymbol{L}$ *of order k if k is greater than or equal to the structural order of* $\boldsymbol{L}$.

The truncated Lie algebra $\mathrm{Trun}_k\, \boldsymbol{L}$ is, roughly speaking, obtained from $\boldsymbol{L}$ by forgetting the structures of $\boldsymbol{L}$ of order higher than k. Thus Theorem I and II reduce the equivalence problem of (in general) infinite dimensional objects to that of finite dimensional ones.

To make precise the statement of Theorem II, we must define the truncated (transitive filtered) Lie algebras. The precise definition will be given in section 3 (Definition 3. 2). This notion is closely related to the structure functions occurring in prolongation schemes of geometric structures. Our formulation of the truncated Lie algebras is better adapted to our prolongation scheme developed in [8] and [9] rather than that of Singer-Sternberg [11], therefore even in the case $\mu=1$ it is somewhat different from the formulation of Guillemin-Sternberg [2]. (See Remarks 3. 1, 3. 2)

In section 3 we shall then investigate how a truncated Lie algebra of order k can be constructed from that of order $k-1$ (Proposition 3. 1 and Proposition 3. 2) and obtain a criterion for two truncated Lie algebras of order k to be isomorphic (Theorem 3. 1).

In secction 4, as immediate cosequences of Proposition 3. 1 and Proposition 3. 2, we obtain our main results (Existence Theorem 4. 1, Embedding Theorem 4. 2, Theorem 4. 3) which give the precise meaning to the above Theorem II and clarify the structure of a transitive filtered Lie algebra of depth μ. In particular it gives us the information how to construct transitive filtered Lie algebras starting from a graded Lie algebra. As a byproduct we also obtain some sufficient conditions for a transitive filtered Lie algebra to be graded (Proposition 4. 1) generalizing the result of Kobayashi-Nagano [4].

Once the notion of truncated Lie algebra is well settled, the proofs of Theorems 4. 1, 4. 2, 4. 3 (or rather those of Proposition 3. 1 and Proposition 3. 2) are somewhat routine. The most cumbersome part would be to verify some cohomological identities (cocycle conditions), which however becomes much simplified and transparent, thanks to the elegant technique of Rim [10], by employing the calculus of skew-graded Lie algebras or super Lie algebras.

Since the contact Lie algebra $C(m, n\,; \nu)$ of order ν with m unknown funcitons and n independent variables is a fundamental example of transitive filtered Lie algebras of depth $\nu+1$, in the last section 5 we shall calculate the cohomology group $H(\mathrm{gr}_{-}C(m, n; \nu), \mathrm{gr}\, C(m, n; \nu))$. The results of the computation, combined with the above theorems, give cohomological interpretation of geometric properties of the higher order contact Lie algebras.

A geometric counterpart of the present algebraic study will be found in the forthcoming paper [9], in which we shall develop a prolongation scheme for those geometric structures that have defferential systems as underlying structures.

§ 1. Transitive filtered Lie algebras of depth μ.

We begin by defining the transitive filtered Lie algebras of depth μ which are the central object of the present study.

Let $\boldsymbol{Z}$ denote the set of integers. Let L be a Lie algebra over a field $\boldsymbol{F}$ of characteristic 0. Let μ be a non-negative integer. A *transitive filtration of L of depth μ* is a sequence $\{L^p\}_{p\in\boldsymbol{Z}}$ of subspaces of L satisfying the following conditions:

(F0) $L=L^{-\mu}$,

(F1) $L^p \supset L^{p+1}$,

(F2) $[L^p, L^q] \subset L^{p+q}$,

(F3) $\dim L^p/L^{p+1} < \infty$,

(F4) $\bigcap_{p\in\boldsymbol{Z}} L^p = 0$,

(F5) $L^{p+1} = \{X \in L^p \mid [X, L^a] \subset L^{p+a+1} \text{ for all } a<0\}$, for any $p \geq 0$.

A *transitive filtered Lie algebra* (abbreviated to TFLA) *of depth* μ is a pair $\boldsymbol{L}=(L, \{L^p\})$ consisting of a Lie algebra L and its transitive filtration $\{L^p\}$ of depth μ.

Morphisms of transitive filtered Lie algebras are defined in the natural manner; a homomorphism of a TFLA $\boldsymbol{L}=(L, \{L^p\})$ into another $\boldsymbol{L}'=(L', \{L'^p\})$ is a Lie homomorphism $f: L \to L'$ which preserves the filtrations, namely $f(L^p) \subset L'^p$ for all p. In particular we call the homomorphism f an *embedding* if f induces an isomorphism of L/L^0 onto L'/L'^0. Note that an embedding is necessarily injective by (F4) and (F5).

If $\boldsymbol{L}=(L, \{L^p\})$ is a TFLA, then L can be endowed with a natural uniform topology and becomes a topological Lie algebra by assigning $\{L^p\}$ as a fundamental system of neighbourhoods of the origin. $\boldsymbol{L}$ is called complete if L is complete with respect to the uniform topology. If we set $\bar{L}=\text{proj lim } L/L^k$, $\bar{L}^p=\text{proj lim } L^p/L^k$, then $\bar{\boldsymbol{L}}=(\bar{L}, \{\bar{L}^p\})$ is a complete TFLA called the completion of $\boldsymbol{L}$ and there is a canonical embedding $\iota: \boldsymbol{L} \to \bar{\boldsymbol{L}}$.

The definitions being settled, several remarks are in order.

REMARK 1.1. If $\boldsymbol{L}$ is a TFLA of depth μ, then it is also of depth ν for any integer $\nu \geq \mu$. We might say that $\boldsymbol{L}$ is *properly of depth* μ if μ is the smallest integer satisfying (F0). But this distinciton will not matter in what follows.

REMARK 1.2. In our terminology, it is the TFLA's of depth 1 that have been customarily called transitive (filtered) Lie algebras and studied by many authors. (cf. [2], [4], [11])

REMARK 1.3. The condition (F5) implies that a transitive filtration $\{L^p\}_{p \in \boldsymbol{Z}}$ is uniquely determined by its lower parts $\{L^p\}_{p<0}$. More generally, given a family $\{L^p\}_{p \leq 0}$ of subspaces of L which satisfies the conditions (F0) $\sim$(F3) only for $p \leq 0$. Then, defining L^p for $p>0$ inductively by (F5), we obtain a filtration $\{L^p\}_{p \in \boldsymbol{Z}}$, which, as easily seen, satisfies all the conditions of a transitive filtration except (F4); in general the ideal $\bigcap_{p \in \boldsymbol{Z}} L^p$ does not vanish.

REMARK 1.4. If $(L, \{L^p\})$ is a TFLA of depth $\mu \geq 1$, then L always admits a transitive filtration of depth 1. In fact, put $\hat{L}^p = L$ for $p<0$ and $\hat{L}^0 = L^0$ and define $\hat{L}^p$ for $p>0$ by

$$\hat{L}^{p+1} = \{x \in \hat{L}^p \mid [x, \hat{L}^{-1}] \subset \hat{L}^p\},$$

then it is easy to verify that

$$L^p \supset \hat{L}^p \supset L^{p\mu} \qquad \text{for } p \geq 0. \tag{1.1}$$

Hence $\bigcap_{p \in \boldsymbol{Z}} \hat{L}^p = 0$, so that $\{\hat{L}^p\}$ is a transitive filtration of depth 1. It follows also from (1.1) that the topology defined by $\{\hat{L}^p\}$ coincides with the one defined by $\{L^p\}$. Hence L, viewed as a topological Lie algebra, is an (abstract) transitive Lie algebra in the sense of Guillemin-Sternberg [2], provided that L is complete.

Now we mention briefly the geometric background of transitive filtered Lie algebras of depth μ. (cf. [12], [9] for more details). Let M be a differentiable manifold and denote TM the tangent bundle of M. A vector subbundle of TM is called a differential system on M. Now suppose that we are given a sequence of differential systems $\{D^p\}_{p<0}$ satisfying:

$$\begin{cases} \text{0)} \quad D^{-\mu} = TM & \\ \text{i)} \quad D^p \supset D^q & \text{for } p<q<0 \\ \text{ii)} \quad [\mathscr{D}^p, \mathscr{D}^q] \subset \mathscr{D}^{p+q} & \text{for } p, q<0, \end{cases}$$

where $\mathscr{D}^p$ denotes the sheaf of sections of D^p.

Let $\mathscr{L}$ be a Lie algebra subsheaf of $\mathscr{T}$M (the Lie algebra sheaf of the sections of TM) and suppose that $\mathscr{L}$ leaves invariant $\{D^p\}$, namely,

$$[\mathscr{L}, \mathscr{D}^p] \subset \mathscr{D}^p \qquad \text{for all } p<0.$$

Then for each stalk $\mathscr{L}_x (x \in M)$ we can introduce the natural filtration $\{\mathscr{L}_x^p\}_{p \in \boldsymbol{Z}}$ by defiining

$$\begin{cases} \mathscr{L}_x^p = \{[X]_x \in \mathscr{L}_x \mid X_x \in D_x^p\} & \text{for } p<0 \\ \mathscr{L}_x^0 = \{[X]_x \in \mathscr{L}_x \mid X_x = 0\} & \\ \mathscr{L}_x^p : \text{by (F5)} & \text{for } p>0 \end{cases}$$

where for a vector field X we have denoted by $[X]_x$ the germ of X at x and by X_x the value (i.e., the tangent vector) of X at x. Then one can verify without difficulty

$$[\mathscr{L}_x^p, \mathscr{L}_x^q] \subset \mathscr{L}_x^{p+q} \qquad \text{for } p, q \leq 0.$$

From this and Remark 1.3 follows that the filtration $\{\mathscr{L}_x^p\}_{p \in \boldsymbol{Z}}$ satisfies the conditions of transitive filtration except (F4). If we pass to the projective limit by setting $L(x) = \text{proj lim } \mathscr{L}_x / \mathscr{L}_x^k$, $L^p(x) = \text{proj lim } \mathscr{L}_x^p / \mathscr{L}_x^k$, we obtain a Lie algebra $L(x)$ and its filtration $\{L^p(x)\}_{p \in \boldsymbol{Z}}$. It is now straightforward:

Proposition 1.1. *Let $\{D^p\}$ and $\mathscr{L}$ be as above, then the pair $\boldsymbol{L}(x) = (L(x), \{L^p(x)\})$ is a complete transitive filtered Lie algebra of depth μ.*

$\boldsymbol{L}(x)$ is called the *formal algebra* of $\mathscr{L}$ at x. It should be remarked that if $\mathscr{L}$ is transitive at x (i.e., the evaluation map $\mathscr{L}_x \ni [X]_x \to X_x \in T_x M$ is surjective) then the formal algebra $\boldsymbol{L}(x)$ inherits all the algebraic information of $\mathscr{L}_x$, in the sense that the kernel of the natural projection $\mathscr{L}_x \to L(x)$ consists of null vector fields X (i.e., in local coordinates the Tayler expansions at x of the coefficients of X vanish identically). This follows easily from (1.1).

§ 2. Transitive graded Lie algebras and their cohomology groups.

In this section we consider the graded Lie algebra associated with a transitive filtered Lie algebra and study its cohomology group.

2.1. Let $\boldsymbol{L} = (L, \{L^p\})$ be a TFLA of depth μ. Let $\text{gr}\,\boldsymbol{L} = \bigoplus_{p \in \boldsymbol{Z}} \text{gr}_p\,\boldsymbol{L}$ be its associated graded Lie algebra, where $\text{gr}_p\,\boldsymbol{L} = L^p / L^{p+1}$, and the bracket operation of gr $\boldsymbol{L}$ is defined in the obvious manner. Then gr $\boldsymbol{L}$ is a transitive graded Lie algebra of depth μ is the following sense:

Definition 2.1. A graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \boldsymbol{Z}} \mathfrak{g}_p$ is called a transitive graded Lie algebra (TGLA) of depth μ if it satisfies the following conditions:

(G1) $\mathfrak{g}_p=0$ for $p<-\mu$,
(G2) $\dim \mathfrak{g}_p<\infty$
(G3) For $i\geq 0$, $x_i\in\mathfrak{g}_i$, if $[x_i, \mathfrak{g}_-]=0$ then $x_i=0$,

where we set: $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$.

REMARK 2.1. TGLA's of depth μ are alternatively called graded Lie algebras of the μ-th kind and studied by Tanaka ([12], [13], [14]), especially with the additional condition that $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$.

Let us now recall the notion of prolongation concerning a TGLA of depth μ, due to Guillemin-Sternberg [2] and Singer-Sternberg [11] for $\mu=1$ and Tanaka (loc cit) for arbitrary $\mu\geq 1$.

For this purpose we begin with the following:

DEFINITION 2.2. Let k be an integer or ∞. A truncated graded Lie algebra of order k is a graded vector space $\mathfrak{g}(k)=\bigoplus_{p\leq k}\mathfrak{g}_p$ equipped with a bracket operation (skew-symmetric bilinear map)

$$[\ ,\ ]:\mathfrak{g}_p\times\mathfrak{g}_q\to\mathfrak{g}_{p+q}$$

defined partially for $p, q, p+q\leq k$, satisfying the partial Jacobi identity:

$$\mathfrak{S}[[x_p, y_p], z_r]=0$$

for $x_p\in\mathfrak{g}_p$, $y_q\in\mathfrak{g}_q$, $z_r\in\mathfrak{g}_r$, whenever $p, q, r, p+q, q+r, r+p, p+q+r\leq k$, where $\mathfrak{S}$ denotes the cyclic sum in x_p, y_q, z_r.

If moreover the conditions (G1) (G2) (G3) of Definition 2.1 are satisfied, $\mathfrak{g}(k)$ is called a truncated transitive graded Lie algebra (truncated TGLA) of order k of depth μ.

Note that a truncated TGLA of order ∞ is just a TGLA. If $\mathfrak{g}(k)=\bigoplus_{p\leq k}\mathfrak{g}_p$ is a truncated TGLA of order k, then for each integer $l\leq k$, $\bigoplus_{p\leq l}\mathfrak{g}_p$ becomes a truncated TGLA of order l with respect to the induced bracket operation, which we will denote by $\mathrm{Trun}_l\,\mathfrak{g}(k)$. Morphisms of truncated TFLA's can be defined in the natural manner. In particular, a homomorphism $\phi:\mathfrak{h}(k)\to\mathfrak{g}(k)$ will be called an *embedding* if ϕ induces an isomorphism of $\mathfrak{h}_-=\bigoplus_{p<0}\mathfrak{h}_p$ onto $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$. Note that an embedding is necessarily injective by (G3).

Now let us define the prolongation of a truncated TGLA $\mathfrak{g}(k)=\bigoplus_{p\leq k}\mathfrak{g}_p$ of order $k\geq -1$. Put $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$, we define $\mathrm{Der}_{k+1}\,\mathfrak{g}(k)$ to be the vector space consisting of all $\alpha\in\mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}(k))$ such that

$$(2.1)\qquad \begin{cases}\alpha(\mathfrak{g}_p)\subset\mathfrak{g}_{p+k+1} & (p<0)\\ \alpha([x, y])=[\alpha(x), y]+[x, \alpha(y)], & x, y\in\mathfrak{g}_-,\end{cases}$$

and we set

$$p\,\mathfrak{g}(k)=\mathfrak{g}(k)\oplus \mathrm{Der}_{k+1}\mathfrak{g}(k).$$

It is then easy to see that there exists a unique bracket operation on $p\,\mathfrak{g}(k)$ which makes $p\,\mathfrak{g}(k)$ into a truncated TGLA of order $k+1$ such that $\mathrm{Trun}_k(p\,\mathfrak{g}(k))=\mathfrak{g}(k)$ and $[\alpha, x]=\alpha(x)$ for $\alpha\in\mathrm{Der}_{k+1}\mathfrak{g}(k)$, $x\in\mathfrak{g}_-$.

