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Lifting modules, extending modules and their applications to generalized uniserial rings

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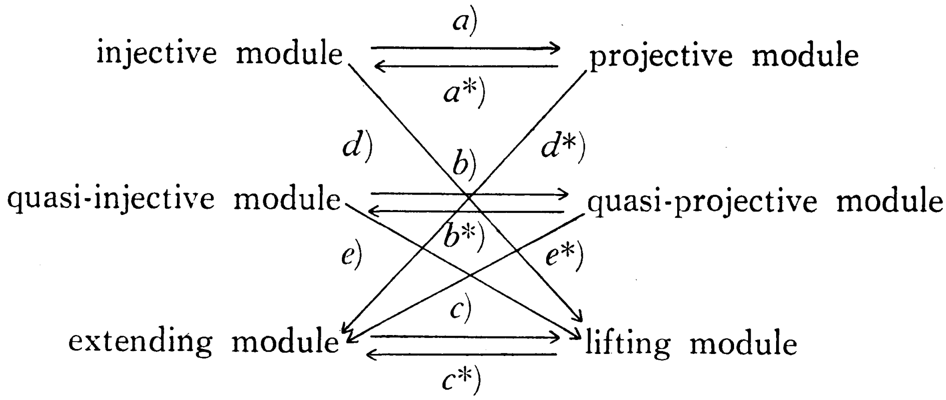
Artinian serial principal ideal rings and artinian serial rings are traditionally called uniserial rings and generalized uniserial rings, respectively. These rings are important classical artinian rings as well as quasi-Frobenius rings. The reader is referred to Faith's Book [4] for these rings. As is well known, a ring R is a quasi-Frobenius ring iff every injective R-module is projective, and if every projective R-module is injective; while R is a uniserial ring iff every quasi-injective R-module is quasi-projective, and iff every quasi-projective R-module is quasi-injective ([2], [3], [5]).

The purpose of this paper is to give similar characterizations of a generalized uniserial ring R in terms of extending and lifting modules. More specifically, consider the following implications:



As just noted above, R is quasi-Frobenius $\Leftrightarrow a) \Leftrightarrow a^{*})$; while R is uniserial $\Leftrightarrow b) \Leftrightarrow b^{*})$. The conditions $d)$ and $d^{*})$ are recently studied by Harada ([6]～[8]) and Oshiro ([15]). In this paper, we study $c)$, $c^{*})$, $e)$ and $e^{*})$ and show the following result: R is generalized uniserial $\Leftrightarrow e) \Leftrightarrow e^{*}) \Leftrightarrow c) \Leftrightarrow R$ is a right perfect ring with $c^{*})$.

NOTATION. Throughout this paper, we assume that R is an associative ring with identity and all R-modules are unitary right R-modules. Let M be an R-module. We use $E(M)$, $J(M)$ $Soc(M)$ to denote the injective hull, Jacobson radical and socle of M, respectively. Further, by $\{J_i(M)\}_I$ and $\{S_i(M)\}_I$, we denote the descending Loewy chain and ascending Loewy chain of M, respectively;

$$J_0(M)=M \qquad S_0(M)=0$$
$$J_1(M)=J(M) \qquad S_1(M)=Soc(M)$$
$$\cdots\cdots \qquad \cdots\cdots$$
$$J_{b+1}(M)=J\big(J_b(M)\big) \qquad S_{b+1}(M)/S_b(M)=Soc\big(M/S_b(M)\big)$$
$$J_a(M)=\bigcap_{b<a} J_b(M) \qquad S_a(M)=\bigcup_{b<a} S_b(M)$$
$$(a: \text{limit ordinal})$$

For a submodule N of M, we use $N\subseteq_e M$ to mean that M is an essential extension of N. If M has the finite composition length, we denote the length by $c(M)$. We say that M satisfies $(M-I)$ if every monomorphism of M to M is an isomorphism.

