Row removal for graded homomorphisms between Specht modules and for graded decomposition numbers

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Abstract

Graded Specht modules over KLR algebras in affine type A have been recently constructed by Brundan, Kleshchev and Wang. We will present joint work with Matthew Fayers and Chris Bowman in which we studied row removal results for reducing the calculation of graded homomorphism spaces between Specht modules, as well as their graded decomposition numbers.

1 Introduction

A key open problem in the representation theory of the symmetric group and the related Ariki–Koike algebra is to determine the decomposition numbers. More generally, we would like to determine the *graded* decomposition numbers for the Ariki–Koike algebra, or KLR algebra. The Specht modules for these algebras give a complete set of ordinary (graded) irreducible modules up to isomorphism (and grading shift) and we would like to determine their structure in the modular case. The graded decomposition numbers encode which graded simple modules appear as composition factors of the Specht modules.

A related problem in understanding the structure of Specht modules is the study of homomorphisms between Specht modules.

Here we address both problems, and present some reduction theorems which generalise row removal theorems of James and Donkin.

2 KLR algebras and their Specht modules

2.1 KLR algebras

We fix a field $\mathbb{F}$ throughout. Let $e \in \{2,3,\dots\} \cup \{\infty\}$ and $I = \mathbb{Z}_{\geqslant 0}$ if $e = \infty$ and $I = \mathbb{Z}/e\mathbb{Z} = \{0,1,2,\dots,e-1\}$ otherwise. We will work with a Cartan datum of type A_∞ if $e=\infty$, or type $A^{(1)}_{e-1}$ otherwise. In particular, $(a_{ij})_{i,j\in I}$ is the Cartan matrix, we let $\{\alpha_i \mid i \in I\}$ denote the set of simple roots, and $Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$ denote the positive cone of the root lattice. If $\alpha = \sum_i c_i\alpha_i$ with $\sum_i c_i = n$, then we say that α has *height* n. We denote by Γ the corresponding quiver, with vertex set I and an arrow from i to $i-1$ for each $i \in I$. We write $i \rightarrow j$ if $e \neq 2$ and $j = i-1$, or $i \rightleftarrows j$ if $e=2$ and $j=i-1$.

Suppose $\alpha \in Q^+$ has height n, and set

$$I^\alpha = \{\, i \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \alpha \}\,.$$

We define R_α to be the unital associative $\mathbb{F}$-algebra with generating set

$$\{\, e(i) \mid i \in I^\alpha\} \cup \{y_1,\dots,y_n\} \cup \{\psi_1,\dots,\psi_{n-1}\}$$

and relations

$$
\begin{aligned}
e(i)e(j) &= \delta_{i,j}e(i);\\
\sum_{i\in I^\alpha} e(i) &= 1;\\
y_r e(i) &= e(i)y_r;\\
\psi_r e(i) &= e(s_r i)\psi_r;\\
y_r y_s &= y_s y_r;\\
\psi_r y_s &= y_s \psi_r && \text{if } s \neq r, r+1;\\
\psi_r \psi_s &= \psi_s \psi_r && \text{if } |r-s|>1;\\
y_r\psi_r e(i) &= (\psi_r y_{r+1} - \delta_{i_r,i_{r+1}})e(i);\\
y_{r+1}\psi_r e(i) &= (\psi_r y_r + \delta_{i_r,i_{r+1}})e(i);\\
\psi_r^2 e(i) &= \begin{cases} 0 & \text{if } i_r = i_{r+1},\\ e(i) & \text{if } i_{r+1} \neq i_r, i_r \pm 1,\\ (y_{r+1}-y_r)e(i) & \text{if } i_r \rightarrow i_{r+1},\\ (y_r - y_{r+1})e(i) & \text{if } i_r \leftarrow i_{r+1},\\ (y_{r+1}-y_r)(y_r-y_{r+1})e(i) & \text{if } i_r \rightleftarrows i_{r+1};\end{cases}\\
\psi_r\psi_{r+1}\psi_r e(i) &= \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1}+1)e(i) & \text{if } i_{r+2} = i_r \rightarrow i_{r+1},\\ (\psi_{r+1}\psi_r\psi_{r+1}-1)e(i) & \text{if } i_{r+2} = i_r \leftarrow i_{r+1},\\ (\psi_{r+1}\psi_r\psi_{r+1}+y_r-2y_{r+1}+y_{r+2})e(i) & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\ (\psi_{r+1}\psi_r\psi_{r+1})e(i) & \text{otherwise;}\end{cases}
\end{aligned}
$$

for all admissible r, s, i, j.