By iterating this construction, we obtain a truncated TGLA of order $k+i$, $p^i\mathfrak{g}(k)(=p(p^{i-1}\mathfrak{g}(k)))$, and a TGLA $p^\infty\mathfrak{g}(k)(=\mathrm{ind}\lim p^i\mathfrak{g}(k))$, which are characterized by the following universal properties:

PROPOSITION 2.1. *For a truncated TGLA $\mathfrak{g}(k)$ of order $k\geq -1$, there exists, uniquely up to isomorphism, a truncated TGLA $p^i\mathfrak{g}(k)$ of order $k+i$ $(0\leq i\leq\infty)$, which satisfies the following conditions :*

i) $\mathrm{Trun}_k(p^i\mathfrak{g}(k))=\mathfrak{g}(k)$

ii) *If $\mathfrak{h}(k+i)$ is a truncated TGLA of order $k+i$ and if there is an embedding $\psi_k:\mathrm{Trun}_k\,\mathfrak{h}(k+i)\rightarrow\mathfrak{g}(k)$, then there exists a unique embedding $\psi_{k+i}:\mathfrak{h}(k+i)\rightarrow p^i\mathfrak{g}(k)$ such that $\psi_{k+i}|_{\mathrm{Trun}_k\mathfrak{h}(k+i)}=\psi_k$.*

The proof is straightforward from the above construction.

The truncated TGLA $p^i\mathfrak{g}(k)$ is called the prolongation of $\mathfrak{g}(k)$. $p^\infty\mathfrak{g}(k)$ will be often denoted by $\mathrm{Prol}\,\mathfrak{g}(k)$. We say also that a TGLA $\mathfrak{g}$ is the prolongation of $\mathrm{Trun}_k\,\mathfrak{g}$ if $\mathfrak{g}=\mathrm{Prol}\,\mathrm{Trun}_k\,\mathfrak{g}$. Note that, by the above proposition $\mathfrak{g}$ can be always identified with a graded subalgebra of $\mathrm{Prol}\,\mathrm{Trun}_k\,\mathfrak{g}$.

2.2. Now we proceed to the study of the cohomology group of a TGLA $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$. We set

$$\mathfrak{m}=\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p,$$

which is a nilpotent subalgebra of $\mathfrak{g}$, and consider the cohomology group associated with the adjoint representation of $\mathfrak{m}$ on $\mathfrak{g}$, namely the cohomology group $H(\mathfrak{m}, \mathfrak{g})=\oplus H^p(\mathfrak{m}, \mathfrak{g})$ of the cochain complex $(C(\mathfrak{m}, \mathfrak{g})=\oplus C^p(\mathfrak{m}, \mathfrak{g}), \partial)$, where

$$C^p(\mathfrak{m}, \mathfrak{g})=\mathrm{Hom}(\Lambda^p\mathfrak{m}, \mathfrak{g})$$

and the coboundary operator $\partial:\mathrm{Hom}(\Lambda^p\mathfrak{m}, \mathfrak{g})\rightarrow\mathrm{Hom}(\Lambda^{p+1}\mathfrak{m}, \mathfrak{g})$ is defined by

$$(2.2)\qquad (\partial\omega)(X_1, X_2,\cdots, X_{p+1})=\sum_{i=1}^{n+1}(-1)^{i-1}[X_i, \omega(X_1,\cdots, \hat{X}_i,\cdots, X_{p+1})] + \sum_{1\leq i<j\leq p+1}(-1)^{i+j}\omega([X_i, X_j], X_1,\cdots, \hat{X}_i,\cdots, \hat{X}_j,\cdots, X_{p+1})$$

for $\omega\in\mathrm{Hom}(\Lambda^p\mathfrak{m}, \mathfrak{g})$, $X_1, X_2,\cdots, X_{p+1}\in\mathfrak{m}$.

Since both $\mathfrak{m}$ and $\mathfrak{g}$ are graded, we can define a bigradation $\oplus H_r^p(\mathfrak{m}, \mathfrak{g})$ of $H(\mathfrak{m}, \mathfrak{g})$ as follows: Denote by $\mathrm{Hom}(\Lambda^p\mathfrak{m}, \mathfrak{g})_r$ the set of all homogeneous p-cochains ω of degree r (i.e., $\omega(\mathfrak{g}_{a_1}\wedge\cdots\wedge\mathfrak{g}_{a_p})\subset\mathfrak{g}_{a_1+\cdots+a_p+r}$ for any $a_1,\cdots, a_p<0$), and set

$$C_r(\mathfrak{m}, \mathfrak{g})=\mathrm{Hom}(\Lambda\mathfrak{m}, \mathfrak{g})_r=\bigoplus_p \mathrm{Hom}(\Lambda^p\mathfrak{m}, \mathfrak{g})_r$$

Note that ∂ preserves the degree. Hence $C_r(\mathfrak{m}, \mathfrak{g})$ is a subcomplex and the direct sum decomposition

$$C(\mathfrak{m}, \mathfrak{g})=\oplus C_r(\mathfrak{m}, \mathfrak{g})$$

yields that of the cohomology group:

$$H(\mathfrak{m}, \mathfrak{g})=\oplus H_r(\mathfrak{m}, \mathfrak{g})=\oplus H_r^p(\mathfrak{m}, \mathfrak{g}).$$

This cohomology group $H(\mathfrak{m}, \mathfrak{g})$ was introduced by Tanaka [14] with another gradation:

$$H^{s,p}(\mathfrak{m}, \mathfrak{g})=H_{s+p-1}^p(\mathfrak{m}, \mathfrak{g}).$$

It should be remarked that if μ(the depth of $\mathfrak{g}$)$=1$ then $\mathfrak{m}$ is abelian and $H^{s,p}(\mathfrak{m}, \mathfrak{g})$ is known as the Spencer cohomology group. Thus the following theorem is a generalization of the result known in the case $\mu=1$ to the case of arbitrary μ (cf. [2], [11]).

THEOREM 2.1. *Let $\mathfrak{g}$ be a TGLA of depth μ. Then there exists an integer r_0 such that $H_r(\mathfrak{m}, \mathfrak{g})=0$, for all $r\geq r_0$.*

The proof of the theorem is based on the fact that the universal enveloping algebra $U(\mathfrak{l})$ of a finite dimensional Lie algebra $\mathfrak{l}$ is noetheian (See for example Dixmier [1] pp. 76). We shall give the proof by dividing it into several steps.

1) First of all we introduce for $X\in\mathfrak{m}$ the operators

$$i(X),\ \theta(X): C(\mathfrak{m}, \mathfrak{g})\rightarrow C(\mathfrak{m}, \mathfrak{g})$$

by defining, for $\omega\in C^p(\mathfrak{m}, \mathfrak{g})$, $X_1,\cdots, X_p\in\mathfrak{m}$,

$$(2.3)\qquad (i(X)\omega)(X_1,\cdots, X_{p-1})=\omega(X, X_1,\cdots, X_{p-1})$$

$$(2.4)\qquad (\theta(X)\omega)(X_1, \cdots, X_p)=[X, \omega(X_1,\cdots, X_p)] - \sum_{i=1}^{p}\omega(X_1, \cdots, [X, X_i], \cdots, X_p).$$

Then we have the following standard formulae:

(2.5) $\theta(X)=i(X)\cdot\partial+\partial\cdot i(X)$.
(2.6) $\partial\cdot\theta(x)=\theta(X)\cdot\partial$.
(2.7) $\theta([X, Y])=\theta(X)\cdot\theta(Y)-\theta(Y)\cdot\theta(X)$, for $X, Y\in\mathfrak{m}$.

Note that $C(\mathfrak{m}, \mathfrak{g})$ has a left $U(\mathfrak{m})$-module structure by the representation θ.

2) Passage to the dual. Let $\mathfrak{g}^*=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p^*$ be the dual of $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$, which is a right $U(\mathfrak{m})$-module by the coadjoint reperesentation of $\mathfrak{m}$. Identifying $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$ with the dual of $C(\mathfrak{m}, \mathfrak{g})=\mathrm{Hom}(\Lambda\mathfrak{m}, \mathfrak{g})$, we obtain the chain comlex $(\Lambda\mathfrak{m}\otimes\mathfrak{g}^*, \partial^*)$ and the associated homology group denoted by $H^*(\mathfrak{m}, \mathfrak{g})$, where the boundary operater (the dual of ∂)

$$\partial^* : \Lambda^p\mathfrak{m}\otimes\mathfrak{g}^*\rightarrow\Lambda^{p-1}\mathfrak{m}\otimes\mathfrak{g}^*$$

can be written explicitly as: for $X_i\in\mathfrak{m}$, $\alpha\in\mathfrak{g}^*$,

(2.2*) $$\partial^*(X_1\wedge\cdots\wedge X_p\otimes\alpha)=\sum(-1)^{i-1}X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_p\otimes\alpha X_i + \sum_{i<j}(-1)^{i+j}[X_i, X_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge\hat{X}_j\wedge\cdots\wedge X_p\otimes\alpha.$$

Put $(\Lambda\mathfrak{m}\otimes\mathfrak{g}^*)_r=\sum(\Lambda^{p_1}\mathfrak{g}_{a_1})\wedge\cdots\wedge(\Lambda^{p_t}\mathfrak{g}_{a_t})\otimes\mathfrak{g}_q^*$, where the sum is taken over all $t\geq 0$, $p_i\geq 0$, $a_i<0$ and $q\in Z$ such that $q-\sum_{i=1}^{t}p_i a_i=r$. Since ∂^* preserve the degree, we have the direct sum decomposition of the complex $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$ into subcomplexes:

$$\Lambda\mathfrak{m}\otimes\mathfrak{g}^*=\bigoplus(\Lambda\mathfrak{m}\otimes\mathfrak{g}^*)_r.$$

which yields the decomposition of the homology group:

$$H^*(\mathfrak{m}, \mathfrak{g})=\bigoplus H_r^*(\mathfrak{m}, \mathfrak{g}).$$

Since $(\Lambda\mathfrak{m}\otimes\mathfrak{g}^*)_r$ can be identified with the dual of $C_r(\mathfrak{m}, \mathfrak{g})$, $H_r^*(\mathfrak{m}, \mathfrak{g})$ is isomorphic to the dual space of $H_r(\mathfrak{m}, \mathfrak{g})$.

Observe also that the duals of $i(X)$ and $\theta(X)$:

$$i^*(X),\ \theta(X)^* : \Lambda\mathfrak{m}\otimes\mathfrak{g}^*\rightarrow\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$$

are given by: for $X_i\in\mathfrak{m}$, $\alpha\in\mathfrak{g}^*$,

(2.3*) $i^*(X)(X_1\wedge\cdots\wedge X_p\otimes\alpha)=X\wedge X_1\wedge\cdots\wedge X_p\otimes\alpha$.

(2.4*) $$\theta^*(X)(X_1\wedge\cdots\wedge X_p\otimes\alpha)=X_1\wedge\cdots\wedge X_p\otimes\alpha X - \sum_{i=1}^{p}X_1\wedge\cdots\wedge[X, X_i]\wedge\cdots\wedge X_p\otimes\alpha.$$

Clearly we have

(2.5*) $\theta^*(X)=\partial^*\cdot i^*(X)+i(X)^*\cdot\partial^*$

(2.6*) $\theta^*(X)\cdot\partial^*=\partial^*\cdot\theta^*(X)$

(2.7*) $\theta^*([X, Y])=-\theta^*(X)\cdot\theta^*(Y)+\theta^*(Y)\cdot\theta^*(X)$, for $X, Y\in\mathfrak{m}$.

Note that $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$ has a right $U(\mathfrak{m})$-module structure by the (anti-)representation θ^*.

3) LEMMA 2.1. *$\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$ is finitely generated as a right $U(\mathfrak{m})$-module.*

PROOF. First we observe that $\mathfrak{g}^*$ is finitely generated as a right $U(\mathfrak{m})$-module. In fact, the map

$$\mathfrak{g}_p\to\mathrm{Hom}(\mathfrak{m}, \mathfrak{g}_{p-\mu}\oplus\cdots\oplus\mathfrak{g}_{p-1})$$

is injective for $p\geq 0$ by (G3), therefore the dual map

$$(\mathfrak{g}^*_{p-\mu}\oplus\cdots\oplus\mathfrak{g}^*_{p-1})\otimes\mathfrak{m}\to\mathfrak{g}^*_p$$

is surjective. Noting that this map is just the multiplication of $\mathfrak{m}$ by the right, we see

$$\mathfrak{g}^*=(\bigoplus_{a<0}\mathfrak{g}^*_a)U(\mathfrak{m}).$$

From this follows the first assertion.

In order to see $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$ is finitely generated, we put

$$(\Lambda\mathfrak{m})^j=\sum(\Lambda^{p_1}\mathfrak{g}_{a_1})\wedge\cdots\wedge(\Lambda^{p_t}\mathfrak{g}_{a_t}),$$

where the sum runs over all $t\geq 0$, $p_i\geq 0$, $a_i<0$ such that $-\sum_{i=1}^{t}p_ia_i\geq j$. By the formula (2.4*) for $\theta^*(X)$, it follows that $(\Lambda\mathfrak{m})^j\otimes\mathfrak{g}^*$ is a $U(\mathfrak{m})$-submodule and that the action of $\theta^*(X)$ on the quotient module $((\Lambda\mathfrak{m})^j\otimes\mathfrak{g}^*)/((\Lambda\mathfrak{m})^{j+1}\otimes\mathfrak{g}^*)$ is exactly the multiplication by X by the right. Hence this quotient module is finitely generated, therefore so is $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$ because $(\Lambda\mathfrak{m})^0=\Lambda\mathfrak{m}$ and $(\Lambda\mathfrak{m})^j=0$ for large j.

4) Now we can finish the proof of the theorem: Let Z^* be the cycles of $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$. Since $\theta^*(X)\cdot\partial^*=\partial^*\cdot\theta^*(X)$, Z^* is a right $U(\mathfrak{m})$-submodule of $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$. As $U(\mathfrak{m})$ is noetherian and $\Lambda\mathfrak{m}\otimes\mathfrak{g}^*$ is finitely generated by Lemma 2.1, Z^* is also finitely generated. Hence there exists an integer r_0 such that Z^* is generated as a right $U(\mathfrak{m})$-module by $\bigoplus_{t\leq r_0}Z^*_t$, where Z^*_t denotes the cycles of degree t. Observe now that if ξ is a cycle then $\theta^*(X)\xi$ is a boundary because of the Stokes' fromula (2.5*). It then follows that $\mathrm{H}^*_r(\mathfrak{m}, \mathfrak{g})=0$ for $r>r_0$. Hence $H_r(\mathfrak{m}, \mathfrak{g})=0$ for $r>r_0$, which proves the theorem.

COROLLARY 2.1. *Let $\mathfrak{g}$ be a TGLA, then there exists an integer $k\geq 0$ such that $\mathfrak{g}$ is the prolongation of* $\mathrm{Trun}_{k-1}\,\mathfrak{g}$.

In fact, according to the definition of the cohomology group $H(\mathfrak{m}, \mathfrak{g})$, we see that for $k\geq 0$

$$H^1_k(\mathfrak{m}, \mathfrak{g})=\mathrm{Der}_k(\mathrm{Trun}_{k-1}\,\mathfrak{g})/\mathfrak{g}_k.$$

Hence $\mathrm{Trun}_k\,\mathfrak{g}$ is the prolongation of $\mathrm{Trun}_{k-1}\,\mathfrak{g}$ if and only if $H^1_k(\mathfrak{m}, \mathfrak{g})=0$. Therefore the corollary follows from Theorem 2.1.

§ 3. Truncated transitive filtered Lie algebras.

In this section we define the truncated transitive filtered Lie algebras and study the structures of these algebras.

3.1. Let A be a vector space. For $\alpha, \beta\in\mathrm{Hom}(\Lambda^2 A, A)$ define $\alpha\circ\beta\in\mathrm{Hom}(\Lambda^3 A, A)$ by

$$(\alpha\circ\beta)(x, y, z)=\mathfrak{S}\ \alpha(\beta(x, y), z),$$

where $\mathfrak{S}$ denotes the cyclic sum in $x, y, z\in A$. Define then a quadratic map

$$J: \mathrm{Hom}(\Lambda^2 A, A)\to\mathrm{Hom}(\Lambda^3 A, A).$$

by $J(\gamma)=\gamma\circ\gamma$ for $\gamma\in\mathrm{Hom}(\Lambda^2 A, A)$. Note that to define a Lie algebra structure on A is to pick out a $\gamma\in\mathrm{Hom}(\Lambda^2 A, A)$ satisfying $J(\gamma)=0$.