DEFINITION. Let M be an R-module. A submodule A of M is said to be *essentially extendible* if there exists a direct summand $A^*\langle\oplus M$ such that $A\subseteq_e A^*$. Dually, A is said to be *small liftable* if there exists a direct summand $A^*\langle\oplus M$ such that $A^*\subseteq A$ and A/A^* is a small submodule of M/A^*. We say that M satisfies the *extending property of uniform modules* if every uniform submodule of M is essentially extendible, and that M satisfies the *lifting property of hollow modules* if every submodule A of M with M/A hollow is small liftable. Further, we say that M is an *extending* module if every submodule of M is essentially extendible, and that M is a *lifting* module if every submodule of M is small liftable. M is said to be a *quasi-semiperfect* module if M is a lifting module and satisfies the following condition: For any two direct summands A_1 and A_2 of M with $M=A_1+A_2$, if $A_1\cap A_2$ is small in M then $M=A_1\oplus A_2$. We note that quasi-semiperfect R-modules are closed under taking direct summands. We know from [13] and [14] that injective$\Rightarrow$quasi-injective$\Rightarrow$extending; while projective $\Rightarrow$quasi-projective$\Rightarrow$(when R is a right perfect ring) lifting. M is said to be *uniserial* if its submodules are linearly ordered by inclusion. R is said to be a *right serial* ring if it is expressed as a direct sum of uniserial right ideals. When R is both right and left serial, R is said to be a *serial* ring. Artinian serial rings and artinian serial principal ideal rings are traditionally called *generalized uniserial* rings and *uniserial* rings, respectively ([1], [11], [12]).

Let $\{A_\alpha\}_J$ be an independent set of submodules of an R-module M.

$\sum_I \oplus A_\alpha$ is said to be a locally direct summand of M if, for any finite subset F of I, $\sum_F \oplus A_\beta$ is a direct summand of M. We use the following condition:

(K) For any independent set $\{A_\alpha\}_I$ of submodules of M, if $\sum_I \oplus A_\alpha$ is a locally direct summand of M then $\sum_I \oplus A_\alpha$ is just a direct summand.

A set $\{M_\alpha\}_I$ which consists of completely indecomposable R-modules is said to be a *locally semi-T-nilpotent* set if, for any family of countable non-isomorphisms $\{f_n: M_{\alpha_n} \to M_{\alpha_{n+1}}\}$ with $\alpha_n \neq \alpha_m$ for $n \neq m$ and any x in M_{α_1}, there exists k (depending on x) such that $f_k f_{k-1} \cdots f_1(x) = 0$. It is well known ([9]) that $\{M_\alpha\}_I$ is locally semi-T-nilpotent iff $M = \sum_I \oplus M_\alpha$ satisfies the above condition (K).

We first show the following theorem.

THEOREM 1. *Let M be a quasi-semiperfect R-module. Then M satisfies the following condition: For any exact sequence: $P \xrightarrow{\phi} M \to 0$ such that $\ker \phi$ is small in P, every decomposition $P = \sum_I \oplus P_\alpha$ implies the decomposition $M = \sum_I \oplus \phi(P_\alpha)$.*