The *affine Khovanov–Lauda–Rouquier algebra* or *quiver Hecke algebra* R_n is defined to be the direct sum $\bigoplus_\alpha R_\alpha$, where the sum is taken over all $\alpha \in Q^+$ of height n.

The relations above immediately yield the following result.

Lemma 2.1. *There is a $\mathbb{Z}$-grading on the algebra R_α such that for all admissible r and i,*

$$\deg(e(i)) = 0, \quad \deg(y_r) = 2, \quad \deg(\psi_r e(i)) = -a_{i_r i_{r+1}}.$$

Given $\alpha \in Q^+$ and an *e-multicharge* $\kappa = (\kappa_1, \dots, \kappa_l) \in I^l$, we define R^κ_α to be the quotient of R_α by the *cyclotomic relations*

$$y_1^{\langle \Lambda_\kappa, \alpha_{i_1}\rangle} e(i) = 0 \quad \text{for } i \in I^\alpha.$$

The *cyclotomic KLR algebra* R^κ_n is then defined to be the sum $\bigoplus_\alpha R^\kappa_\alpha$. Here we sum over all $\alpha \in Q^+$ of height n, though in fact only finitely many of the summands will be non-zero, so (even when $e = \infty$) R^κ_n is a unital algebra.

Theorem 2.2 ([BK09]). *If $e = \infty$, or if $\mathbb{F}$ contains a primitive eth root of unity, then R^κ_n is isomorphic to an Ariki–Koike algebra of level l, defined at an eth root of unity.*

Similarly, if $e = \operatorname{char}(\mathbb{F})$, then R^κ_n is isomorphic to a degenerate Ariki–Koike algebra; in particular, when $l = 1$, R^κ_n is isomorphic to the group algebra $\mathbb{F}\mathfrak{S}_n$.

In particular, the Ariki–Koike algebras are non-trivially $\mathbb{Z}$-graded.

2.2 Multipartitions and tableaux

Definition 2.3. A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $|\lambda| := \sum \lambda_i = n$. We write $\varnothing$ for the unique partition of 0.

An *l-multipartition* of n is an l-tuple of partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ such that $|\lambda| := \sum |\lambda^{(i)}| = n$. We write $\varnothing$ again for the unique l-multipartition of 0, and $\mathscr{P}^l_n$ for the set of l-multipartitions of n.

For $\lambda \in \mathscr{P}^l_n$, the *Young diagram* $[\lambda]$ is the set

$$\{(r, c, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, l\} \mid c \leqslant \lambda^{(m)}_r\}.$$

For each component $\lambda^{(m)}$ of λ, we depict $[\lambda^{(m)}]$ using the English convention. The Young diagram $[\lambda]$ is then depicted as a column vector of these Young diagrams.

A *λ-tableau* is a bijection $\mathtt{T} : [\lambda] \to \{1, \dots, n\}$, which we depict by drawing $[\lambda]$ and filling each box with its image under $\mathtt{T}$. A λ-tableau $\mathtt{T}$ is *standard* if entries increase from left to right along each row of the diagram, and down each column. We write $\mathrm{Std}(\lambda)$ for the set of standard λ-tableaux. In $\mathrm{Std}(\lambda)$, we have a distinguished tableau, which we denote by $\mathtt{T}^\lambda$. This is the tableau obtained by writing entries $1, \dots, n$ into $[\lambda]$ in order along rows from top to bottom.

Example. Let $\lambda = ((3, 1), (2, 2)) \in \mathscr{P}^2_8$. Then

$$[\lambda] = \begin{array}{l} \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \\ \\ \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \end{array} \end{array}$$

and

$$\mathtt{T}^\lambda = \begin{array}{l} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \\ \\ \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array} \end{array}$$

Another example of a standard tableau is

$$\mathtt{T} = \begin{array}{l} \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \\ \\ \begin{array}{|c|c|}\hline 2 & 7 \\ \hline 4 & 8 \\ \hline \end{array} \end{array}$$

Now fix an e-multicharge $\kappa = (\kappa_1, \dots, \kappa_l) \in I^l$. Then for $\lambda \in \mathscr{P}^l_n$, we define the *residue* of a node $A = (r, c, m) \in [\lambda]$ to be $\mathrm{res}\, A = \kappa_m + c - r \mod e$. If a node $A \in [\lambda]$ has residue i, we call A an *i-node*. We may then obtain a residue sequence $i(\mathtt{T})$ for any λ-tableau $\mathtt{T}$ by reading off residues of nodes in the order the entries of $\mathtt{T}$, i.e. $i(\mathtt{T}) = (\mathrm{res}\, \mathtt{T}^{-1}(1), \dots, \mathrm{res}\, \mathtt{T}^{-1}(n))$. In the special case of the tableau $\mathtt{T}^\lambda$, we write $i^\lambda := i(\mathtt{T}^\lambda)$.