Now suppose that A is endowed with a descending filtration $\{A^p\}_{p\in\boldsymbol{Z}}$, that is, a sequence of subspaces of A such that $A^p\supset A^{p+1}$. Then it induces on $\mathrm{Hom}(\Lambda^r A, A)$ a natural filtration $\{\mathrm{Hom}(\Lambda^r A, A)^k\}_{k\in\boldsymbol{Z}}$, where $\mathrm{Hom}(\Lambda^r A, A)^k$ consists of all $\alpha\in\mathrm{Hom}(\Lambda^r A, A)$ satisfying $\alpha(A^{p_1}\wedge\cdots\wedge A^{p_r})\subset A^{p_1+\cdots+p_r+k}$ for any $(p_1, \cdots, p_r)\in\boldsymbol{Z}^r$.

Let us define on $\mathrm{Hom}(\Lambda^r A, A)^0$ another filtration $\{I^k\mathrm{Hom}(\Lambda^r A, A)^0\}_{k\in\boldsymbol{Z}}$ which will play an important rôle in the sequel. We set, for $k, p_1,\cdots, p_r\in\boldsymbol{Z}$,

$$\tau(k\,; p_1,\cdots, p_r)=\mathrm{Min}\{k, k-p_{i_1}-\cdots-p_{i_l}\ (1\leq i_1<\cdots<i_l\leq r,\ 1\leq l\leq r)\} \tag{3.1}$$

We then define $I^k\mathrm{Hom}(\Lambda^r A, A)^0$ to be the subspace of $\mathrm{Hom}(\Lambda^r A, A)^0$ consisting of all $\alpha\in\mathrm{Hom}(\Lambda^r A, A)^0$ such that

$$\alpha(A^{p_1}\wedge\cdots\wedge A^{p_r})\subset A^{p_1+\cdots+p_r+\tau(k\,; p_1\cdots p_r)}$$

for any $(p_1,\cdots, p_r)\in\boldsymbol{Z}^r$. It is easy to check that if $\alpha-\beta\in I^k\mathrm{Hom}(\Lambda^2 A, A)^0$ for $\alpha, \beta\in\mathrm{Hom}(\Lambda^2 A, A)^0$ then $J(\alpha)-J(\beta)\in I^k\mathrm{Hom}(\Lambda^3 A, A)^0$. Therefore if we put

$$[\mathrm{Hom}(\Lambda^r A, A)^0]^k = \mathrm{Hom}(\Lambda^r A, A)^0 / I^{k+1}\mathrm{Hom}(\Lambda^r(A, A)^0,$$

we have the induced map

$$J : [\mathrm{Hom}(\Lambda^2 A, A)^0]^k \to [\mathrm{Hom}(\Lambda^3 A, A)^0]^k$$

defined by $J[\alpha]^k = [J(\alpha)]^k$ for $\alpha \in \mathrm{Hom}(\Lambda^2 A, A)^0$, where $[\beta]^k$ denotes the equivalence class of $\beta \in \mathrm{Hom}(\Lambda^r A, A)^0$ modulo $I^{k+1}\mathrm{Hom}(\Lambda^r A, A)^0$.

DEFINITION 3.1. A truncated filtered Lie algebra of order k $(0 \leq k \leq \infty)$ is a vecter space A endowed with a descending filtration $\{A^p\}_{p \in Z}$ and a "truncated bracket" $[\gamma]^k \in [\mathrm{Hom}(\Lambda^2 A, A)^0]^k$ satisfying the following conditions:

1) $A^{k+1} = 0$
2) $J([\gamma]^k) = 0$ (truncated Jacobi identity).

Note that if $A(k) = (A, \{A^p\}, [\gamma]^k)$ is a truncated filtered Lie algebra then gr A has the induced structure of a truncated graded Lie algebra, which will be denoted by gr $A(k)$.

DEFINITION 3.2. A truncated filtered Lie algebra $A(k)$ is called a truncated transitive filtered Lie algebra (truncated TFLA) if gr $A(k)$ is transitive.

Note that a truncated TFLA of order ∞ is just a TFLA.

Let $A(k) = (A, \{A^p\}, [\gamma]^k)$ be a truncated TFLA of order k $(0 \leq k \leq \infty)$. For an integer l $(0 \leq l \leq k)$ we set $A(l) = A/A^{l+1}$ and define on it a filtration $\{A(l)^p\}$ by $A(l)^p = A^p / A^p \cap A^{l+1}$. If we identify $A(l)$ with a subspace W of A complementary to A^{l+1} and define $\gamma_W \in \mathrm{Hom}(\Lambda^2 A(l), A(l))^0$ by

$$\gamma_W(x, y) = W\text{-component of } \gamma(x, y) \qquad \text{for } x, y \in W,$$

where γ is a representative of $[\gamma]^k$, then $[\gamma_W]^l \in [\mathrm{Hom}(\Lambda^2 A(l), A(l))^0]^l$ does not depend on the choice of W and γ, and satisfies $J([\gamma_W]^l) = 0$. Thus $(A(l), \{A(l)^p\}, [\gamma_W]^l)$ becomes a truncated TFLA of order l, which will be denoted by $\mathrm{Trun}_l\, A(k)$.

Morphisms of truncated TFLA's can be defined in the natural manner. Let $A(k) = (A, \{A^p\}, [\gamma]^k)$, $B(k) = (B, \{B^p\}, [\delta]^k)$ be truncated TFLA's of order k. A filtration preserving linear map $\psi : A \to B$ is called a homomorphism of $A(k)$ to $B(k)$ if

$$\psi(\gamma[x_p, y_p]) \equiv \delta(\psi(x_p), \psi(y_q)) \bmod B^{p+q+\tau(k+1;p,q)}$$

for $x_p \in A^p$, $y_q \in A^q$. Note that this condition does not depend on the choice of representatives γ, δ. Two homomorphisms ψ, ψ' of $A(k)$ to $B(k)$ are said to be congruent (denoted as $\psi\equiv\psi'$) if

$$\psi(x_p)\equiv\psi'(x_p) \pmod{B^{p+\tau(k+1;p)}}, \text{ for any } x_p\in A^p.$$

If $\psi:A(k)\to B(k)$ is a homomorphism then it induces for each $0\leq l\leq k$ a homomorphis: $\mathrm{Trun}_l\, A(k)\to\mathrm{Trun}_l\, B(k)$ denoted by $\mathrm{Trun}_l\,\psi$ and also a homomorphism $\mathrm{gr}\, A(k)\to\mathrm{gr}\, B(k)$ denoted by $\mathrm{gr}\,\psi$. A homomorphism ψ: $A(k)\to B(k)$ is called an embedding if $\mathrm{gr}\,\psi$ is an embedding.

3.2. For a truncated TFLA $A(k)$, the cohomology group $H^p_r((\mathrm{Prol\ gr}\, A(k))_-, \mathrm{Prol\ gr}\, A(k))$ associated to the TGLA $\mathrm{Prol\ gr}\, A(k)$ will be simply denoted by $H^p_r(\mathrm{gr}\, A(k))$. Our main goal of this section is to prove the following propositions:

Proposition 3.1 *Let $A(k)$ be a truncated TFLA of order k $(0\leq k<\infty)$ satisfying:*

$$\begin{cases} H^2_{k+1}(\mathrm{gr}\, A(k))=0, & \\ H^3_{k+1}(\mathrm{gr}\, A(k))=0 & (\text{if } k\geq 1). \end{cases}$$

Then there exists a truncated TFLA $A(k+1)$ of order $k+1$ such that $\mathrm{Trun}_k A(k+1)=A(k)$ *and that* $\mathrm{gr}\, A(k+1)$ *is the prolongation of* $\mathrm{gr}\, A(k)$.

Proposition 3.2. *Let $A(k+1)$, $B(k+1)$ be truncated TFLA's of order $k+1$ $(k\geq 0)$ and assume that*

$$H^1_{k+1}(\mathrm{gr}\, B(k+1))=H^2_{k+1}(\mathrm{gr}\, B(k+1))=0.$$

If there is an embedding $\psi_k:\mathrm{Trun}_k A(k+1)\to\mathrm{Trun}_k B(k+1)$ then there exists an embedding $\psi_{k+1}: A(k+1)\to B(k+1)$ such that $\mathrm{Trun}_k\psi_{k+1}\equiv\psi_k$.

Before entering into the proof of the above propositions, we shall first recall, following Rim [10], several formulae in skewgraded Lie algebras, and then we shall prepare a lemma for the proof of Proposition 3.1.

Let V be a vector space. For $f^{(m)}\in\mathrm{Hom}(\Lambda^{m+1}V, V)$ and $f^{(n)}\in\mathrm{Hom}(\Lambda^{n+1}V, V)$, define $f^{(n)}\circ f^{(m)}\in\mathrm{Hom}(\Lambda^{m+n+1}V, V)$ by the following rule:

$$\begin{aligned} &f^{(n)}\circ f^{(m)}(x_1,\cdots, x_{m+n+1}) \\ &\quad=\sum\mathrm{sign}(\sigma)f^{(n)}(f^{(m)}(x_{\sigma(1)}\cdots x_{\sigma(m+1)}), x_{\sigma(m+2)}\cdots x_{\sigma(m+n+1)}) \end{aligned}$$

where σ runs through all permutations of $(1, 2,\cdots, m+n+1)$ such that $\sigma(1)<\cdots<\sigma(m+1)$ and $\sigma(m+2)<\cdots<\sigma(m+n+1)$. Then we put

$$[f^{(n)}, f^{(m)}]=f^{(n)}\circ f^{(m)}-(-1)^{nm}f^{(m)}\circ f^{(n)}.$$

By this bracket $\mathrm{Hom}(\Lambda V, V)=\oplus\,\mathrm{Hom}(\Lambda^n V, V)$ becomes a skew-graded Lie algebras; we have

(3.2) $\quad [f^{(n)}, f^{(m)}]=-(-1)^{nm}[f^{(m)}, f^{(n)}]$

(3.3) $\quad (-1)^{mp}[[f^{(m)}, f^{(n)}], f^{(p)}]+(-1)^{nm}[[f^{(n)}, f^{(p)}], f^{(m)}] +(-1)^{pn}[[f^{(p)}, f^{(m)}], f^{(n)}]=0.$

If V is endowed with a Lie algebra structure by $\gamma\in\mathrm{Hom}(\Lambda^2 V, V)$, the coboundary operator $\partial:\mathrm{Hom}(\Lambda^{n+1}V, V)\to\mathrm{Hom}(\Lambda^{n+2}V, V)$ can be written as,

(3.4) $\quad \partial f^{(n)}=(-1)^n[\gamma, f^{(n)}]$, for $f^{(n)}\in\mathrm{Hom}(\Lambda^{n+1}V, V)$.

As to the contraction $i(x)$ by $x\in V$, we have

(3.5) $\quad \begin{cases} i(x)(f^{(n)}\circ f^{(m)})=(-1)^m(i(x)f^{(n)})\circ f^{(m)}+f^{(n)}\circ i(x)f^{(m)}. \\ i(x)[f^{(n)}, f^{(m)}]=(-1)^m[i(x)f^{(n)}, f^{(m)}]+[f^{(n)}, i(x)f^{(m)}]. \end{cases}$

In what follows we will often use the following notation: For graded vector spaces $V=\oplus V_p$, $W=\oplus W_p$, $\mathrm{Hom}(\Lambda^m V, W)_l$ will denote the set of all $\alpha_l\in\mathrm{Hom}(\Lambda^m V, W)$ such that, for any $p_1,\cdots, p_m\in Z$,

$$\alpha_l(V_{p_1}\wedge\cdots\wedge V_{p_m})\subset W_{p_1+\cdots+p_m+l}.$$

For $\alpha\in\mathrm{Hom}(\Lambda^m V, W)$, we will denote by α_l the $\mathrm{Hom}(\Lambda^m V, W)_l$-component of α and write formally as $\alpha=\sum_{l\in Z}\alpha_l$.

Now let $A(k)=(A, \{A^p\}, [\gamma]^k)$ be a truncated TFLA of order k. Let us study its structure a little more closely. We write $\mathfrak{a}=\bigoplus_{p\in Z}\mathfrak{a}_p=\mathrm{Prol\ gr}\,A(k)$. By choosing a direct sum decomposition $A=\bigoplus_{p\le k}A_p$ such that $A^i=\bigoplus_{i\le p\le k}A_p$, we identify $\mathfrak{a}_p$ with A_p, thus $A=\bigoplus_{p\le k}\mathfrak{a}_p$. Let γ be a representative of $[\gamma]^k$ and consider it as an element of $\mathrm{Hom}(\Lambda^2\mathfrak{a}, \mathfrak{a})^0$. Write

$$\gamma=\sum_{l\ge 0}\gamma_l, \qquad \text{with } \gamma_l\in\mathrm{Hom}(\Lambda^2\mathfrak{a}, \mathfrak{a})_l.$$

We may assume that γ_0 coincides with the bracket of $\mathfrak{a}$. The components of γ which are effectively determined by $[\gamma]^k$ are the following:

(3.6) $\quad \gamma_l(x_p, y_q)$, for $x_p\in\mathfrak{a}_p$, $y_q\in\mathfrak{a}_q$, $l\le\tau(k\,;p, q)$.

The truncated Jacobi identity may be written as

(3.7) $\quad (\gamma\circ\gamma)_l(x_p, y_q, z_r)=0 \quad$ for $x_p\in\mathfrak{a}_p$, $y_q\in\mathfrak{a}_q$, $z_r\in\mathfrak{a}_r$, $l\le\tau(k\,;p, q, r)$.

More explicitly it may be expressed as

$$(3.8)\quad \begin{cases} \text{i)} & \iota_{\mathfrak{m}}^{*}(\gamma\circ\gamma)_l=0 & (l\le k) \\ \text{ii)} & \iota_{\mathfrak{m}}^{*}i(x_i)(\gamma\circ\gamma)_l=0 & (l\le k-i) \\ \text{iii)} & \iota_{\mathfrak{m}}^{*}i(x_i)i(y_j)(\gamma\circ\gamma)_l=0 & (l\le k-i-j) \\ \text{iv)} & (\gamma\circ\gamma)_l(x_i,y_j,z_m)=0 & (l\le k-i-j-m) \end{cases}$$

for any $x_i\in\mathfrak{a}_i$, $y_j\in\mathfrak{a}_j$, $z_m\in\mathfrak{a}_m$, $i, j, m\ge 0$, where we put $\mathfrak{m}=\bigoplus_{p<0}\mathfrak{a}_p$ and denote by $\iota_{\mathfrak{m}}^{*}$ the canonical restriction map: $\mathrm{Hom}(\Lambda\mathfrak{a},\mathfrak{a})\to\mathrm{Hom}(\Lambda\mathfrak{m},\mathfrak{a})$ and denote by $i(x)$ the contraction by $x\in\mathfrak{a}$.