PROOF. Let $P \xrightarrow{\phi} M \to 0$ be an exact sequence with $\ker \phi$ small in P. To prove the statement, we may show the following: Let $P = P_1 \oplus P_2$. Then $M = \phi(P_1) \oplus \phi(P_2)$. Putting $A_i = \phi(P_i)$, we first show $A_i \langle \oplus M$, $i = 1, 2$. Since M is quasi-semiperfect, we can take direct summands A_1^* and A_2^* of M such that $M = A_1^* + A_2^*$ and $A_1^* \subseteq A_1$ and $A_2^* \subseteq A_2$. Then $P = ((\phi^{-1}(A_1^*) \cap P_1) \oplus (\phi^{-1}(A_2^*) \cap P_2)) + \ker \phi$ and hence $P = \phi^{-1}(A_1^*) \cap P_1 \oplus \phi^{-1}(A_2^*) \cap P_2$. Therefore we see that $A_i = A_i^* \langle \oplus M$ for $i = 1, 2$. Next, putting $X = A_1 \cap A_2$ we show that X is small in M. Since M is quasi-semiperfect, we get a decomposition $M = X^* \oplus X^{**}$ such that $X^* \subseteq X$ and $X \cap X^{**}$ is small in M. Then $P = ((\phi^{-1}(X^{**}) \cap P_1) \oplus P_2) + \ker \phi$; whence $P = (\phi^{-1}(X^{**}) \cap P_1) \oplus P_2$ and it follows $P_1 = \phi^{-1}(X^{**}) \cap P_1$. This implies $\phi(P_1) = A_1 \subseteq X^{**}$. Hence $X = X \cap X^{**}$ and X is small in M. Thus we have situations: $M = \phi(P_1) + \phi(P_2)$, $\phi(P_i) \langle \oplus M$ for $i = 1, 2$ and $\phi(P_1) \cap \phi(P_2)$ is small in M. Since M is quasi-semiperfect, this shows that $M = \phi(P_1) \oplus \phi(P_2)$.

Theorem 1 and [14, Theorem 2. 1] we have

COROLLARY 1. *Assume that R is a right perfect ring whose projective R-module are expressed as a direct sums of uniserial modules with finite (composition) length. Then every quasi-projective R-module is also expressed as a direct sum of uniserial modules with finite length.*

PROPOSITION 1. *If M is a quasi-injective R-module then so is $J_a(M)$ for all ordinal a. Dually, if M is a quasi-projective R-module then so is $M/S_a(M)$ for all ordinal a.*

PROOF. Straightforward.

PROPOSITION 2. *Every indecomposable extending R-module is a uniform module. Dually, when R is a right perfect ring, every indecomposable lifting R-module is a cyclic hollow module.*

PROOF. The proof of the first statement is obvious. Assume that R is a right perfect ring and let M $(\neq 0)$ be an indecomposable lifting R-module. Then, since R is right perfect, we have a proper maximal submodule A in M. Since M is an indecomposable lifting module, every proper submodule of M is small in M. As a result, M has a unique maximal submodule A and hence M is a cyclic hollow module.

LEMMA 1. *Consider situations of R-modules:*

$$M = M_1 \oplus M_2 \oplus M_3\,,$$
$$M_1 \oplus M_2 \supseteq A_1 \oplus A_2\,,$$
$$M_1\,{}_e\!\supseteq A_1\,.$$

Then, if M_2 is a uniform module, $A_1 \oplus A_2 \subseteq_e M_1 \oplus M_2$.

PROOF. Let π be the projection: $M = M_1 \oplus M_2 \oplus M_3 \to M_2$. Then, $A_1 \oplus A_2 \subseteq_e M_1 \oplus A_2 = M_1 \oplus \pi(A_2) \subseteq_e M_1 \oplus M_2$ and hence $A_1 \oplus A_2 \subseteq_e M_1 \oplus M_2$.

LEMMA 2. *Let M be an R-module which has the extending property of uniform submodules and satisfies the condition (K). Then, every submodule A of M which is expressed as a direct sum of uniform modules is essentially extendible to a direct summand of M.*

PROOF. This is easily shown by Lemma 1 and Zorn's lemma.

PROPOSITION 3. *Let $\{M_\alpha\}_I$ be a family of uniserial R-modules with finite length. If $\{c(M_\alpha)\}$ is an upper bound, then $M = \sum_I \oplus M_\alpha$ satisfies the condition (K).*

PROOF. This is clear by [9, Lemma 12].

PROPOSITION 4. *Let R be a generalized uniserial ring and let M be an R-module. If M has the lifting property of hollow modules then M has the extending property of uniform modules.*

PROOF. In view of [10, Theorem 10] (cf. [12]), we may show the following: Let M be an R-module expressed as $M=M_1\oplus M_2$ with each M_i indecomposable and let A_i be a submodule of M_i, $i=1, 2$. Then, every homomorphism ϕ from A_1 to A_2 with $\ker\phi\neq 0$ is extendible to a homomorphism from M_1 to M_2.