Next, we recall from [BKW11] the degree of a standard tableau. Suppose $\lambda \in \mathscr{P}^l_n$. We call an i-node of $[\lambda]$ *removable* if we may remove it from $[\lambda]$ and still leave a valid Young diagram for a multipartition. Similarly, a node not in $[\lambda]$ is called *addable* if we may add it to $[\lambda]$ to yield a valid Young diagram for a multipartition.

Suppose A is an i-node of λ. Set

$$d_A(\lambda) := \big|\{\text{addable } i\text{-nodes of } [\lambda] \text{ strictly below } A\}\big| - \big|\{\text{removable } i\text{-nodes of } [\lambda] \text{ strictly below } A\}\big|.$$

For $\mathtt{T} \in \mathrm{Std}(\lambda)$ we define the *degree* of $\mathtt{T}$ recursively, setting $\deg(\mathtt{T}) := 0$ when $\mathtt{T}$ is the unique $\varnothing$-tableau. If $\mathtt{T} \in \mathrm{Std}(\lambda)$ with $|\lambda| > 0$, let $A = \mathtt{T}^{-1}(n)$, let $\mathtt{T}_{<n}$ be the tableau obtained by removing this node and set

$$\deg(\mathtt{T}) := d_A(\lambda) + \deg(\mathtt{T}_{<n}).$$

Example. Suppose $e = 3$, $\kappa = (1,1)$ and $\mathtt{T}$ is the $((2),(2,1))$-tableau

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}$$
$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

which has residue sequence $i(\mathtt{T}) = (1,0,1,2,2)$. Letting $A = \mathtt{T}^{-1}(5) = (1,2,2)$, we find that $d_A(\lambda) = 1$. Recursively one finds that for the tableau

$$\mathtt{T}_{<5} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

we have $\deg(\mathtt{T}_{<5}) = 2$, so that $\deg(\mathtt{T}) = 3$.

2.3 Specht modules

We recall a presentation of Specht modules given in [KMR12]. Let $\lambda \in \mathscr{P}^l_n$ and $\kappa \in I^l$. The graded Specht module $\mathrm{S}^\lambda_\kappa$ is the (left) R_n-module with cyclic generator z^λ of degree $\deg \mathtt{T}^\lambda$ subject to the homogeneous relations

1. $e(i)z^\lambda = \delta_{i,i^\lambda} z^\lambda$ for all $i \in I^n$;
2. $y_r z^\lambda = 0$ for all $r = 1, 2, \dots, n$;
3. $\psi_r z^\lambda = 0$ for all $r = 1, 2, \dots, n-1$ such that $r+1$ appears to the right of r in $\mathtt{T}^\lambda$;
4. 'homogeneous Garnir relations'.

If the e-multicharge κ is clear, we may drop it from our notation and simply write S^λ.

Remark. The Specht module $\mathrm{S}^\lambda_\kappa$ for $\lambda \in \mathscr{P}^l_n$ factors through the quotient map $R_n \twoheadrightarrow R^\kappa_n$. In other words, we may define $\mathrm{S}^\lambda_\kappa$ over R^κ_n by the same presentation above.

For any λ-tableau $\mathtt{T}$, we define $w^\mathtt{T} \in \mathfrak{S}_n$ to be the permutation such that $w^\mathtt{T}\mathtt{T}^\lambda = \mathtt{T}$. For each $\mathtt{T} \in \mathrm{Std}(\lambda)$, we fix a reduced expression $w^\mathtt{T} = s_{i_1} \dots s_{i_r}$, and define $\psi^\mathtt{T} := \psi_{i_1} \dots \psi_{i_r}$. Finally, we define $v^\mathtt{T} := \psi^\mathtt{T} z^\lambda$. We emphasise that these basis elements are dependent on the chosen reduced expressions.