LEMMA 4.1. *Let the notation be as above and denote by ∂ the coboundary operator of* $\mathrm{Hom}(\Lambda\mathfrak{m},\mathfrak{a})$, *then we have :*

i) $\partial\iota_{\mathfrak{m}}^{*}(\gamma\circ\gamma)_{k+1}=0$

If moreover $\iota_{\mathfrak{m}}^{*}(\gamma\circ\gamma)_{k+1}=0$, then for $x_i\in\mathfrak{a}_i$ $(0\le i\le k+1)$,

ii) $\partial\iota_{\mathfrak{m}}^{*}i(x_i)(\gamma\circ\gamma)_{k+1-i}=0$

PROOF. We set ; $J_l=(\gamma\circ\gamma)_l$. Denoting by $\partial_{\mathfrak{a}}$ the coboundary operator of $\mathrm{Hom}(\Lambda\mathfrak{a},\mathfrak{a})$, and using the notation of skew-graded Lie algebras, we have :

$$\begin{aligned} J_l &= \sum_{\substack{s+t=l\\ s,t\ge 0}}\gamma_s\circ\gamma_t \\ &=[\gamma_0,\gamma_l]+\frac{1}{2}\sum_{\substack{s+t=l\\ s,t>0}}[\gamma_s,\gamma_t] \\ &=-\partial_{\mathfrak{a}}\gamma_l+\frac{1}{2}\sum_{\substack{s+t=l\\ s,t>0}}[\gamma_s,\gamma_t]. \end{aligned}$$

Then we have :

$$\begin{aligned} \partial\iota_{\mathfrak{m}}^{*}J_{k+1} &= \iota_{\mathfrak{m}}^{*}\partial_{\mathfrak{a}}J_{k+1} \\ &=\frac{1}{2}\iota_{\mathfrak{m}}^{*}\partial_{\mathfrak{a}}\sum_{\substack{s+t=k+1\\ s,t>0}}[\gamma_s,\gamma_t] \\ &=\frac{1}{2}\iota_{\mathfrak{m}}^{*}\sum_{\substack{s+t=k+1\\ s,t>0}}[\gamma_0,[\gamma_s,\gamma_t]] \\ &=\frac{1}{2}\iota_{\mathfrak{m}}^{*}\sum_{\substack{s+t=k+1\\ s,t>0}}\{[[\gamma_0,\gamma_t],\gamma_s]+[[\gamma_0,\gamma_s],\gamma_t]\} \end{aligned}$$

(by skew Jacobi identity (3. 3) and (3. 2)).

$$=\iota_{\mathfrak{m}}^{*}\sum_{\substack{s+t=k+1\\ s,t>0}}[[\gamma_0,\gamma_s],\gamma_t]$$

$$\begin{aligned}
&= \iota_{\mathfrak{m}}^* \sum_{\substack{s+t=k+1\\ s,t>0}} [J_s, \gamma_t] - \frac{1}{2}\iota_{\mathfrak{m}}^* \sum_{\substack{r+s+t=k+1\\ r,s,t>0}} [[\gamma_r, \gamma_s], \gamma_t] \\
&= \iota_{\mathfrak{m}}^* \sum_{\substack{s+t=k+1\\ s,t>0}} [J_s, \gamma_t] \quad \text{(by skew Jacobi identity).}
\end{aligned}$$

But on account of the hypothesis $\iota_{\mathfrak{m}}^* J_l = 0$ $(l \leq k)$ and $\iota_{\mathfrak{m}}^* i(x_i) J_l = 0$ $(l \leq k-1-i)$, we see that

$$\iota_{\mathfrak{m}}^*[J_s, \gamma_t] = 0 \qquad \text{if } s+t=k+1,\ s, t>0.$$

In fact, for $a, b, c, d<0$, $[J_s, \gamma_t](x_a, y_b, z_c, w_d)$ is a sum of the following kind of terms:

$$\gamma_t(J_s(x_a, y_b, z_c), w_d) \text{ and } J_s(\gamma_t(x_a, y_b), z_c, w_d).$$

Since $s \leq k$, from $\iota_{\mathfrak{m}}^* J_l = 0$ $(l \leq k)$ it follows that $J_s(x_a, y_b, z_c) = 0$ and that $J_s(\gamma_t(x_a, y_b), z_c, w_d) = 0$ if $t+a+b<0$. If $t+a+b \geq 0$, it follows from $\iota_{\mathfrak{m}}^* i(x_i) J_l = 0$ $(l \leq k-1-i)$ that $J_s(\gamma_t(x_a, y_b), z_c, w_d) = 0$ because $k-1-(a+b+t) = s-2-(a+b) \geq s$. Hence $\iota_{\mathfrak{m}}^*[J_s, \gamma_t] = 0$, which proves $\partial \iota_{\mathfrak{m}}^* J_{k+1} = 0$.

Next we have

$$\begin{aligned}
&\partial \iota_{\mathfrak{m}}^* i(x_i) J_{k+1-i} \\
&\quad = \iota_{\mathfrak{m}}^* \partial_{\mathfrak{a}} i(x_i) J_{k+1-i} \\
&\quad = \iota_{\mathfrak{m}}^* (\theta(x_i) - i(x_i)\partial_{\mathfrak{a}}) J_{k+1-i} \quad \text{(by Stokes' formula (2.5))} \\
&\quad = \iota_{\mathfrak{m}}^* \theta(x_i) J_{k+1-i} - \frac{1}{2}\iota_{\mathfrak{m}}^* i(x_i) \sum_{\substack{s+t=k+1-i\\ s,t>0}} [\gamma_0, [\gamma_s, \gamma_t]] \\
&\quad = \iota_{\mathfrak{m}}^* \theta(x_i) J_{k+1-i} - \iota_{\mathfrak{m}}^* i(x_i) \sum_{\substack{s+t=k+1-i\\ s,t>0}} [J_s, \gamma_t].
\end{aligned}$$

By a similar arguement as above, we see that the last terms vanish because of the hypothesis $J_l(x_p, y_q, z_r) = 0$ $(l \leq \tau(k\,; p, q, r))$ and $\iota_{\mathfrak{m}}^* J_{k+1} = 0$, which proves ii).

3.3. Proof of Proposition 3.1. Let $A(k) = (A, \{A^p\}, [\gamma]^k)$ be a truncated TFLA of order k satisfying the assumption of the proposition. Keeping the same notation as before we denote by $\mathfrak{a} = \bigoplus_{p \in \boldsymbol{Z}} \mathfrak{a}_p$ the prolongation of $\operatorname{gr} A(k)$, and identify A with $\bigoplus_{p \leq k} \mathfrak{a}_p$. We take a representative $\gamma = \sum_{l \geq 0} \gamma_l$ of $[\gamma]^k$ as an element of $\operatorname{Hom}(\Lambda^2\mathfrak{a}, \mathfrak{a})^0$ so that γ_0 coincides with the bracket of $\mathfrak{a}$. To prove the proposition, we will define on $\bigoplus_{p \leq k+1} \mathfrak{a}_p$ a truncated TFLA $A(k+1)$ of order $k+1$ satisfying the assertion of the proposition. In order that, it suffices to show that there exists $\hat{\gamma} = \sum_{l \geq 0} \hat{\gamma}_l \in \operatorname{Hom}(\Lambda^2\mathfrak{a}, \mathfrak{a})^0$ satisfying the following conditions:

$$(3.9)\quad \begin{cases} \text{i)} & \hat{\gamma}_0=\gamma_0=\text{the bracket of } \mathfrak{a}. \\ \text{ii)} & \hat{\gamma}_l(x_p, y_q)=\gamma_l(x_p, y_q) \qquad \text{for } l\leq\tau(k\,;p, q) \\ \text{iii)} & (\hat{\gamma}\circ\hat{\gamma})_l(x_p, y_q, z_r)=0 \qquad \text{for } l\leq\tau(k+1\,;p, q, r) \end{cases}$$

From i) and ii), we may express $\hat{\gamma}$ as

$$(3.10)\quad \hat{\gamma}=\gamma+\xi+\sum_{0\leq i\leq k}\eta_i+\sum_{\substack{i+j\leq k\\ i,j\geq 0}}\zeta_{ij}, \text{ with } \xi\in\mathrm{Hom}(\Lambda^2\mathfrak{m}, \mathfrak{a})_{k+1},$$
$$\eta_i\in\mathrm{Hom}(\mathfrak{a}_i, \mathrm{Hom}(\mathfrak{m}, \mathfrak{a})_{k+1}),\ \zeta_{ij}\in\mathrm{Hom}(\mathfrak{a}_i\wedge\mathfrak{a}_j, \mathfrak{a}_{k+1}).$$

Now we are going to determine ξ, η_i, ζ_{ij} so as to satisfy (3.9)iii). Since it is obviously satisfied for $l<\tau(k+1\,;p, q, r)$, we have only to examine the case $l=\tau(k+1\,;p, q, r)$, that is, the following conditions must be satisfied:

$$(3.11)\quad \begin{cases} \text{i)} & \iota_{\mathfrak{m}}^*(\hat{\gamma}\circ\hat{\gamma})_{k+1}=0 \\ \text{ii)} & \iota_{\mathfrak{m}}^*i(x_i)(\hat{\gamma}\circ\hat{\gamma})_{k+1-i}=0 \qquad (0\leq i\leq k+1) \\ \text{iii)} & \iota_{\mathfrak{m}}^*i(x_i)i(y_j)(\hat{\gamma}\circ\hat{\gamma})_{k+1-i-j}=0 \qquad (i, j\geq 0,\ i+j\leq k+1) \\ \text{iv)} & (\hat{\gamma}\circ\hat{\gamma})_{k+1-i-j-m}(x_i, y_j, z_m)=0 \qquad (i, j, m\geq 0,\ i+j+m\leq k+1) \end{cases}$$

Note that since $\hat{\gamma}_0$ is the bracket of $\mathfrak{a}$ the above conditions ii), iii), or iv) are satisfied if $i=k+1$, $i+j=k+1$, or $i+j+m=k+1$ respectively.

To check the above condition i), we note that, taking account of (3.10), we have:

$$(3.12)\quad \iota_{\mathfrak{m}}^*(\hat{\gamma}\circ\hat{\gamma})_{k+1}=-\partial\xi+\iota_{\mathfrak{m}}^*(\gamma\circ\gamma)_{k+1}.$$

But by Lemma 3.1 $\iota_{\mathfrak{m}}^*(\gamma\circ\gamma)_{k+1}$ is a cocycle (in case $k=0$ $\iota_{\mathfrak{m}}^*(\gamma\circ\gamma)_1(=-\partial\iota_{\mathfrak{m}}^*\gamma_1)$ is a fortiori a coboundary). Hence our hypothesis $H_{k+1}^3(\mathfrak{a})=0$ (if $k\geq 1$) allows us to find ξ so as to satisfy (3.11) i).

Now choose such a ξ and put $\tilde{\gamma}=\gamma+\xi$. Then, noting that $\iota_{\mathfrak{m}}^*(\tilde{\gamma}\circ\tilde{\gamma})_{k+1}=0$, we have, for $x_i\in\mathfrak{a}_i$ $(0\leq i\leq k)$;

$$(3.12')\quad \iota_{\mathfrak{m}}^*i(x_i)(\hat{\gamma}\circ\hat{\gamma})_{k+1-i}=\partial\eta_i(x_i)+\iota_{\mathfrak{m}}^*i(x_i)(\tilde{\gamma}\circ\tilde{\gamma})_{k+1-i}.$$

In fact,

$$\begin{aligned} &\iota_{\mathfrak{m}}^*i(x_i)((\hat{\gamma}\circ\hat{\gamma})_{k+1-i}-(\tilde{\gamma}\circ\tilde{\gamma})_{k+1-i}) \\ &\quad=-\iota_{\mathfrak{m}}^*i(x_i)\partial(\hat{\gamma}_{k+1-i}-\gamma_{k+1-i}) \\ &\quad=-\iota_{\mathfrak{m}}^*\theta(x_i)(\hat{\gamma}_{k+1-i}-\gamma_{k+1-i})+\partial\iota_{\mathfrak{m}}^*i(x_i)(\hat{\gamma}_{k+1-i}-\gamma_{k+1-i}) \\ &\quad=\partial\eta_i(x_i). \end{aligned}$$

Hence, by lemma 3.1 and our hypothesis $H_{k+1}^2(\mathfrak{a})=0$, we can find η_i $(0\leq i\leq k)$ so as to satisfy (3.11) ii).

Finally let us define $\zeta_{i,j}$ by induction on $h=i+j$. For an integer $h\geq 0$, put

$$\gamma^{(h)}=\gamma+\xi+\sum_{0\leq i\leq k}\eta_i+\sum_{\substack{i,j\leq 0\\ i+j<h}}\zeta_{i,j}$$

and suppose $\gamma^{(h)}$ satisfies:

$$(3.13)\quad\begin{cases}\iota_{\mathfrak{m}}^*(\gamma^{(h)}\circ\gamma^{(h)})_l=0 & (l\leq k+1)\\ \iota_{\mathfrak{m}}^* i(x_i)(\gamma^{(h)}\circ\gamma^{(h)})_l=0 & (l\leq k+1-i)\\ \iota_{\mathfrak{m}}^* i(x_i)i(y_j)(\gamma^{(h)}\circ\gamma^{(h)})_l=0 & \\ \qquad (l\leq k-i-j, \text{ or } l=k+1-i-j \text{ and } i+j<h) & \\ (\gamma^{(h)}\circ\gamma^{(h)})_l(x_i, y_j, z_m)=0 & (l\leq k-i-j-m)\end{cases}$$

for $x_i\in\mathfrak{a}_i$, $y_j\in\mathfrak{a}_j$, $z_m\in\mathfrak{a}_m$, $i, j, m\geq 0$.

Again similar arguements as in the proofs of (3.12′) and of Lemma 3.1 will show that, for $i+j=h$, $i, j\geq 0$,

$$\begin{aligned}&\iota_{\mathfrak{m}}^* i(x_i)i(x_j)(\gamma^{(h+1)}\circ\gamma^{\cdot h+1)})_{k+1-i-j}\\ &\quad=\partial\zeta_{ij}(x_i, y_j)+\iota_{\mathfrak{m}}^* i(x_i)i(y_j)(\gamma^{(h)}\circ\gamma^{(h)})_{k+1-i-j}\end{aligned}$$

and that $\partial\iota_{\mathfrak{m}}^* i(x_i)i(y_j)(\gamma^{(h)}\circ\gamma^{(h)})_{k+1-i-j}=0$. Therefore, noting that $H^1_{k+1}(\mathfrak{a})=0$ since $\mathfrak{a}$ is the prolongation of $\bigoplus_{p\leq k}\mathfrak{a}_p$, we can determine ζ_{ij} $(i+j=h$, $i, j\geq 0)$ so that we have

$$\iota_{\mathfrak{m}}^* i(x_i)i(x_j)(\gamma^{(h+1)}\circ\gamma^{(h+1)})_l=0$$

for $l=k+1-i-j$, $i+j<h+1$, which completes the induction.

Thus we have shown that we can find $\tilde{\gamma}$ satisfying i), ii) and iii) of (3.11). It remains to check iv). But this is a direct consequence of i), ii), iii). In fact, we see, by repeated application of the truncated Jacobi dientity (i.e., i), ii), iii)), that

$$\tilde{\gamma}((\tilde{\gamma}\circ\tilde{\gamma})(x_i, y_j, z_m), v_a)\equiv 0 \qquad (\mathrm{mod} \bigoplus_{p\geq k+2+a}\mathfrak{a}_p)$$

for $i+j+m\leq k$, $i, j, m\geq 0$, $a<0$, which, by virtue of (G3), implies iv). Thus we have completed the proof of Proposition 3.1.

3.4. Proof of Proposition 3.2. Write $\mathrm{Trun}_k A(k+1)=A(k)$, $\mathrm{Trun}_k B(k+1)=B(k)$. For a given embedding $\psi_k: A(k)\to B(k)$, take any filtration preserving injective linear map $\phi_{k+1}: A(k+1)\to B(k+1)$ such that $\mathrm{Trun}_k\,\phi_{k+1}\equiv\psi_k$. Our task is to modify it so as to obtain an embedding $\psi_{k+1}: A(k+1)\to B(k+1)$.

Write: $\mathrm{Prol}\ \mathrm{gr}\, A(k+1)=\mathfrak{a}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{a}_p$, $\mathrm{Prol}\ \mathrm{gr}\, B(k+1)=\mathfrak{b}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{b}_p$. By fixing a direct sum decomposition of $A(k+1)$ compatible with the filtration, we identify once and for all $A(k+1)$ with $\bigoplus_{p\leq k+1}\mathfrak{a}_p$. Furthermore, by choosing a direct sum decomposition $B(k+1)=\bigoplus_{p\leq k+1}B_p$ compatible with the filtration such that $\phi_{k+1}(\mathfrak{a}_p)\subset B_p$ for $p\leq k+1$, we identify $\mathfrak{b}_p$ with B_p. We also identify $A(k+1)$ with $\phi_{k+1}(A(k+1))$ via ϕ_{k+1}. Thus by these identifications (depending on ϕ_{k+1}), we have

$$(3.14)\quad \begin{cases} A(k+1)=\bigoplus_{p\leq k+1}\mathfrak{a}_p, \quad B(k+1)=\bigoplus_{p\leq k+1}\mathfrak{b}_p, \\ \mathfrak{a}_p=\mathfrak{b}_p \ (p<0), \quad \mathfrak{a}_p\subset\mathfrak{b}_p \ (0\leq p\leq k+1) \end{cases}$$

Let γ, δ be representatives of the truncated brackets of $A(k+1)$ and $B(k+1)$ respectively and write (depending on the above identifications) $\gamma=\sum_{l\geq 0}\gamma_l$, $\delta=\sum_{l\geq 0}\delta_l$, with $\gamma_l\in\mathrm{Hom}(\Lambda^2\mathfrak{a},\mathfrak{a})_l$, $\delta_l\in\mathrm{Hom}(\Lambda^2\mathfrak{b},\mathfrak{b})_l$.