Actually, let ϕ be a homomorphism: $A_1\to A_2$ with $\ker\phi\neq 0$. We extend ϕ to a homomorphism ψ: $\mathrm{E}(M_1)\to\mathrm{E}(M_2)$. Assume that $\psi(M_1)\not\subseteq M_2$. Then, we see from [12] that $\psi(M_1)\supsetneq M_2$. If we put $T=\{x\in\mathrm{E}(M_1)\mid\psi(x)\in M_2\}$, we see that $A\subseteq T\subsetneqq M_1$. If we put $X=\{x+\psi(x)\mid x\in T\}$, then $M_1/(X\cap M_1)\simeq M/X$; whence M/X is hollow. Since X is uniform and is not small in M, we see from the lifting property for M that X is a direct summand of M. Hence $M=X\oplus M_1$ or $M=X\oplus M_2$ by the exchange property of X (cf. [16]). Since $X\cap M_1\neq 0$, we get $M=X\oplus M_2$ which shows $T=M_1$, a contradiction. Thus we must have $\psi(M_1)\subseteq M_2$ and hence $\psi| M_1$ is a desired extension of ϕ.

We are now in a position to show our main result.

THEOREM 2. *The following conditions are equivalent for a given ring* R:

1) *Every extending R-module is a lifting module.*
2) *Every quasi-injective R-module is a lifting module.*
3) *R is a right perfect ring and every lifting R-module is an extending module.*
4) *Every quasi-projective R-module is an extending module.*
5) *R is a generalized uniserial ring.*

PROOF. 1)$\Rightarrow$2) is clear and 3)$\Rightarrow$4) follows from [14, Theorem 2. 1].

2)$\Rightarrow$5). We see from [15, Theorem 2. 11] that R is a right artinian ring. Therefore, in view of [4, 25. 4. 2], we may show that every finitely generated R-module is expressed as a direct sum of uniserial modules. Note that every finitely generated R-module has the finite (Goldie) dimension. By Proposition 2, it is easy to see that every uniform R-module is indeed uniserial. We show our assertion by induction on the dimension. So, assume that every finitely generated R-module with dimension $<n$ is expressed as a direct sum of uniserial modules and let M be an R-module with the dimension n. Then, $\mathrm{E}(M)$ is expressed as a direct sum of just n indecomposable injective modules; say $E(M)=E_1\oplus\cdots\oplus E_n$. Then each E_i is uniserial as noted above. We can take a number k such that

$$M\subseteq J_k(E(M))=J_k(E_1)\oplus\cdots\oplus J_k(E_n)$$

but

$$M \subsetneq J_{k+1}(E(M)) = J_{k+1}(E_1) \oplus \cdots \oplus J_{k+1}(E_n)\,.$$

By Proposition 1, $J_k(E(M))$ is quasi-injective and hence, by the assumption, it is a lifting module. Since $M \not\subseteq J(J_k(E(M))) = J_{k+1}(E(M))$, M is not small in $J_k(E(M))$. Hence, we have a decomposition $J_k(E(M)) = A \oplus B$ such that $M = A \oplus (B \cap M)$ and $B \cap M$ is small in $J_k(E(M))$. If $B \cap M = 0$ then $M = J_k(E(M))$ and hence M is a direct sum of uniserial modules $J_k(E_1), \cdots, J_k(E_n)$. If $B \cap M \neq 0$ then both dimensions of A and $B \cap M$ are smaller than n; whence, by induction hypothesis, A and $B \cap M$ are expressed as direct sums of uniserial modules. Thus, in any case, M is expressed as a direct sum of uniserial modules.