Theorem 2.4 ([BKW11]). *For $\lambda \in \mathscr{P}^l_n$, the set $\left\{ v^\mathtt{T} \;\middle|\; \mathtt{T} \in \mathrm{Std}(\lambda) \right\}$ is a basis of S^λ. Moreover, each $v^\mathtt{T}$ is homogeneous of degree $\deg \mathtt{T}$ and for any $i \in I^n$, we have $e(i)v^\mathtt{T} = \delta_{i,i(\mathtt{T})} v^\mathtt{T}$.*

Remark. Inherently in this setup, we now have an algorithm for calculating homomorphisms $\mathrm{S}^\lambda \to \mathrm{S}^\mu$ for $\lambda, \mu \in \mathscr{P}^l_n$ (over R_n, or equivalently, over R^κ_n). The Specht module S^λ is generated by z^λ, so we need only determine what this generator maps to. We have a basis for S^μ, so we may write the image of z^λ as some linear combination of this basis. Furthermore, it suffices to search for homogenous homomorphisms, as $\mathrm{Hom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$ has a homogenous basis. So we may check linear combinations of basis elements of a fixed degree. Thus, it remains to check which linear combinations satisfy the four sets of relations above. For the first, the above theorem tells us that if we have some homogenous $\varphi \in \mathrm{Hom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$, then

$$\varphi(z^\lambda) = \sum_{\substack{\mathtt{T} \in \mathrm{Std}(\lambda) \\ \deg \mathtt{T} = x \\ i(\mathtt{T}) = i^\lambda}} a_\mathtt{T} v^\mathtt{T} \quad \text{for some } a_\mathtt{T} \in \mathbb{F},\ x \in \mathbb{Z}.$$

However, in practice, this quickly becomes a large computation; we may have many basis vectors in the same degree, and need to compute actions of many elements of R_n on each, and then perform some linear algebra to determine linear combinations which satisfy all necessary relations. The next section seeks to enable quicker computation of homomorphisms as well as presenting a 'row removal' result for calculating $\mathrm{Hom}_{R_n}(S^\lambda, S^\mu)$ up to graded vector space isomorphism.

Example 2.5. Let $\lambda = (3^5, 1)$, $\mu = (4^3, 2, 1^2)$ and $e = 3$. Then $\{\mathtt{T} \in \mathrm{Std}(\mu) \mid i(\mathtt{T}) = i^\lambda\} =$

$$\left\{ \begin{smallmatrix}1&2&3&9\\4&5&6&12\\7&8&14&15\\10&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&9\\4&5&6&12\\7&8&14&16\\10&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&3&9\\4&5&6&13\\7&8&14&15\\10&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&9\\4&5&6&13\\7&8&14&16\\10&11\\12\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&6&12\\7&8&14&15\\9&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&6&12\\7&8&14&16\\9&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&6&13\\7&8&14&15\\9&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&6&13\\7&8&14&16\\9&11\\12\\15\end{smallmatrix},\right.$$

$$\begin{smallmatrix}1&2&3&9\\4&5&7&12\\6&8&14&15\\10&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&9\\4&5&7&12\\6&8&14&16\\10&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&3&9\\4&5&7&13\\6&8&14&15\\10&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&9\\4&5&7&13\\6&8&14&16\\10&11\\12\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&7&12\\6&8&14&15\\9&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&7&12\\6&8&14&16\\9&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&7&13\\6&8&14&15\\9&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&3&10\\4&5&7&13\\6&8&14&16\\9&11\\12\\15\end{smallmatrix},$$

$$\begin{smallmatrix}1&2&4&9\\3&5&6&12\\7&8&14&15\\10&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&9\\3&5&6&12\\7&8&14&16\\10&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&4&9\\3&5&6&13\\7&8&14&15\\10&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&9\\3&5&6&13\\7&8&14&16\\10&11\\12\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&6&12\\7&8&14&15\\9&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&6&12\\7&8&14&16\\9&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&6&13\\7&8&14&15\\9&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&6&13\\7&8&14&16\\9&11\\12\\15\end{smallmatrix},$$

$$\left.\begin{smallmatrix}1&2&4&9\\3&5&7&12\\6&8&14&15\\10&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&9\\3&5&7&12\\6&8&14&16\\10&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&4&9\\3&5&7&13\\6&8&14&15\\10&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&9\\3&5&7&13\\6&8&14&16\\10&11\\12\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&7&12\\6&8&14&15\\9&11\\13\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&7&12\\6&8&14&16\\9&11\\13\\15\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&7&13\\6&8&14&15\\9&11\\12\\16\end{smallmatrix},\ \begin{smallmatrix}1&2&4&10\\3&5&7&13\\6&8&14&16\\9&11\\12\\15\end{smallmatrix}\right\}.$$

We have $|\{\mathtt{T} \in \mathrm{Std}(\mu) \mid i(\mathtt{T}) = i^\lambda\}| = 32$. Even if we break this up according to the degrees of the 32 tableaux which may index basis elements in the image of z^λ, we have 10 tableaux of degree 0 and 10 of degree 2. Thus the necessary computation is quite difficult!