Then, since ψ_k is a homomorphism and $\mathrm{Trun}_k\,\phi_{k+1}\equiv\psi_k$, we have

$$(3.15)\qquad \gamma_l(x_p, y_q)=\delta_l(x_p, y_q) \qquad \text{if } l<\tau(k+1\,;p, q),$$

for $x_p\in\mathfrak{a}_p$, $y_q\in\mathfrak{a}_q$, $p, q\leq k+1$.

Let ψ_{k+1} be another filtration preserving linear map: $A(k+1)\to B(k+1)$ such that $\mathrm{Trun}_k\,\psi_{k+1}\equiv\psi_k$. Then there is a linear map $f: A(k+1)\to B(k+1)$ such that

$$(3.16)\quad \begin{cases} f(\mathfrak{a}_p)\subset\mathfrak{b}_{\tau(k+1\,;p)+p} & (p\leq k), \\ \psi_{k+1}(x_p)\equiv x_p+f(x_p) & (\mathrm{mod}\ A(k+1)^{p+\tau(k+1\,;p)+1}) \text{ for } x_p\in\mathfrak{a}_p. \end{cases}$$

If we define $K\in\mathrm{Hom}(\Lambda^2A(k+1), B(k+1))$ by

$$K(x, y)=\psi_{k+1}(\gamma(x, y))-\delta(\psi_{k+1}(x), \psi_{k+1}(y)),$$

then $K(x_p, y_q)\in B(k+1)^{\tau(k+1\,;p,q)+p+q}$ for any $x_p\in A(k+1)^p$, $y_q\in A(k+1)^q$. Therefore ψ_{k+1} is an embedding if and only if

$$(3.17)\qquad K_{\tau(k+1\,;p,q)}(x_p, y_q)=0, \text{ for } x_p\in\mathfrak{a}_p,\ y_q\in\mathfrak{a}_q,\ p, q<k+1.$$

On the other hand it follows easily from (3.15) and (3.16) that we have, for $p, q<0$,

$$\begin{aligned} K_{\tau(k+1\,;p,q)}(x_p, y_q)&=\gamma_{k+1}(x_p, y_q)-\delta_{k+1}(x_p, y_q) \\ &\quad -\{\delta_0(x_p, f(x_q))+\delta_0(f(x_p), y_q)-f(\gamma_0(x_p, y_q))\}, \end{aligned}$$

or

$$\iota_{\mathfrak{m}}^*K_{k+1}=\iota_{\mathfrak{m}}^*(\gamma_{k+1}-\delta_{k+1})-\partial\iota_{\mathfrak{m}}^*f,$$

where $\mathfrak{m}=\bigoplus_{p<0}\mathfrak{a}_p=\bigoplus_{p<0}\mathfrak{b}_p$ and ∂ stands for the coboundary operator of

$\mathrm{Hom}(\Lambda\mathfrak{m}, \mathfrak{b})$.

Let us show that $\iota_{\mathfrak{m}}^*(\gamma_{k+1}-\delta_{k+1})$ is a cocycle. In fact, recalling that

$$(\gamma\circ\gamma)_l=-\partial_{\mathfrak{a}}\gamma_l+\sum_{\substack{s+t=l\\ s,t>0}}\gamma_s\circ\gamma_t,$$

we have

$$\iota_{\mathfrak{m}}^*(\gamma\circ\gamma)_{k+1}=-\partial\iota_{\mathfrak{m}}^*\gamma_{k+1}+\iota_{\mathfrak{m}}^*\sum_{\substack{s+t=k+1\\ s,t>0}}\gamma_s\circ\gamma_t.$$

But $\iota_{\mathfrak{m}}^*(\gamma\circ\gamma)_{k+1}=0$, for γ defines a truncated Lie algebra of order $k+1$. Hence we have

$$\partial\iota_{\mathfrak{m}}^*\gamma_{k+1}=\iota_{\mathfrak{m}}^*\sum_{\substack{s+t=k+1\\ s,t>0}}\gamma_s\circ\gamma_t,$$

and the same formula for δ. But it follows from (3.15) that

$$\iota_{\mathfrak{m}}^*\sum_{\substack{s+t=k+1\\ s,t>0}}\gamma_s\circ\gamma_t=\iota_{\mathfrak{m}}^*\sum_{\substack{s+t=k+1\\ s,t>0}}\delta_s\circ\delta_t.$$

which implies that $\partial\iota_{\mathfrak{m}}^*(\gamma_{k+1}-\delta_{k+1})=0$.

Hence, on account of our assumption that $H_{k+1}^2(\mathrm{gr}\,B(k+1))=0$, there exists an $f_-\in\mathrm{Hom}(\mathfrak{m}, \mathfrak{b})_{k+1}$ such that $\partial f_-=\iota_{\mathfrak{m}}^*(\gamma_{k+1}-\delta_{k+1})$.

Now put $\bar{\phi}_{k+1}=id+f_-$ and make the identifications (3.14) via $\bar{\phi}_{k+1}$ instead of ϕ_{k+1}. Then we have, in addition to (3.15),

(3.18) $\quad \gamma_{k+1}(x_p, y_q)=\delta_{k+1}(x_p, y_q) \qquad$ for $p, q<0$.

We are again looking for $\psi_{k+1}: A(k+1)\to B(k+1)$ that satisfies (3.17) and that has, in this time, the following form:

(3.19) $\quad \psi_{k+1}=id+f_+$, with $f_+\in\mathrm{Hom}(\bigoplus_{0\le p\le k+1}\mathfrak{a}_p, \mathfrak{b}_{k+1})$.

It then follows from (3.15), (3.18) and (3.19) that (3.17) is already satisfied for $p, q<0$, while for $0\le i\le k+1$, $q<0$ we have

$$\begin{aligned}&K_{\tau(k+1;\,i,\,q)}(x_i, y_q)\\ &\quad=\gamma_{k+1-i}(x_i, y_q)-\delta_{k+1-i}(x_i, y_q)-\gamma_0(f_+(x_i), y_q),\end{aligned}$$

or

$$\iota_{\mathfrak{m}}^*i(x_i)K_{k+1-i}=\iota_{\mathfrak{m}}^*i(x_i)(\gamma_{k+1-i}-\delta_{k+1-i})-\partial f_+(x_i).$$

Again let us show that $\iota_{\mathfrak{m}}^*i(x_i)(\gamma_{k+1-i}-\delta_{k+1-i})$ is a cocycle: Recall that

$$\begin{aligned}\partial\iota_{\mathfrak{m}}^*i(x_i)\gamma_{k+1-i}&=\iota_{\mathfrak{m}}^*i(x_i)(\gamma\circ\gamma))_{k+1-i}\\ &\quad+\iota_{\mathfrak{m}}^*\theta(x_i)\gamma_{k+1-i}-\iota_{\mathfrak{m}}^*i(x_i)\sum_{\substack{s+t=k+1-i\\ s,t>0}}\gamma_s\circ\gamma_t,\end{aligned}$$

which, γ replaced by δ, also holds. But clearly $\iota_{\mathrm{m}}^{*}i(x_i)(\gamma\circ\gamma)_{k+1-i}=\iota_{\mathrm{m}}^{*}i(x_i)(\delta\circ\delta)_{k+1-i}=0$. Furthermore from (3.15) and (3.18) we see that

$$\begin{cases}\iota_{\mathrm{m}}^{*}\theta(x_i)(\gamma_{k+1-i}-\delta_{k+1-i})=0\\ \iota_{\mathrm{m}}^{*}i(x_i)\sum\limits_{\substack{s+t=k+1-i\\ s,t>0}}(\gamma_s\circ\gamma_t-\delta_s\circ\delta_t)=0.\end{cases}$$

It then follows that $\partial\iota_{\mathrm{m}}^{*}i(x_i)(\gamma_{k+1-i}-\delta_{k+1-i})=0$. Hence, if $H_{k+1}^{1}(\mathrm{gr}\,B(k+1))=0$, one can find $f_{+}\in\mathrm{Hom}(\bigoplus_{0\leq p\leq k+1}\mathfrak{a}_p, \mathfrak{b}_{k+1})$ so that $\iota_{\mathrm{m}}^{*}i(x_i)K_{k+1-i}=0$ for $x_i\in\mathfrak{a}_i$ $(0\leq i\leq k+1)$. Thus we have found ψ_{k+1} such that

$$(3.20)\qquad K_{\tau(k+1;\,p,\,q)}(x_p, y_q)=0 \qquad \text{for } p<0 \text{ or } q<0.$$

It is not difficult to verify that (3.17) is a consequence of (3.20) and therefore ψ_{k+1} is an embedding of $A(k+1)$ into $B(k+1)$. q. e. d.

3.5. The proof of Proposition 3.2 also yields the following criterion for two truncated TFLA's to be isomorphic.

THEOREM 3.1 *Let $A(k+1)$, $B(k+1)$ be truncated TFLA's of order $k+1$ $(k\geq 0)$. Suppose that there exist isomorphisms $\psi_k:\mathrm{Trun}_k\,A(k+1)\rightarrow\mathrm{Trun}_k\,B(k+1)$ and $h_{k+1}:\mathrm{gr}\,A(k+1)\rightarrow\mathrm{gr}\,B(k+1)$ such that h_{k+1} is an extension of $\mathrm{gr}\,\psi_k$. Then, denoting $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ a TGLA isomorphic to $\mathrm{Prol}\,\mathrm{gr}\,A(k+1)$, we may assume $A(k+1)$ and $B(k+1)$ are defined on $\mathfrak{g}(k+1)=\bigoplus_{p\leq k+1}\mathfrak{g}_p$ respectively by $\gamma=\sum_{l\geq 0}\gamma_l$ and $\delta=\sum_{l\geq 0}\delta_l$ $(\gamma_l, \delta_l\in\mathrm{Hom}(\Lambda^2\mathfrak{g}, \mathfrak{g})_l)$ satisfying the following conditions :*

i) $\gamma_l(x_p, y_q)=\delta_l(x_p, y_q)$, for $l\leq\tau(k\,;p, q)$, $x_p\in\mathfrak{g}_p$, $y_q\in\mathfrak{g}_q$.

ii) $\gamma_0=\delta_0=$ *the bracket of* $\mathfrak{g}$.

Under this condition, $A(k+1)$ and $B(k+1)$ are isomorphic if and only if the cocycles :

$$\begin{cases}\iota_{\mathfrak{g}_-}^{*}(\gamma_{k+1}-\delta_{k+1})\in Hom(\Lambda^2\mathfrak{g}_-, \mathfrak{g})_{k+1},\\ \iota_{\mathfrak{g}_-}^{*}i(x_i)(\gamma_{k+1-i}-\delta_{k+1-i})\in Hom(\mathfrak{g}_-, \mathfrak{g})_{k+1}\ (x_i\in\mathfrak{g}_i,\ 0\leq i\leq k)\end{cases}$$

are exact.

Finally, to illustrate the preceding discussion, let us examine truncated TFLA's of lower orders.

It is clear that a truncated TFLA of order 0 is determined, up to isomorphism, by a truncated TGLA of order 0, namely a finite dimensional graded Lie algebra $\bigoplus_{p\leq 0}\mathfrak{g}_p$ such that the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$ is faithful.

To examine truncated TFLA's of order 1 and 2, we make somewhat general observation which follows immediately from the preceding discussion, especially from the proof of Proposition 3. 1.

Let $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ be a TGLA and let $\gamma=\sum_{l\geq 0}\gamma_l\in\mathrm{Hom}(\Lambda^2\mathfrak{g},\mathfrak{g})$ be such that γ_0 coincides with the bracket operation of $\mathfrak{g}$. Suppose that for an integer $k\geq 1$, $[\gamma]^{k-1}$ defines on $\mathfrak{g}(k-1)=\bigoplus_{p\leq k-1}\mathfrak{g}_p$ a truncated TFLA of order $k-1$. Then $[\gamma]^k$ defines on $\mathfrak{g}(k)$ a truncated TFLA of order k if and only if the following conditions are satisfied;

$$(2.21)_k\quad\begin{cases}\text{i)}\quad \partial\iota_{\mathrm{m}}^*\gamma_k=\iota_{\mathrm{m}}^*\sum_{\substack{s+t=k\\ s,t>0}}\gamma_s\circ\gamma_t,\\ \text{ii)}\quad \partial\iota_{\mathrm{m}}^*i(x_i)\gamma_{k-i}=\iota_{\mathrm{m}}^*\theta(x_i)\gamma_{k-i}-\iota_{\mathrm{m}}^*i(x_i)\sum_{\substack{s+t=k-i\\ s,t>0}}\gamma_s\circ\gamma_t,\\ \qquad\qquad\text{for } x_i\in\mathfrak{g}_i\ (0\leq i\leq k-1),\\ \text{iii)}\quad \partial i(x_i)i(y_j)\gamma_{k-i-j}=\iota_{\mathrm{m}}^*(\theta(x_i)i(y_j)-i(x_i)\theta(y_j))\gamma_{k-i-j}\\ \qquad +\iota_{\mathrm{m}}^*i(x_i)i(y_j)\sum_{\substack{s+t=k-i-j\\ s,t>0}}\gamma_s\circ\gamma_t\\ \qquad\text{for } x_i\in\mathfrak{g}_i,\ y_j\in\mathfrak{g}_j\ (i,j\geq 0,\ i+j\leq k-1).\end{cases}$$

Note that the right hand side of i) is a cocycle and that of ii) (resp. iii)), involving $\iota_{\mathrm{m}}^*\gamma_k$ (resp, $\iota_{\mathrm{m}}^*i(x_i)\gamma_{k-i}$), is a cocycle provided that i) (resp. ii)) is satisfied.

Now in the case $k=1$, $(2.21)_k$ reduces to:

$$(2.21)_1\quad\begin{cases}\text{i)}\quad \partial\iota_{\mathrm{m}}^*\gamma_1=0\\ \text{ii)}\quad \partial\iota_{\mathrm{m}}^*i(x_0)\gamma_1=\theta(x_0)\iota_{\mathrm{m}}^*\gamma_1\\ \text{iii)}\quad \partial i(x_0)i(y_0)\gamma_1=\iota_{\mathrm{m}}^*(\theta(x_0)i(y_0)-i(x_0)\theta(y_0))\gamma_1.\end{cases}$$

Thus, to define a truncated TFLA of order 1, we have to choose a cocycle $\iota_{\mathrm{m}}^*\gamma_1$ so that $\theta(x_0)\iota_{\mathrm{m}}^*\gamma_1$ is a coboundary for any $x_0\in\mathfrak{g}_0$. Then determine $\iota_{\mathrm{m}}^*i(x_0)\gamma_1$ by ii) (not necessarily uniquely). By this choice, if the right hand side of iii) is a coboundary (for instant it is always the case if $\mathfrak{g}(1)$ is the prolongation of $\mathfrak{g}(0)$), we can determine $i(x_0)i(y_0)\gamma_1$ by iii), to obtain a truncated TFLA of order 1.

In the case $k=2$, the first condition i) reduces to:

$$(3.21)_2\quad \text{i)}\quad \partial\iota_{\mathrm{m}}^*\gamma_2=(\iota_{\mathrm{m}}^*\gamma_1)\circ(\iota_{\mathrm{m}}^*\gamma_1)$$

Note that, if μ(the depth of $\mathfrak{g}$)$=1$, $(3.21)_1$ i) is always satisfied and $(3.21)_1$ ii), $(3.21)_2$ i) coincide with (4.5) and (4.6) of [2].

REMARK 3. 1. In the definition of the truncated Lie algebras, there are several reasonable choices for truncation. For instance set

$$\sigma(k\,;p_1,\cdots,p_r)$$
$$=\operatorname{Min}\{k+1,\,k+1-l-p_1-\cdots-p_{i_l}\ (1\leq i_l\leq r,\,1\leq l\leq r)\}$$

for $k, p_1, \cdots, p_r \in \boldsymbol{Z}$ (cf. (3.1)), and define the truncated Lie algebras by using σ instead of τ. Then we can as well develop the theory with this σ-truncation, to which the notation $H^{s,p}(=H^p_{s+p-1})$, rather than H^p_r, is well adapted.