4)$\Rightarrow$5). By [15, Theorem 3.18], R is a left and right perfect ring. Let e be a primitive idempotent of R. Since $eR/S_1(eR)$ is quasi-projective (cf. Proposition 1), it is an extending module by the assumption. So, $S_2(eR)/S_1(eR)$ is simple. By similar inductive argument, we can conclude that all $S_{b+1}(eR)/S_b(eR)$ and $S_a(eR)/\bigcup_{c<a} S_c(eR)$ (a: limit ordinal) are simple module. This implies that eR is uniserial. Since R is a left perfect ring, it follows that R is right artinian and right serial. Now, as in the proof of 2)$\Rightarrow$5), it is enough to show that every finitely generated R-module is expressed as a direct sum of uniserial modules. Let M be a finitely generated R-module and consider a projective cover:

$$P \xrightarrow{\ \phi\ } M \longrightarrow 0\,.$$

Put $K = \ker \phi$. Inasmuch as P is expressed as a direct sum of uniserial modules and is an extending module, we can assume that $K \subseteq_e P$. Then, $S_1(P) \subseteq K$. Hence ϕ induces an epimorphism:

$$P/S_1(P) \xrightarrow{\ \phi_1\ } M \longrightarrow 0\,.$$

Here, using Proposition 1 and Corollary 1, we see that $P/S_1(P)$ is a quasi-projective module which is expressed as a direct sum of uniserial modules. Since $P/S_1(P)$ is also extending, we can also assume that $\ker \phi_1 \subseteq_e P/S_1(P)$; whence $S_2(P)/S_1(P) \subseteq \ker \phi_1$. As a result, ϕ_1 induces an epimorphism:

$$P/S_2(P) \xrightarrow{\ \phi_2\ } M \longrightarrow 0\,.$$

This procedure must terminate; so we see that M is expressed as a direct sum of uniserial modules.

5)$\Rightarrow$3). Let M be a lifting R-module and A a submodule of M. Then,

A is expressed as a direct sum of uniserial modules ([12]); say $A=\sum_I \oplus A_\alpha$. Consider the family $\mathscr{S}$ of all pairs $(J, \sum_J \oplus M_\beta)$ such that J is a subset of I and $\{M_\alpha\}_J$ is an independent family of direct summands of M such that $\sum_J \oplus M_\alpha$ is a locally direct summand of M with

$$\sum_J \oplus A_\beta \subseteq_e \sum_J \oplus M_\beta .$$

Then $\mathscr{S} \neq \phi$ by Proposition 4. Using Zorn's lemma, we can take a maximal pair $(J_0, \sum_{J_0} \oplus M_\beta)$ in the sense that if $J_0 \subseteq J$ and $\{M_\beta\}_{J_0} \subseteq \{M_\gamma\}_J$ then $J_0 = J$. Using Proposition 3, $M=\sum_{J_0} \oplus M_\beta \oplus M'$ for some submodule M'. Note that M' is also a lifting module. Now, let π be the projection: $M=\sum_{J_0} \oplus M_\beta \oplus M' \to M'$. Assume that $J_0 \neq I$ and take $\alpha \in J_0$. Since $\pi(A_\alpha) \simeq A_\alpha$, we see that $\pi(A_\alpha)$ is a uniform module. Hence, using proposition 4, we have a direct summand M_α of M such that $\pi(A_\alpha) \subseteq_e M$. By Lemma 1, we see that

$$\sum_{J_0} \oplus A_\beta \oplus A_\alpha \subseteq_e \sum_{J_0} \oplus M_\beta \oplus M_\alpha \langle \oplus M .$$

This contradicts the maximality of $(J_0, \sum_{J_0} \oplus M_\beta)$. Thus $I=J_0$ and hence 3) holds.