3 Row removal for homomorphisms

In this section, we discuss our reduction results, which generalise analogous ungraded results for the symmetric group [FL03] and the Hecke algebra [LM05].

3.1 Dominated tableaux and dominated homomorphisms

Definition 3.1. We say that a standard μ-tableau $\mathtt{T}$ is λ-*dominated* if each entry of $\mathtt{T}$ is at least as *high* as it appears in $\mathtt{T}^\lambda$. We denote the set of all λ-dominated μ-tableaux by $\mathrm{Std}(\mu)^\lambda$.

Example.

1. If $\lambda \rhd \mu$ then $\mathrm{Std}(\mu)^\lambda$ is empty.
2. $\mathrm{Std}(\lambda)^\lambda = \{\mathtt{T}^\lambda\}$.
3. If $\lambda = (3, 2)$ and $\mu = (4, 1)$, then $\mathrm{Std}(\mu)^\lambda = \left\{ \begin{smallmatrix}1&2&3&4\\5\end{smallmatrix},\ \begin{smallmatrix}1&2&3&5\\4\end{smallmatrix} \right\}$.

Definition 3.2. We define $\mathrm{DHom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$ to be the space of homomorphisms $\varphi : \mathrm{S}^\lambda \to \mathrm{S}^\mu$ for which $\varphi(z^\lambda) \subseteq \langle v^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\mu)^\lambda \rangle_{\mathbb{F}}$. We call such maps *dominated homomorphisms*.

Proposition 3.3 ([FS16])**.** *$\langle v^{\mathtt{T}} \mid \mathtt{T} \in \operatorname{Std}(\mu)^\lambda \rangle_{\mathbb{F}}$ is well-defined (i.e. does not depend on our choice of preferred expression for each $v^{\mathtt{T}}$), and therefore so is $\operatorname{DHom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$.*

Proposition 3.4 ([FS16])**.** *$\operatorname{DHom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$ is a graded subspace of $\operatorname{Hom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$. That is, $\operatorname{DHom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$ is spanned by homogeneous homomorphisms.*

Theorem 3.5 ([FS16])**.** *Suppose $e \neq 2$ and $\kappa_i \neq \kappa_j$ whenever $i \neq j$. Then $\operatorname{DHom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu) = \operatorname{Hom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$.*

Remark. When $e = 2$, this is certainly not true in general. An easy counter example arises with the trivial and sign representations being isomorphic: If $\lambda = (2)$ and $\mu = (1^2)$, there is a homomorphism $z^\lambda \mapsto z^\mu$, but $\mathtt{T}^\mu$ is clearly not λ-dominated.

Similarly, if $\kappa = (0, 0)$, the Specht modules indexed by $(1, \varnothing)$ and $(\varnothing, 1)$ are isomorphic for any e.

Example. Continuing with Example 2.5, we see that

$$\left\{ \mathtt{T} \in \operatorname{Std}(\mu)^\lambda \;\middle|\; i(\mathtt{T}) = i^\lambda \right\} = \left\{ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 9 \\ \hline 4 & 5 & 6 & 12 \\ \hline 7 & 8 & 14 & 15 \\ \hline 10 & 11 \\ \cline{1-2} 13 \\ \cline{1-1} 16 \\ \cline{1-1} \end{array} \right\}$$

The computation is now much more reasonable, and one can check that there is indeed a degree 1 homomorphism $\mathrm{S}^\lambda \to \mathrm{S}^\mu$ mapping z^λ to $v^{\mathtt{T}}$, where $\mathtt{T}$ is the above tableau.

Corollary 3.6. *If $e \neq 2$, then $\operatorname{Hom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu) \neq 0 \Rightarrow \lambda \trianglelefteq \mu$. Also, for any $\lambda \in \mathscr{P}^l_n$, $\operatorname{Hom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\lambda)$ is one-dimensional and so S^λ is indecomposable.*

In general, we have seen that Theorem 3.5 does not always hold when $e = 2$ or $\kappa_i = \kappa_j$ for some $i \neq j$. However, we make the following conjecture for cases when we expect the result to remain true.

Conjecture 3.7. *If λ is a* Kleshchev multipartition*, then $\operatorname{DHom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu) = \operatorname{Hom}_{R_n}(\mathrm{S}^\lambda, \mathrm{S}^\mu)$.*

3.2 Row removal

We now restrict our attention to working with dominated homomorphisms and note that there are often no other homomorphisms between Specht modules, by Theorem 3.5.