REMARK 3.2. In the case when $\mu=1$, the notion of truncated Lie algebra (of order 1) was introduced by Guillemin-Sternberg [2] and extended to higher order by Hayashi [3]. A truncated Lie algebra in the sense of [2] is, in our terminology, a σ-truncated TFLA of order 1 of depth 1 which is extendable to order 2. A truncated Lie algebra of order k in the sense of [3] is a σ-truncated TFLA of order k of depth 1.

§ 4. Main theorems.

We are now ready to obtain our main theorems:

THEOREM 4.1. *Let $A(k)$ be a truncated TFLA of order $k\geq 0$, and assume that*

$$H^2_r(\operatorname{gr} A(k))=0\ (r\geq k+1),\ \textit{and}\ H^3_r(\operatorname{gr} A(k))=0\ (r\geq \operatorname{Max}\{k,1\}+1).$$

Then there exists, uniquely up to isomorphism, a complete TFLA $\boldsymbol{L}$, such that $\operatorname{Trun}_k \boldsymbol{L}=A(k)$ *and that* $\operatorname{gr}\boldsymbol{L}=\operatorname{Prol}\operatorname{gr} A(k)$.

PROOF. The existence follows from Proposition 3.1 and uniqueness follows from Proposition 3.2.

THEOREM 4.2. *Let $\boldsymbol{L}$ be a complete TFLA. Let k be a non-negative integer such that*

$$H^1_r(\operatorname{gr}\boldsymbol{L})=H^2_r(\operatorname{gr}\boldsymbol{L})=0,\quad \text{for } r\geq k+1.$$

For a TFLA $\boldsymbol{K}$, if there is an embedding $\psi_k:\operatorname{Trun}_k\boldsymbol{K}\to\operatorname{Trun}_k\boldsymbol{L}$, there exists an embedding $\phi:\boldsymbol{K}\to\boldsymbol{L}$ such that $\operatorname{Trun}_k\phi\equiv\psi_k$. Moreover two such extensions differ by an inner automorphism of $\boldsymbol{L}$ which fixes $\operatorname{Trun}_k\boldsymbol{L}$.

PROOF. The existence follows from Proposition 3.2. As regards the rigidity, we first note that for an element $X\in L^{l+1}(l\geq 0)$, $\exp\operatorname{ad} X=\sum_{n=0}^{\infty}\frac{1}{n!}(\operatorname{ad} X)^n$ is a well-defined automorphism of $\boldsymbol{L}$ and $\operatorname{Trun}_l(\exp\operatorname{ad} X)\equiv\operatorname{id.}$ of $\operatorname{Trun}_l\boldsymbol{L}$. If ψ_{k+1}, ψ'_{k+1} are embeddings: $\operatorname{Trun}_{k+1}\boldsymbol{K}\to\operatorname{Trun}_{k+1}\boldsymbol{L}$, covering ψ_k, then from the proof of Proposition 3.2

and from our hypothesis $H^1_{k+1}(\mathrm{gr}\, \boldsymbol{L})=0$. We see that there exists $X_{k+1}\in L^{k+1}$ such that

$$\psi'_{k+1}\equiv \mathrm{Trun}_{k+1}(\exp \mathrm{ad}\, X_{k+1})\circ\psi_{k+1}.$$

Therefore if ψ, ψ' are embeddings: $\boldsymbol{K}\to\boldsymbol{L}$, covering ψ_k, then there exists a sequence $\{X_l\}_{l\geq k+1}$ $(X_l\in L^l)$ such that $\psi'=\alpha\circ\psi$, where $\alpha=\lim_{l\to\infty}(\exp \mathrm{ad}\, X_l)\circ\cdots\circ(\exp \mathrm{ad}\, X_{k+1})$ is a well-defined automorphism of $\boldsymbol{L}$ and satisfies $\mathrm{Trun}_k\alpha\equiv \mathrm{id}$.

As an immediate consequence of Theorem 4.2, we have:

THEOREM 4.3. *Let $\boldsymbol{L}_i$ $(i=1,2)$ be complete TFLA's, and let k be an integer ≥ 0 such that*

$$H^1_r(\mathrm{gr}\, \boldsymbol{L}_i)=H^2_r(\mathrm{gr}\, \boldsymbol{L}_i)=0, \qquad \text{for } i=1,2,\ r\geq k+1.$$

Then $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are isomorphic if and only if $\mathrm{Trun}_k\, \boldsymbol{L}_1$ *and* $\mathrm{Trun}_k\, \boldsymbol{L}_2$ *are isomorphic.*

COROLLARY 4.1. *Let $\boldsymbol{L}$ be a complete TFLA satisfying:*

$$H^1_r(\mathrm{gr}\, \boldsymbol{L})=H^2_{r+1}(\mathrm{gr}\, \boldsymbol{L})=0 \qquad \text{for } r\geq 0.$$

For a TFLA $\boldsymbol{K}$, if there is an embedding $\psi_{-1}: \mathrm{gr}_-(\boldsymbol{K})\to\mathrm{gr}_-(\boldsymbol{L})$, there exists, uniquely up to inner automorphism of $\boldsymbol{L}$, an embedding $\phi: \boldsymbol{K}\to\boldsymbol{L}$ such that $\mathrm{Trun}_{-1}(\mathrm{gr}\, \phi)=\psi_{-1}$.

PROOF. By Proposition 2.1 and by the assumption $H^1_0(\mathrm{gr}\, \boldsymbol{L})$, the embedding ψ_{-1} is uniquely extended to an embedding $\psi_0: \mathrm{Trun}_0\, \boldsymbol{K}\to\mathrm{Trun}_0\, \boldsymbol{L}$, so that the assertion follows from Theorem 4.2.

REMARK 4.1. Examples of the TFLA's satisfying the assumption of Corollary 4.1 are the Lie algebra of all formal vector fields in n variables $(\mu=1)$, the contact Lie algebras $(\mu=2)$ and some special class of higher order contact Lie algebras. (See Corollary 5.3. iii))

As a special case of Theorem 4.3, we have

COROLLARY 4.2. *Let $\boldsymbol{L}$ be TFLA satisfying* $H^1_r(\mathrm{gr}\, \boldsymbol{L})=H^2_r(\mathrm{gr}\, \boldsymbol{L})=0$ for $r\geq 1$, *then $\boldsymbol{L}$ is graded, that is, $\boldsymbol{L}$ can be embedded into the completion of the graded Lie algebra* $\mathrm{gr}\, \boldsymbol{L}$.

We have also, from Theorem 3.1, the following

PROPOSITION 4.1. *Let $\boldsymbol{L}$ be a TFLA. If $\mathrm{gr}_0 L$ contains an element E such that* $[E, x_p]=p\, x_p$ for $x_p\in\mathrm{gr}_p\, \boldsymbol{L}$, *then $\boldsymbol{L}$ is graded.*

PROOF Write $\mathfrak{g}=\oplus\mathfrak{g}_p=\mathrm{gr}\, \boldsymbol{L}$. If we choose $\{G_p\}_{p\in \boldsymbol{Z}}$ such that $L^p=$

$G_p \oplus L^{p+1}$, we can identify $\mathfrak{g}_p$ with G_p and we can decompose the bracket γ of L as $\gamma = \sum_{l \geq 0} \gamma_l$ with $\gamma_l \in \mathrm{Hom}(\Lambda^2 \mathfrak{g}, \mathfrak{g})_l$. To prove the proposition, we shall show by induction on $k \geq 0$ that we can choose $\{G_p\}$ so that we have:

(4.1) $\quad \gamma_l(x_p, y_q) = 0, \qquad$ for $x_p \in \mathfrak{g}_p,\ y_q \in \mathfrak{g}_q,\ 0 < l \leq \tau(k\,; p, q)$.

Assume this valid for a k. To show it for $k+1$, it suffices, by virtue of Theorem 3.1, to see that $\iota_{\mathfrak{m}}^* \gamma_{k+1}$, $\iota_{\mathfrak{m}}^* i(x_i) \gamma_{k+1-i}$ $(0 \leq i \leq k,\ x_i \in \mathfrak{g}_i)$ are coboundaries.

First of all note that $\theta(E)\alpha = l\alpha$ for $\alpha \in \mathrm{Hom}(\Lambda \mathfrak{g}, \mathfrak{g})_l$. Now from Jacobi identity and from (4.1) we see:

$$\begin{aligned} 0 &= \iota_{\mathfrak{m}}^* i(E)(\gamma \circ \gamma)_{k+1} \\ &= \iota_{\mathfrak{m}}^* i(E)(-\partial_{\mathfrak{g}} \gamma_{k+1} + \sum_{\substack{s+t=k+1 \\ s,t>0}} \gamma_s \circ \gamma_t) \\ &= -\iota_{\mathfrak{m}}^* i(E) \partial_{\mathfrak{g}} \gamma_{k+1} \\ &= -\iota_{\mathfrak{m}}^* \theta(E) \gamma_{k+1} + \partial \iota_{\mathfrak{m}}^* i(E) \gamma_{k+1} \end{aligned}$$

Hence $(k+1)\iota_{\mathfrak{m}}^* \gamma_{k+1} = \partial \iota_{\mathfrak{m}}^* i(E) \gamma_{k+1}$.
We, therefore, replace $\{G_p\}$ by $\{G'_p\}$, where

$$\begin{cases} G'_p = \{ v_p - \dfrac{1}{k+1} \gamma_{k+1}(E, v_p) \mid v_p \in G_p\}, & \text{for } p < 0, \\ G'_p = G_p, \quad \text{for } p \geq 0, \end{cases}$$

and we decompose γ as $\gamma = \Sigma \gamma'_l$ according to the identification via $\{G'_p\}$. Then, recalling the proof of Proposition 3.2, we have (4.1) (γ replaced by γ') and $\iota_{\mathfrak{m}}^* \gamma'_{k+1} = 0$. Moreover we have:

(4.2) $\quad \iota_{\mathfrak{m}}^* i(E) \gamma'_{k+1} = 0$.

In fact, for $v'_p = v_p - \dfrac{1}{k+1} \gamma_{k+1}(E, v_p) \in G'_p$, we have

$$\begin{aligned} \gamma(E, v'_p) &= \sum_{l \geq 0} \{\gamma_l(E, v_p) - \frac{1}{k+1} \gamma_l(E, \gamma_{k+1}(E, v_p))\} \\ &\equiv [E, v_p]' + \frac{1}{k+1} \gamma_{k+1}(E, v_p) + \gamma_{k+1}(E, v_p) \\ &\quad - \frac{1}{k+1} [E, \gamma_{k+1}(E, v_p)] \qquad (\mathrm{mod}\ L^{k+2+p}) \\ &\equiv [E, v_p]' \qquad (\mathrm{mod}\ L^{k+2+p}) \end{aligned}$$

where we set $[E, v_p]' = [E, v_p] - \dfrac{1}{k+1} \gamma(E, [E, v_p])$. This implies (4.2).

Next let us show that $\iota_{\mathfrak{m}}^* i(x_i) \gamma'_{k+1-i}$ $(0 \leq i \leq k)$ are coboundaries. We have:

$$
\begin{aligned}
0 &= \iota_{\mathrm{m}}^{*} i(x_i) i(E) (\gamma' \circ \gamma')_{k+1-i} \\
&= -\iota_{\mathrm{m}}^{*} i(x_i) i(E) \partial_{\mathfrak{g}} \gamma'_{k+1-i} \\
&= -\partial \iota_{\mathrm{m}}^{*} i(x_i) i(E) \gamma'_{k+1-i} + \iota_{\mathrm{m}}^{*} \theta(x_i) i(E) \gamma'_{k+1-i} - (k+1-i) \iota_{\mathrm{m}}^{*} i(x_i) \gamma'_{k+1-i}.
\end{aligned}
$$

But we see that $\iota_{\mathrm{m}}^{*}\theta(x_i)i(E)\gamma'_{k+1-i}=0$, for this follows from (4.1) if $i>0$ and form (4.2) if $i=0$. Hence $\iota_{\mathrm{m}}^{*}i(x_i)\gamma'_{k+1-i}$ are coboundaries. Therefore by Theorem 3.1 (or proof of Proposition 3.2), we can choose $\{G_p\}$ so that (4.1) is satisfied for $0<l\leq\tau(k+1\,;p,q)$, completing the induction. q. e. d.

REMARK 4.1. If we restrict ourselves to the transitive filtered Lie algebras of depth 1 (i. e., $\mu=1$), the results obtained in this section reduce to the well-known ones: If $\mu=1$, Theorem 4.1 essentially coincides with the Existence and Uniqueness Theorem of Guillemin-Sternberg [2], and Theorem 4.2 coincides with the Embedding Theorem of Hayashi [3] which improves the Realization theorem of Guillemin-Sternberg [2]. Proposition 4.1 is also known in the case $\mu=1$ by Kobayashi-Nagano [4] (See also C. Buttin, C. R. Acad. Sc. Paris. 264 (1967), 496—498).

§ 5. Contact Lie algebras of higher order.

5.1 In higher order contact geometry there can be found many examples of filtered Lie algebras of depth greater than one. Among them the most fundamental are the contact Lie algebras of higher order; in this section we shall calculate their cohomology groups.

Let $\pi: M\to N$ be a fibred manifold and let $J^{\nu}(M,N)$ the bundle of ν-jets of cross-sections of $\pi: M\to N$. As well known, $J^{\nu}(M,N)$ has the contact structure of order ν, that is, the flag of differential systems:

$$TJ^{\nu}(M,N)=D^{-\nu-1}\supset D^{-\nu}\supset\cdots\supset D^{-1}.$$

In local coordinates, it is defined as follows: Let $(u^1,\cdots,u^n)$ be a local coordinate system of N and $(u^1,\cdots,u^n,\ w^1,\cdots,w^m)$ that of M (we write π^*u^i just as u^i), then we have a local coordinate system $(u^1,\cdots,u^n,\cdots,p^{\rho}_{\alpha}\cdots)$ of $J^{\nu}(M,\ N)$, where $1\leq\rho\leq m$, $\alpha=(\alpha_1,\cdots,\alpha_n)$ with $|\alpha|=\Sigma\alpha_i\leq\nu$ and $p^{\rho}_{0}=w^{\rho}$. Put

$$\omega^{\rho}_{\alpha}=dp^{\rho}_{\alpha}-\sum_{i=1}^{n}p^{\rho}_{\alpha+1_i}dx^i \quad (1\leq\rho\leq m,\ 0\leq|\alpha|\leq\nu-1)$$

where $\alpha+1_i=(\alpha_1,\cdots,\alpha_i+1,\cdots,\alpha_n)$. Then the differential system D^a $(a\leq-1)$ is defined by

$$D^a=\{\omega^{\rho}_{\alpha}=0,\ \rho=1,\cdots,m,\ |\alpha|\leq\nu+a\}.$$

Of course the definition does not depend on the choice of local coordinates, and one can verify easily:

$$[\mathscr{D}^a, \mathscr{D}^b] \subset \mathscr{D}^{a+b},$$

where $\mathscr{D}^a$ denotes the sheaf of sections of D^a.

A (local) vector field X on $J^\nu(M, N)$ is called a contact vector field of order ν if it infinitesimally leaves invariant each D^a (namely if Y is a section of D^a so is $[X, Y]$). We denote by $\mathscr{C}(J^\nu(M, N))$ the Lie algebra sheaf of germs of contact vector fields of order ν. Note that $\mathscr{C}(J^0(M, N))$ is just the sheaf $\mathscr{T}M$ of all germs of vector fields on M.

It is well known that every contact vector field X of order ν can be lifted to a contact vector field of order $\nu+l$ (denoted by $p_{\nu,\nu+l}(X)$ or $X^{(l)}$); it is defined as follows: Let (ϕ_t) be a local one parameter transformation group generated by X and let σ be a section of $\pi^\nu_N: J^\nu(M, N)\to N$, which, regarded as a submanifold of $J^\nu(M,N)$, is transformed by ϕ_t to another section $\phi_t\sigma$ (for t small enough). Thus ϕ_t induces the map $\phi_t{}^{(l)}: J^l(J^\nu(M, N), N)\to J^l(J^\nu(M, N), N)$ by $\phi_t^{(l)}(j^l_x\sigma)=j^l_{x_t}(\phi_t\sigma)$ where $x_t=(\pi^\nu_N\circ\phi_t\circ\sigma)(x)$. Then $\phi_t^{(l)}$ leaves invariant the submanifold $J^{\nu+l}(M, N)$ of $J^l(J^\nu(M, N), N)$ and moreover the contact structure on it, thus yielding a contact vector field $X^{(l)}$ on $J^{\nu+l}(M, N)$. In this way we obtain an injective homomorphisn of Lie algebra sheaves:

$$p_{\nu,\nu+l}: (\pi^{\nu+l}_\nu)^{-1}\mathscr{C}(J^\nu(M, N))\to \mathscr{C}(J^{\nu+l}(M, N)),$$

where $\pi^{\nu+l}_\nu$ denotes the natural projection $J^{\nu+l}(M, N)\to J^\nu(M, N)$. Moreover we have:

PROPOSITION 5.1. *$p_{\nu,\nu+l}$ is an isomorphism for all $\nu, l\geq 0$ unless* $\dim M-\dim N=1$ *and* $\nu=0$.