5)$\Rightarrow$1). Let M be an extending R-module and A a submodule of M. As above, A is expressed as a direct sum of uniform modules; say $A=\sum_I \oplus A_\alpha$. By Zorn's lemma, we can take a maximal subset J_0 of I such that $\sum_{J_0} \oplus A_\beta$ is a locally direct summand of M. (Of course, 'maximal' means that if $J_0 \subseteq J \subseteq I$ and $\sum_J \oplus A_\gamma$ is a locally direct summand of M then $J_0 = J$.) Then, by Proposition 3,

$$M = \sum_{J_0} \oplus A_\beta \oplus M'$$

for some submodule M. It follows that

$$A = \sum_{J_0} \oplus A_\beta \oplus (M' \cap A) .$$

Let π be the projection: $A=\sum_{J_0} \oplus A_\beta \oplus (M' \cap A) \to M' \cap A$. Now, we may show that $\sum_{I-J_0} \oplus A_\alpha$ is small in M. If $J_0 = I$, there is nothing to prove. So, assume that $I-J_0 \neq \phi$. Let $\alpha \in I-J_0$. Then, we see that $A_\alpha \simeq \pi(A_\alpha)$ and hence $\pi(A_\alpha)$ is a uniform module. Using [13, Proposition 1. 4], we can take a direct summand M_α of M' such that $\pi(A_\alpha) \subseteq_e M_\alpha$. If $\pi(A_\alpha) = M_\alpha$ then $\sum_{J_0} \oplus A_\beta \oplus A_\alpha \langle \oplus M$. This contradicts the choice of J_0. As a result,

$\pi(A_\alpha)\subsetneqq M_\alpha$ and hence $\pi(A_\alpha)$ is small in M. For each $\beta\in J_0$, let π_β be the projection: $M=\sum_{J_0}\oplus A_\beta\oplus M'\to A_\beta$. Then, $\pi_\beta(A_\alpha)$ is a homomorphic image of $\pi(A_\alpha)$ and hence $\pi_\beta(A_\alpha)$ is small in M. Therefore, A_α is small in M and hence so is $\sum_{I-J_0}\oplus A_\alpha$ as desired. The proof is now completed.

References

[1] K. ASANO: Uber Haupidealringe mit Kettensatz, Osaka Math. J. 1 (1949), 52–61.

[2] K. A. BYRD: Some characterizations of uniserial rings, Math. Ann. 186 (1970), 163–170.

[3] C. G. FAITH and A. A. WALKER: Direct sum representations of injective modules, J. Algebra 5 (1967), 203–221.

[4] C. G. FAITH: Algebra II Ring Theory, New York Springer-Verlag, 1976.

[5] K. R. FULLER: On representations of quasi-injectives and quasi-projectives, Arch. Math. 20 (1969), 495–502.

[6] M. HARADA: On one sided QF-2 rings I, Osaka J. Math. 17 (1980), 422–431.

[7] M. HARADA: On one sided QF-2 rings II, Osaka J. Math. 17 (1980), 433–438.

[8] M. HARADA: Non-small modules and non-cosmall modules, In Ring Theory: Proceedings of the 1978 Antwerp Conference, F. Van Oystaeyen, ed. New York: Marcel Dekker, 1979, pp. 669–689.

[9] M. HARADA and Y. SAI: On categories of indecomposable modules I, Osaka, J. Math. 7 (1970), 323–344.

[10] M. HARADA and K. OSHIRO: On extending property on direct sums of uniform modules, Osaka J. Math. 18 (1981), 767–785.

[11] G. KÖTHE: Verallgemeinerte abelsche gruppen mit hyperkomplexem operaterring. Math. Z. 39 (1935), 31–44.

[12] T. NAKAYAMA: On Frobenius algebra II, Ann of Math. 42 (1941), 1–21.

[13] K. OSHIRO: Continuous modules and quasi-continuous modules, Osaka J. Math. 20 (1983), 681–694.

[14] K. OSHIRO: Semiperfect modules and quasi-semiperfect modules, Osaka J. Math. 20 (1983), 337–372.

[15] K. OSHIRO: Lifting modules, extending modules and their applications to QF-rings, to appear.

[16] R. B. WARFIELD, Jr.: Decompositions of injective modules, Pacific J. Math. 31 (1979), 263–276.

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