Definition 3.8. Suppose λ is a partition of n. For any $r \geqslant 0$, we define $\lambda_{\mathrm{T}} = \lambda_{\mathrm{T}}(r) = (\lambda_1, \lambda_2, \dots, \lambda_r)$ and $\lambda_{\mathrm{B}} = \lambda_{\mathrm{B}}(r) = (\lambda_{r+1}, \dots)$, the top and bottom pieces of λ with respect to r. More generally, let $\lambda \in \mathscr{P}^l_n$. For any $r \geqslant 0$ and $1 \leqslant m \leqslant l$ we define

$$\lambda_{\mathrm{T}} = \lambda_{\mathrm{T}}(r, m) = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m-1)}, \lambda^{(m)}_{\mathrm{T}}(r)) \quad \text{and} \quad \lambda_{\mathrm{B}} = \lambda_{\mathrm{B}}(r, m) = (\lambda^{(m)}_{\mathrm{B}}, \lambda^{(m+1)}, \dots).$$

We write $n_{\mathrm{T}} := |\lambda_{\mathrm{T}}|$ and $n_{\mathrm{B}} := |\lambda_{\mathrm{B}}|$.

Now, let $\lambda, \mu \in \mathscr{P}^l_n$, and suppose that for some $r \geqslant 0$ and $1 \leqslant m \leqslant l$, we have $|\lambda_{\mathrm{T}}| = |\lambda_{\mathrm{T}}(r, m)| = |\mu_{\mathrm{T}}| = |\mu_{\mathrm{T}}(r, m)| =: n_{\mathrm{T}}$ and $|\lambda_{\mathrm{B}}| = |\lambda_{\mathrm{B}}(r, m)| = |\mu_{\mathrm{B}}| = |\mu_{\mathrm{B}}(r, m)| =: n_{\mathrm{B}}$. Then we say that the pair (λ, μ) admits a horizontal cut at (r, m).

Theorem 3.9 (First row removal)**.** *Suppose $\lambda, \mu \in \mathscr{P}^l_n$, $\kappa \in I^l$ and $\lambda^{(1)} = \cdots = \lambda^{(m-1)} = \mu^{(1)} = \cdots = \mu^{(m-1)} = \varnothing$ and $\lambda^{(m)}_1 = \mu^{(m)}_1 = k$ for some $1 \leqslant m \leqslant l$ and some k. Let $\lambda_{\mathrm{B}} = \lambda_{\mathrm{B}}(1, m)$, $\mu_{\mathrm{B}} = \mu_{\mathrm{B}}(1, m)$ and $\kappa_{\mathrm{B}} = (\kappa_m - 1, \kappa_{m+1}, \dots, \kappa_l)$. Then $\operatorname{DHom}_{R_n}(\mathrm{S}^\lambda_\kappa, \mathrm{S}^\mu_\kappa) \cong \operatorname{DHom}_{R_{n-k}}(\mathrm{S}^{\lambda_{\mathrm{B}}}_{\kappa_{\mathrm{B}}}, \mathrm{S}^{\mu_{\mathrm{B}}}_{\kappa_{\mathrm{B}}})$ as graded vector spaces.*

Some tricks involving taking duals and twisting by a 'sign automorphism' yield that the above row removal result is equivalent to a certain column removal result. These in turn allow us to obtain a 'final row removal' result analogous to the above first row removal. These are all used in conjunction in order to prove the following generalisation.

Theorem 3.10. *Let $\lambda, \mu \in \mathscr{P}_n^l$, and suppose that for some $r \geqslant 0$ and $1 \leqslant m \leqslant l$, the pair (λ, μ) admits a horizontal cut at (r, m). Define $\kappa_{\mathrm{T}} = (\kappa_1, \dots, \kappa_m)$ and $\kappa_{\mathrm{B}} = (\kappa_m - r, \kappa_{m+1}, \dots, \kappa_l)$. Then, as graded vector spaces, we have*

$$\mathrm{DHom}_{R_n}(\mathrm{S}_\kappa^\lambda, \mathrm{S}_\kappa^\mu) \cong \mathrm{DHom}_{R_{n_{\mathrm{T}}}}(\mathrm{S}_{\kappa_{\mathrm{T}}}^{\lambda_{\mathrm{T}}}, \mathrm{S}_{\kappa_{\mathrm{T}}}^{\mu_{\mathrm{T}}}) \otimes \mathrm{DHom}_{R_{n_{\mathrm{B}}}}(\mathrm{S}_{\kappa_{\mathrm{B}}}^{\lambda_{\mathrm{B}}}, \mathrm{S}_{\kappa_{\mathrm{B}}}^{\mu_{\mathrm{B}}}).$$

Once again, there is also an equivalent column removal result. As we will refer to vertical cuts later, we now define them.