For a geometric formulation and proof refer to Yamaguchi [15], [16]. We will denote by $\mathscr{C}(J^\nu(M, N))^{(l)}$ the sheaf $p_{\nu,\nu+l}(\pi^{\nu+l}_\nu)^{-1}\mathscr{C}(J^\nu(M, N))$.

Let $\mathscr{O}\in M$ and take a local coordinate system $(u^1,\cdots, u^n, w^1,\cdots, w^m)$ of the fibred manifold (M, N, π) centered at $\mathscr{O}$. Let $\mathscr{O}^\nu$ a point of $J^\nu(M, N)$ such that $\pi^\nu_0(\mathscr{O}^\nu)=\mathscr{O}$. Then we have easily:

LEMMA 5.1. *The vectors* $\left\{\left(u^\alpha\frac{\partial}{\partial w^\rho}\right)^{(\nu)}_{\mathscr{O}^\nu}, \left(\frac{\partial}{\partial u^i}\right)^{(\nu)}_{\mathscr{O}^\nu}\right\}$ $(\nu+a+1\leq|\alpha|\leq\nu,\ \rho=1,\cdots m,\ i=1,\cdots n)$ *form a basis of* $(D^a)_{\mathscr{O}^\nu}$ *for* $-\nu-1\leq a\leq -1$. *Furthermore*

$$\begin{cases} \left(u^{\alpha}w^{\beta}\dfrac{\partial}{\partial w^{\rho}}\right)^{(\nu)}_{\mathcal{O}^{\nu}}=0 & \text{if } |\alpha|\geq\nu+1 \text{ or } |\beta|\geq 1, \\ \left(u^{\alpha}w^{\beta}\dfrac{\partial}{\partial u^{i}}\right)^{(\nu)}_{\mathcal{O}^{\nu}}=0 & \text{if } |\alpha|+|\beta|>0. \end{cases}$$

From this lemma it follows immediately that $\mathscr{C}(J^{0}(M,N))^{(\nu)}$ (and hence $\mathscr{C}(J^{\nu}(M,N))$ is transitive. By Proposition 1.1, the stalk $\mathscr{C}(J^{\nu}(M,N))_{\mathcal{O}^{\nu}}$ can be endowed with a filtration $\{\Psi^{p}\mathscr{C}(J^{\nu}(M,N))_{\mathcal{O}^{\nu}}\}_{p\in Z}$ compatible with the contact structure $\{D^{a}\}$. Hence it yields the formal algebra at $\mathcal{O}^{\nu}$, denoted by $C(m,n;\nu)$ (since this depends up to isomorphism only on dim $M=n+m$, dim $N=n$ and the order ν), and it induces on $C(m,n;\nu)$ a filtration $\{\Psi^{p}C(m,n;\nu)\}$. Then the pair $(C(m,n;\nu), \{\Psi^{p}C(m,n;\nu)\})$ is a transitive filtered Lie algebra of depth $\nu+1$. Similarly from $\mathscr{C}(J^{\nu-l}(M,N))^{(l)}$ we obtain a transitive filtered Lie algebra

$$(C(m,n,\nu-l)^{(l)}, \{\Psi^{p}C(m,n;\nu-l)^{(l)}\}),$$

which is a subalgebra of $C(m,n;\nu)$.

In order to obtain the cohomology group $H(\operatorname{gr} C(m,n;\nu))$, we shall calculate $H(\operatorname{gr} C(m,n;0)^{(\nu)})$ and $H(\operatorname{gr} C(1,n;1)^{(\nu-1)})$ since Proposition 5.1 implies that

$$\begin{cases} C(m,n;\nu)=C(m,n;0)^{(\nu)} & \text{for } \nu\geq 0, \text{ if } m\geq 2 \\ C(1,n;\nu)=C(1,n,1)^{(\nu-1)} & \text{for } \nu\geq 1. \end{cases}$$

This fact, however, will be obtained as a byproduct of our calculation (Corollary 5.3. i)). In what follows, we write as

$$C(m,n;0)=A(m,n)=A,\ \ C(1,n;1)=C(n)=C.$$

Note that $A(m,n)$ is nothing but the Lie algebra of the formal vector fields in $(m+n)$-variables and $C(n)$ the contact Lie algebra of a $(2n+1)$-dimensional contact manifold in the usual sense.

5.2. The cohomology group $H(\operatorname{gr} A(m,n)^{(\nu)})$.

First of all let us write down explicitly the graded structure of $\operatorname{gr} A(m,n)^{(\nu)}$. Let V be an $(m+n)$-dimensional vector space and $V=U\oplus W$ a direct sum decomposition with dim $U=n$, dim $W=m$. Take a basis $\left\{\dfrac{\partial}{\partial u^{1}},\cdots \dfrac{\partial}{\partial u^{n}}, \dfrac{\partial}{\partial w^{1}},\cdots \dfrac{\partial}{\partial w^{m}}\right\}$ of $U\oplus W$ and the dual basis $\{u^{1},\cdots,u^{1}, w^{1},\cdots,w^{m}\}$ of $U^{*}\oplus W^{*}$, then any element

$$\sum a^{i}_{\alpha\beta}u^{\alpha}\otimes w^{\beta}\otimes\frac{\partial}{\partial u^{i}}+\sum b^{\rho}_{\alpha\beta}u^{\alpha}\otimes w^{\beta}\otimes\frac{\partial}{\partial w^{\rho}}$$

of $S(V^*)\otimes V=S(U^*)\otimes S(W^*)\otimes(U\oplus W)$ being regarded as a polynomial vector field, $S(V^*)\otimes V$ carries the natural Lie algebra structure. Now for an integer $\nu\geq 0$, put

$$\mathfrak{a}(W, U)_p^{(\nu)}=\bigoplus_{i+(\nu+1)j-1=p} S^i(U^*)\otimes S^j(W^*)\otimes U \oplus \bigoplus_{i+(\nu+1)j-(\nu+1)=p} S^i(U^*)\otimes S^j(W^*)\otimes W$$

Then it is easy to verify:

$$[\mathfrak{a}(W, U)_p^{(\nu)}, \mathfrak{a}(W, U)_q^{(\nu)}]\subset\mathfrak{a}(W, U)_{p+q}^{(\nu)}.$$

Therefore $\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{a}(W, U)_p^{(\nu)}$ becomes a graded Lie algebra. From Lemma 5.1 and the definition of the filtration $\{\Psi^p A(m, n)^{(\nu)}\}$, we have immediately:

PROPOSITION 5.2. gr $A(m, n)^{(\nu)}\cong\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{a}(W, U)_p^{(\nu)}$ *as graded Lie algebras.*

We shall therefore identify gr $A(m, n)^{(\nu)}$ with $\oplus\,\mathfrak{a}(W, U)_p^{(\nu)}$.

To state our results we need still some notation. We put,

$$\begin{cases} F_a(=F_a^{(\nu)})=S^{\nu+1+a}U^*\otimes \mathrm{W} \quad (\nu\leq a\leq -1), \\ F(=F^{(\nu)})=\bigoplus_{a=-\nu}^{-1} F_a^{(\nu)} \end{cases}$$

If $\nu=0$, it is understood that $F^{(\nu)}=\boldsymbol{F}$ (the ground field).

Let us define an operator

(5.1) $\qquad \delta:\mathrm{Hom}(\Lambda^pF, U)\rightarrow\mathrm{Hom}(\Lambda^{p+1}F, W).$

For that, we first define

$$\delta:\mathrm{Hom}(\Lambda^q(U^*\otimes W), U)\rightarrow\mathrm{Hom}(\Lambda^{q+1}(U^*\otimes W), W)$$

by $(\delta\omega)(A_1,\cdots, A_{q+1})=\sum_{i=1}^{q+1}(-1)^{i+1}A_i(\omega(A_1,\cdots,\hat{A}_i,\cdots, A_{q+1}))$ for $A_1,\cdots, A_{q+1}\in$ $\mathrm{U}^*\otimes W=\mathrm{Hom}(U, W)$. If $\nu\geq 1$, recalling that $F_{-\nu}=U^*\otimes W$, we extend this to the operator $\delta:\mathrm{Hom}(\Lambda^pF, U)\rightarrow\mathrm{Hom}(\Lambda^{p+1}F, W)$ in the obvious manner. Note that δ preserves the degree r, that is, if we put

$$\mathrm{Hom}(\Lambda^pF, U)_r=\bigoplus_{\substack{\Sigma p_a=p \\ -1-\Sigma ap_a=r}}\mathrm{Hom}(\bigotimes_{a=-\nu}^{-1}\Lambda^{p_a}F_a, U) \tag{5.2}$$

$$\mathrm{Hom}(\Lambda^pF, W)_r=\bigoplus_{\substack{\Sigma p_a=p \\ -\nu-1-\Sigma ap_a=r}}\mathrm{Hom}(\bigotimes_{a=-\nu}^{-1}\Lambda^{p_a}F_a, W), \tag{5.3}$$

then $\delta(\mathrm{Hom}(\Lambda^pF, U)_r\subset\mathrm{Hom}(\Lambda^{p+1}F, W)_r$. If $\nu=0$, we understand that $\delta=0$ and that $\mathrm{Hom}(\Lambda F, U)=\mathrm{Hom}(\Lambda^0F, U)_{-1}=U$, $\mathrm{Hom}(\Lambda F, W)=\mathrm{Hom}(\Lambda^0F,$

$W)_{-1}=W$.

Now we can state:

THEOREM 5.1. *One has the following long exact sequence:*

$$\cdots\to \mathrm{Hom}(\Lambda^p F^{(\nu)}, W)_r \to H_r^p(\mathrm{gr}\, A^{(\nu)}) \to \mathrm{Hom}(\Lambda^p F^{(\nu)}, U)_r \xrightarrow{\delta} \mathrm{Hom}(\Lambda^{p+1} F^{(\nu)}, W)_r \to \cdots.$$

For the proof we use the following well-known facts:

PROPOSITION 5.3. *To an exact sequence of cohain complex* $0\to K'\to K\to K''\to 0$, *there corresponds a long exact sequence*:

$$\cdots\to H^p(K')\to \mathrm{H}^p(K)\to \mathrm{H}^p(K'')\to \mathrm{H}^{p+1}(K')\to\cdots.$$

PROPOSITION 5.4. *The Koszul complex* $(\mathrm{Hom}(\Lambda V, SV^*, \partial))$ *is exact except at* $\mathrm{Hom}(\Lambda V, \mathrm{S}^0 V^*)$, *where* V *is a finite dimensonal vector space and the boundary operator*

$$\begin{aligned}\partial: &\mathrm{Hom}(\Lambda^p V, S^i V^*)\ (\cong S^i V^*\otimes\Lambda^p V^*)\\ &\to \mathrm{Hom}(\Lambda^{p+1} V, S^{i-1} V^*)\ (\cong S^{i-1} V^*\otimes \Lambda^{p+1} V^*)\end{aligned}$$

is just defined by skew-symmetrization.

Now let us proceed to the proof of Theorem 5.1. Using the above notation, the cochain complex in question can be expressed as

$$\begin{aligned}C(&=C(\mathrm{gr}_- A^{(\nu)}, \mathrm{gr}\, A^{(\nu)})\\ &=\mathrm{Hom}(\Lambda U\otimes \Lambda W\otimes \Lambda F, SU^*\otimes \mathrm{S}W^*\otimes(U\oplus W)).\end{aligned}$$

Recalling the bracket rule in $SV^*\otimes V$, we have the following exact sequence of cochain complexes:

$$0\to K\otimes W\to C\to K\otimes U\to 0,$$

where $K=\mathrm{Hom}(\Lambda U\otimes\Lambda W\otimes \Lambda F, SU^*\otimes SW^*)$ with the coboundary operator ∂ defined in the natural manner.

Now let us compute the cohomology group of K. For $\gamma=(\gamma_{-\nu},\cdots,\gamma_{-1})$ we put $\Lambda^\gamma F=\Lambda^{\gamma_{-\nu}} F_{-\nu}\otimes\cdots\otimes\Lambda^{\gamma_{-1}} F^{-1}$. Then we have:

$$\begin{aligned}&\partial \mathrm{Hom}(\Lambda^p U\otimes \Lambda^q W\otimes \Lambda^\gamma F, S^i U^*\otimes S^j W^*)\\ &\quad\subset \mathrm{Hom}(\Lambda^{p+1} U\otimes \Lambda^q W\otimes\Lambda^\gamma F, S^{i-1} U^*\otimes S^j W^*)\\ &\quad\oplus \mathrm{Hom}(\Lambda^p U\otimes \Lambda^{q+1} W\otimes \Lambda^\gamma F, S^i U^*\otimes S^{j-1} W^*)\\ &\quad\oplus \bigoplus_{a=-\nu}^{-1} \mathrm{Hom}(\Lambda^p U\otimes \Lambda^q W\otimes \Lambda^{\gamma+1_a} F, S^{i+\nu+1+a} U^*\otimes S^{j-1} W^*)\\ &\quad\oplus \bigoplus_{a=-\nu+1}^{-1} \mathrm{Hom}(\Lambda^{p+1} U\otimes\Lambda^q W\otimes \Lambda^{\gamma+1_{a}-1_{a-1}} F, S^i U^*\otimes \mathrm{S}^j W^*)\\ &\quad\oplus \mathrm{Hom}(\Lambda^{p+1} U\otimes \Lambda^{q-1} W\otimes \Lambda^{\gamma+1_{-\nu}} F, S^i U^*\otimes S^j W^*).\end{aligned}$$

Accordingly we put, for $s, t>0$,

$$K(s,t)=\bigoplus_{q+j\leq s-1}\mathrm{Hom}(\Lambda U\otimes\Lambda^q W\otimes\Lambda F,\ SU^*\otimes S^j W^*)$$
$$\oplus\bigoplus_{\substack{q+j=s\\ p+i\geq t}}\mathrm{Hom}(\Lambda^p U\otimes\Lambda^q W\otimes\Lambda F, S^i U^*\otimes S^j W^*).$$

Then we see that $K(s,t)$ is a subcomplex of K and that the quotient complex $K(s,t)/K(s,t+1)$ is isomorphic to $K_{s,t}(U,W)\otimes(\Lambda F)^*$, where

$$K_{s,t}(U,W)=\bigoplus_{\substack{p+i=t\\ q+j=s}}\mathrm{Hom}(\Lambda^p U\otimes\Lambda^q W, S^i U^*\otimes S^i W^*)$$

is a direct summand of the Koszul complex $\mathrm{Hom}(U\oplus W), S(U\oplus W)^*)$. Hence by Proposition 5.4, $H(K(s,t)/K(s,t+1))=0$ if $(s,t)\neq(0,0)$and $H(K(0,0)/K(0,1))=(\Lambda F)^*$, which, in view of Proposition 5.3, shows $H(K)\cong(\Lambda F)^*$. Therefore we have the isomorphisms $H(K\otimes W)\underset{\cong}{\to}\mathrm{Hom}(\Lambda F,W)$ and $H(K\otimes U)\underset{\cong}{\to}\mathrm{Hom}(\Lambda F,U)$ and obtain the long exact sequence

$$\cdots\to\mathrm{Hom}(\Lambda^p F,W)\to H^p(C)\to\mathrm{Hom}(\Lambda^p F,U)\to\mathrm{Hom}(\Lambda^{p+1}F,W)\to\cdots.$$

It is easy to check that the connecting homomorphism coincides with δ defined previously in (5.1), which proves Theorem 5.1.

As an immediate consequence of the above theorem we have:

COROLLARY 5.1.

(1) $H^0_{-1}(\mathrm{gr}\,A(m,n))\cong U\oplus W$
$H^p_r(\mathrm{gr}\,A(m,n))=0$ if $(p,r)\neq(0,-1)$

(2) *For* $\nu\geq1$, *if we write* $H^p_r=H^p_r(\mathrm{gr}\,A(m,n)^{(\nu)})$, *then we have*:

i) *If* $m=1$, *then* for $r\geq0$
$$H^1_r\cong\begin{cases}0 & r\neq\nu-1\\ S^2U & r=\nu-1\end{cases}.$$
ii) *If* $m\geq2$, *then* $H^1_r=0$ for $r\geq0$,
iii) *If* $m\geq3$, *then* $H^2_r=0$ for $r\geq\nu$.

PROOF. The assertion (1) follows immediately. If $\nu\geq1$, in the exact sequence

$$\cdots\to\mathrm{Hom}(F^{(\nu)},W)_r\to H^1_r\to\mathrm{Hom}(F^{(\nu)},U)_r\xrightarrow{\delta}\mathrm{Hom}(\Lambda^2F^{(\nu)},W)_r$$
$$\to H^2_r\to\mathrm{Hom}(\Lambda^2F^{(\nu)},U)_r\xrightarrow{\delta}\mathrm{Hom}(\Lambda^3F^{(\nu)},W)_r\to\cdots$$

observe that $\mathrm{Hom}(F^{(\nu)},W)_r=0$ for $r\geq0$ and $\mathrm{Hom}(\Lambda^2F^{(\nu)},W)_r=0$ for $r\geq\nu$. Indeed this follows immediately from (5.3). Thus the vanishing of H^1_r for $r\geq0$ (or H^2_r for $r\geq\nu$) is equivalent to the injectivity of

$\delta:\mathrm{Hom}(F, U)_r \to \mathrm{Hom}(\Lambda^2 F, W)_r$ for $r\geq 0$ (resp. $\delta:\mathrm{Hom}(\Lambda^2 F, U)_r\to \mathrm{Hom}(\Lambda^3 F, \mathrm{W})_r$ for $r>\nu$). On the other hand it is easy to see that

$$\delta:\mathrm{Hom}(\Lambda^p(U^*\otimes W), U)\to\mathrm{Hom}(\Lambda^{p+1}(U^*\otimes W), W)$$

is injective if dim $W\geq p+1$, from which the assertion (2) is straightforward.

5. 3. The cohomology group $H(\mathrm{gr}\ C(n)^{(\nu-1)})$ ($\nu\geq 1$). To compute this cohomology group, we first write down explicitly the graded structure of gr $C(n)^{(\nu-1)}$ by means of the bijective correspondence between the contact vector fields and the infinitesimal generating functions. As previously let (M, N, π) be a fibred manifold and here assume dim $M=n+1$, dim $N=n$. Fix a contact form ω on $J^1(M, N)$, then to each function f on $J^1(M, N)$, there corresponds bijectively a contact vector field X_f by the following condition:

$$\begin{cases} <X_f, \omega>=f \\ X_f \rfloor d\omega\equiv -df \qquad (\mathrm{mod}\ \ \omega). \end{cases}$$

Let $(u^1,\cdots, u^n, w)$ be a local coordinate system of the fibred manifold (M, N, π) and $(u^1,\cdots, u^n, w, p_1,\cdots, p_n)$ the associated local coordinates of $J^1(M, N)$. Then by the above bijection $C(n)$ is isomorphic to the formal power series ring in the variables $(u^1,\cdots, u^n, w, p_1,\cdots, p_n)$, which therefore inherits the bracket operation [,]; we have

$$[f, g]=\Big(1-\sum_{i=1}^n p_i\frac{\partial}{\partial p_i}\Big)f\cdot\frac{\partial g}{\partial w}-\Big(1-\sum_{i=1}^n p_i\frac{\partial}{\partial p_i}\Big)g\cdot\frac{\partial f}{\partial w}$$
$$+\sum_{i=1}^n\Big(\frac{\partial f}{\partial u^i}\frac{\partial g}{\partial p_i}-\frac{\partial g}{\partial u^i}\frac{\partial f}{\partial p_i}\Big).$$

Let denote by U^*, W^*, W and P the vector spaces spanned respectively by $\{u^1,\cdots, u^n\}$, $\{w\}$, $\{1\}$ and $\{p_1,\cdots, p_n\}$, and put

$$\mathfrak{c}_p^{(\nu-1)}=\oplus S^i U^*\otimes S^j W^*\otimes S^k P$$

where the sum is taken over all $i, j, k\geq 0$ such that

$$i+(\nu+1)j+\nu k-(\nu+1)=p$$

under the agreement that $S^0 P=W(=\boldsymbol{F}1)$.

Then it is easy to see that

$$[\mathfrak{c}_p^{(\nu-1)}, \mathfrak{c}_q^{(\nu-1)}]\subset\mathfrak{c}_{p+q}^{(\nu-1)}.$$

Hence $\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{c}_p^{(\nu-1)}$ is a graded Lie algebra.

Now observe that for the contact structure D^a $(a<0)$ and $\mathscr{O}^\nu \in J^\nu(M, N)$, $D^a_{\mathscr{O}^\nu} = \{(X_f^{(\nu-1)})_{\mathscr{O}^\nu} | f \in \bigoplus_{p=a}^{-1} \mathfrak{c}_p^{(\nu-1)}\}$ and that $(X_f^{(\nu-1)})_{\mathscr{O}^\nu}=0$ for $f \in \mathfrak{c}_p^{(\nu-1)}$ if $p \geq 0$, then from the definition of the filtration $\Psi^p C^{(\nu-1)}$, we have immediately:

PROPOSITION 5.5. gr $C^{(\nu-1)} \cong \bigoplus_{p \in \boldsymbol{Z}} \mathfrak{c}_p{}^{(\nu-1)}$ *as graded Lie algebras.*

This observation in due to [7] in the case $\nu=1$ and to Yamaguchi and Yatsui for the case $\nu>1$.

Henceforce we identify gr $C^{(\nu-1)}$ with $\oplus \mathfrak{c}_p^{(\nu-1)}$. Thus

$$\begin{aligned} \mathrm{gr}_- C^{(\nu-1)} &= \bigoplus_{p<0} \mathrm{gr}_p C^{(\nu-1)} \\ &= W \oplus U^* \oplus S^2 U^* \oplus \cdots \oplus (S^\nu U^* \oplus P) \end{aligned}$$

where $\mathrm{gr}_a C^{(\nu-1)} = S^{\nu+a+1} U^*$ for $a \leq -2$ and $\mathrm{gr}_{-1} C^{(\nu-1)} = S^\nu U^* \oplus P$.

Now we put

$$\begin{cases} E_a (= E_a^{(\nu-1)}) = S^{\nu+1+a} U^* & (-\nu+1 \leq a \leq -1) \\ E (= E^{(\nu-1)}) = \bigoplus_{-\nu+1}^{-1} E_a \end{cases}$$

where we agree to put $E^{(0)} = \boldsymbol{F}$ (the base field). Then the complex $C = C(\mathrm{gr}_- C^{(\nu-1)}, \mathrm{gr}\, C^{(\nu-1)})$ is expressed as

$$C = \mathrm{Hom}(\Lambda P \otimes \Lambda W \otimes \Lambda U^* \otimes \Lambda E, SU^* \otimes SW^* \otimes SP).$$

Using the following identities:

$$\begin{cases} [p_i, f] = -\dfrac{\partial f}{\partial u^i} \\ [1, f] = \dfrac{\partial f}{\partial w} \\ [h(u), f] = h(u) \dfrac{\partial f}{\partial w} + \sum_{i=1}^n \dfrac{\partial h}{\partial u^i} \dfrac{\partial f}{\partial p_i} \end{cases}$$

we see that, for $p, q, r, i, j, k \geq 0$ and $\alpha = (\alpha_{-\nu+1}, \cdots, \alpha_{-1})$,

$$\begin{aligned} &\partial \mathrm{Hom}(\Lambda^p P \otimes \Lambda^q W \otimes \Lambda^r U^* \otimes \Lambda^\alpha E, S^i U^* \otimes S^j W^* \otimes S^k P) \\ &\quad \subset \mathrm{Hom}(\Lambda^{p+1} P \otimes \cdots, S^{i-1} U^* \otimes \cdots) \\ &\quad \oplus \mathrm{Hom}(\cdots \otimes \Lambda^{q+1} W \otimes \cdots, \cdots \otimes S^{j-1} W^* \otimes \cdots) \\ &\quad \oplus \mathrm{Hom}(\cdots \otimes \Lambda^{r+1} U^* \otimes \cdots, \cdots S^{k-1} P) \\ &\quad \oplus \mathrm{Hom}(\cdots \otimes \Lambda^{r+1} U^* \otimes \cdots, S^{i+1} U^* \otimes S^{j-1} W^* \otimes \cdots) \\ &\quad \oplus \bigoplus_{a=-\nu+1}^{-1} \mathrm{Hom}(\cdots \otimes \Lambda^{\alpha+1_a} E, S^{i+\nu+a} U^* \otimes \cdots \otimes S^{k-1} P) \end{aligned}$$

$$\bigoplus_{a=-\nu+2}^{-1} \mathrm{Hom}(\Lambda^{p+1}P\otimes\cdots\otimes\Lambda^{a+1_a-1_{a-1}}E,\cdots)$$
$$\oplus\mathrm{Hom}(\Lambda^{p+1}P\otimes\cdots\otimes\Lambda^{r-1}U^*\otimes\Lambda^{a+1-\nu+1}E,\cdots)$$
$$\oplus\mathrm{Hom}(\Lambda^{p+1}P\otimes\Lambda^{q-1}W\otimes\Lambda^{r+1}U^*\otimes\cdots,\cdots),$$

where unaltered terms are written by dots. In view of the above formula, we put, for $s, t\geq 0$,

$$K(s, t)=\bigoplus_{q+j\leq s-1}\mathrm{Hom}(\Lambda P\otimes\Lambda^q W\otimes\Lambda U^*\otimes\Lambda E, SU^*\otimes S^j W^*\otimes SP)$$
$$\oplus\bigoplus_{\substack{q+j=s\\ p+i\geq t}}\mathrm{Hom}(\Lambda^p P\otimes\Lambda^q W\otimes\Lambda U^*\otimes\Lambda E, S^i U^*\otimes S^j W^*\otimes SP).$$

Then we see that $K(s, t)$ is a subcomplex of C and for $q+j=s$, $p+i=t$,

$$\partial\mathrm{Hom}(\Lambda^p P\otimes\Lambda^q W\otimes\Lambda^r \mathrm{U}^*\otimes\Lambda^a E, S^i U^*\otimes S^j W^*\otimes S^k P)$$
$$\subset\mathrm{Hom}(\Lambda^{p+1}P\otimes\cdots, S^{i-1}U^*\otimes\cdots)$$
$$\oplus\mathrm{Hom}(\cdots\otimes\Lambda^{q+1}W\otimes\cdots, \cdots\otimes S^{j-1}W^*\otimes\cdots)$$
$$\oplus\mathrm{Hom}(\cdots\otimes\Lambda^{r+1}U^*\otimes\cdots, \cdots\otimes S^{k-1}P)\ \oplus\ K(s, t+1).$$

Noting that P (resp. W) is dual to U^* (resp. W^*) with respect to the bracket, we see easily that the quotient complex $K(s, t)/K(s, t+1)$ is isomorphic to $K'_{s,t}\otimes(\Lambda E)^*$, where

$$K'_{s,t}=\bigoplus_{\substack{p+i=t\\ q+j=s}}\mathrm{Hom}(\Lambda^p U\otimes\Lambda^q W\otimes\Lambda U^*, S^i U^*\otimes S^j \mathrm{W}^*\otimes SU)$$

is a direct summand of the Koszul complex $\mathrm{Hom}(\Lambda(U\oplus W\oplus U^*), S(U\oplus W\oplus U^*)^*)$, whose cohomology group vanishes except at degree 0. Thus we have proved:

THEOREM 5.2. $H(\mathrm{gr}\ C(n)^{(\nu-1)})\cong\mathrm{Hom}(\Lambda E^{(\nu-1)}, W)$, *or more precisely*

$$H^p_r(\mathrm{gr}\ C(n)^{(\nu-1)})\cong\bigoplus_{\substack{\Sigma p_a=p\\ -\Sigma a p_a-\nu-1=r}}\mathrm{Hom}\Big(\bigotimes_{a=-\nu+1}^{-1}\Lambda^{p_a}(S^{\nu+1+a}U^*), W\Big)$$

where U and W are vector spaces of dimension n and 1 *respectively.*

As an immediate consequence of this theorem, we have

COROLLARY 5.2.

i) $H^0_{-2}(\mathrm{gr}\ C(n))=\boldsymbol{F}$, $H^p_r(C(n))=0$ if $(p, r)\neq(0, -2)$
ii) $H^1_r(\mathrm{gr}\ C(n)^{(\nu-1)})=0$ for $r\geq 0$
iii) $H^2_r(\mathrm{gr}\ C(n)^{(\nu-1)})=0$ for $r\geq\nu-2$
iv) $H^2_r(\mathrm{gr}\ C(1)^{(\nu-1)})=0$ for $r\geq\nu-3$.

5.4. Finally we summarize some information on the contact Lie algebras which follows immediately from the above computation.

COROLLARY 5.3.

i) $C(m, n; \nu)=A(m, n)^{(\nu)}$ if $m\geq 2$,
$C(1, n; \nu)=C(n)^{(\nu-1)}$.

ii) gr $C(m, n; \nu)$ *is the prolongation of* $\mathrm{gr}_{-}C(m, n; \nu)$.

iii) $H_r^1=H_{r+1}^2=0$ for $r\geq 0$ *to the following Lie algebras*: $A(m, n)$, $C(m, n; 1)$ $(m\geq 3)$, $C(1, n; \nu)$ $(0\leq\nu\leq 3)$, $C(1, 1; 4)$.

In fact, since $H_r^1(\mathrm{gr}\, A(m, n)^{(\nu)})=0$ for $r\geq 0$ and $m\geq 2$ (Cor. 5.1. ii)) and $H_r^1(\mathrm{gr}\, C(n)^{(\nu-1)})=0$ for $r\geq 0$ (Cor. 5.2. ii)), taking into account $A(m, n)^{(\nu)}$ (resp. $C(n)^{(\nu-1)}$) is an embedded subalgebra of $C(m, n; \nu)$ (resp. $C(1, n; \nu)$), we have the above assertion i). Then ii) and iii) are straightforward from Corollary 5.1 and 5.2.

We remark that Corollary 5.3. ii) is known alternatively from the fact that the contact structure on $J^k(M, N)$ is a standard differential system ([12]).

The contact Lie algebras listed in Corollary. 5.3. iii) have some universal property in the sense that Corollary 4.1 applies to these Lie algebras. (cf. [16] Theorem 1.6).

We also have:

COROLLARY 5.4. *Let* $(L, \{L^p\})$ *be a complete filtered Lie algebra. If* $\mathrm{gr}\, L$ *is isomorphic to* $\mathrm{gr}\, C(m, n; \nu)$, *then* $(L, \{L^p\})$ *is isomorphic to* $C(m, n; \nu)$.

Since $H_0^1(\mathrm{gr}\, C(m, n; \nu))=0$, the derivation $E\in\mathrm{Der}_0(\mathrm{gr}_{-}C(m, n; \nu))$ is contained in $gr_0C(m, n; \nu)$. Hence the corollary above follows from Proposition 4.1.

We remark that Corollary 5.4, as a special case of $m=\nu=1$, gives another (and more conceptual) proof of Proposition 5.1 in [6].

REMARK 5.1. In general it is difficult to compute the cohomology group $H(\mathfrak{g})$ for a given transitive graded Lie algebra $\mathfrak{g}$ of depth μ. If $\mathfrak{g}$ is finite dimensional and simple then the method of Kostant [5] is applicable. If $\mathfrak{g}$ is of depth 1, the theorem of Serre [2], which states that $H^{s,p}(\mathfrak{g})=0$ for $s>0$ if and only if $\mathfrak{g}_0$ has a quasi-regular basis (i.e, involutive), is useful to compute the cohomology group when $\mathfrak{g}$ is infinite dimensional. It would be interesting if this sort of criterion could be obtained also for $\mu\geq 2$.*)

*) Added in proof: Recently we have obtained such a criterion, generalizing the theorem of Serre, which will be published elsewhere.

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