Definition 3.11. Let λ be a partition of n. For any $c \geqslant 0$, we define $\lambda_{\mathrm{L}} = \lambda_{\mathrm{L}}(c)$ to be the partition consisting of all nodes in the first c columns of λ, and $\lambda_{\mathrm{R}} = \lambda_{\mathrm{R}}(c)$ to be the partition consisting of all nodes after the first c columns of λ. Now suppose $\lambda \in \mathscr{P}_n^l$. For any $c \geqslant 0$ and $1 \leqslant m \leqslant l$ we define

$$\lambda_{\mathrm{L}} = \lambda_{\mathrm{L}}(c, m) = (\lambda^{(1)}, \dots, \lambda^{(m-1)}, \lambda_{\mathrm{L}}^{(m)}(c)) \quad \text{and} \quad \lambda_{\mathrm{R}} = \lambda_{\mathrm{R}}(c, m) = (\lambda_{\mathrm{R}}^{(m)}(c), \lambda^{(m+1)}, \dots, \lambda^{(l)})$$

Let $\lambda, \mu \in \mathscr{P}_n^l$ and suppose that for some $c \geqslant 1$ and $1 \leqslant m \leqslant l$, we have $|\lambda_{\mathrm{L}}| = |\lambda_{\mathrm{L}}(c, m)| = |\mu_{\mathrm{L}}| = |\mu_{\mathrm{L}}(c, m)| =: n_{\mathrm{L}}$ and $|\lambda_{\mathrm{R}}| = |\lambda_{\mathrm{R}}(c, m)| = |\mu_{\mathrm{R}}| = |\mu_{\mathrm{R}}(c, m)| =: n_{\mathrm{R}}$. Then we say that the pair (λ, μ) admits a vertical cut at (c, m).

Example. Let $e = 3$, $\kappa = (0, 1, 2)$, $\lambda = ((6, 4, 3^2, 1^3), (5, 4, 2, 1^4), (9^2, 6, 2^2))$ and $\mu = ((10, 4, 3^2, 2^2), (2^5), (13, 9, 6$ Then (λ, μ) admits a horizontal cut at $(3, 2)$, and this cut results in

$$\lambda_{\mathrm{T}} = ((6, 4, 3^2, 1^3), (5, 4, 2)), \qquad \lambda_{\mathrm{B}} = (1^4), (9, 9, 6, 2, 2)),$$
$$\mu_{\mathrm{T}} = ((10, 4, 3^2, 2^2), (2^3)), \qquad \mu_{\mathrm{B}} = ((2^2), (13, 9, 6)).$$

We may perform a further horizontal cut between the final two components, in order to reduce the difficult computation of a large level 2 homomorphism space to two level 1 spaces.

Now suppose that $\operatorname{char} \mathbb{F} = 0$. We may calculate (by computer) that

$$\dim \mathrm{Hom}_{R_{30}}(\mathrm{S}_{\kappa_{\mathrm{T}}}^{\lambda_{\mathrm{T}}}, \mathrm{S}_{\kappa_{\mathrm{T}}}^{\mu_{\mathrm{T}}}) = v^7 \quad \text{and} \quad \dim \mathrm{Hom}_{R_{n_{\mathrm{B}}}}(\mathrm{S}_{\kappa_{\mathrm{B}}}^{\lambda_{\mathrm{B}}}, \mathrm{S}_{\kappa_{\mathrm{B}}}^{\mu_{\mathrm{B}}}) = v \times 2v^5 = 2v^6.$$

Thus, we retrieve that $\dim \mathrm{Hom}_{R_n}(\mathrm{S}_\kappa^\lambda, \mathrm{S}_\kappa^\mu) = 2v^{13}$.

4 Webster's diagrammatic Cherednik algebra

Now, we move on to analogous reduction results for graded decomposition numbers. First we recall a few basic notions.

Recall that for $\kappa \in I^l$, the Specht modules for R_n^κ are indexed by the set $\mathscr{P}_n^l$ of l-multipartitions of n. The simple modules are indexed by the subset of $\mathscr{P}_n^l$ consisting of so called *Kleshchev multipartitions*, which we do not define here. If μ is a Kleshchev multipartition, the simple module D^μ is defined to be the head of S^μ.

Now suppose A is a graded algebra, and M is a graded (left) A-module. Then $M\langle k\rangle$ is the graded module obtained by 'shifting the grading up by k', i.e. $M\langle k\rangle_d = M_{d-k}$. If a graded module M contains a graded simple module L as a composition factor with multiplicity c, we write $[M : L] = c$.

In particular, in our situation we define the graded decomposition number $d_{\lambda\mu}$ to be

$$d_{\lambda\mu} = \sum_{k \in \mathbb{Z}} [\mathrm{S}^\lambda : \mathrm{D}^\mu\langle k\rangle] v^k.$$

In order to obtain row removal results for these graded decomposition numbers, we must leave the world of KLR algebras and instead work with their quasi-hereditary covers, which we shall now introduce.

The *diagrammatic Cherednik algebras* $A(n,\theta,\kappa)$ introduced by Webster [Web12, Web13] are a family of diagram algebras which are graded quasi-hereditary covers of R_n^κ, indexed by an extra parameter $\theta \in \mathbb{R}^l$, which we call the *weighting*. The weighting controls much of the tableau combinatorics of the algebra, and different choices of θ correspond to different labellings $\Theta \subset \mathscr{P}_n^l$ of simple modules of R_n^κ.

Definition 4.1. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_l) \in \mathbb{R}^l$. We call θ *well-separated* if $\theta_i - \theta_j > n$ for all $i > j$.

In the case that θ is a well-separated weighting, we retrieve the labelling of simple modules we have so far discussed (Kleshchev multipartitions). There are other nice weightings which we may consider, and most of the sequel was originally done for arbitrary weightings. For this talk, we focus on the well-separated case and adapt the combinatorics accordingly.

Let $\lambda, \mu \in \mathscr{P}_n^l$. We have a notion of *semistandard tableaux* of shape λ and weight μ, which has a technical definition, which we do not recall here, but is analogous to the classical notion. We denote the set of semistandard tableaux of shape λ and weight μ by $\mathrm{SStd}(\lambda,\mu)$.

Theorem 4.2 ([Web13])**.** *The diagrammatic Cherednik algebra $A(n,\theta,\kappa)$ is a graded cellular algebra with respect to the θ-dominance order and a basis indexed by $\mathrm{SStd}(\lambda,\mu)$ as λ and μ range over $\mathscr{P}_n^l$.*

In particular we have graded standard modules $\Delta(\lambda) = \langle C_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{SStd}(\lambda,-)\rangle_{\mathbb{F}}$ with graded simple heads $\mathrm{L}(\lambda)$ forming a complete set of graded simple modules, up to grading shift.

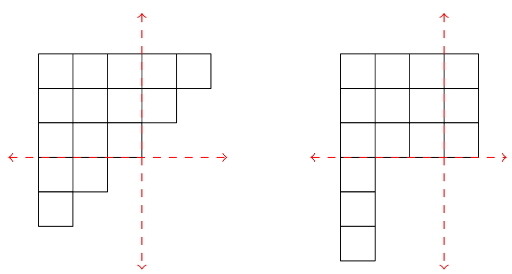
Over $\mathbb{C}$, the module category of $A(n,\theta,\kappa)$ is equivalent to category $\mathcal{O}$ for the rational cyclotomic Cherednik algebra. If θ is well-separated, then $A(n,\theta,\kappa)$ is Morita equivalent to the q-Schur algebra of Dipper–James–Mathas over arbitrary fields.

5 'Row removal' for graded decomposition numbers

With our new conventions in hand, we now seek analogous results to Theorem 3.10 for the graded decomposition numbers $d_{\lambda\mu}$ for $A(n,\theta,\kappa)$ with θ well-separated. This result will also apply to the cyclotomic KLR algebra R_n^κ, and our choice of a well-separated weighting will ensure that the decomposition numbers are compatible with our Specht modules.

Definition 5.1. Let $\theta \in \mathbb{R}^l$ be a well-separated weighting, $\lambda, \mu \in \mathscr{P}_n^l$ and suppose there exist some (r,c,m) such that the pair (λ,μ) admits a horizontal cut at (r,m) and a vertical cut at (c,m). Then we say that the pair (λ,μ) *admits a θ-diagonal cut at (r,c,m)*.

Example. Let $e=3$, $\theta=(0)$ and $(\lambda,\mu) = ((5,4,3,2,1),(4^3,1^3))$. This pair admits a θ-diagonal cut at $(3,3)$.



Definition 5.2. Let $\lambda \in \mathscr{P}_n^l$. We define the top of λ (with respect to a θ-diagonal cut at (r,c,m)) to be the l-multipartition $\lambda_{\mathrm{T}} = (\lambda_{\mathrm{T}}(r,m), \varnothing, \ldots, \varnothing)$, and the bottom of λ to be the l-multipartition $\lambda_{\mathrm{B}} = (\varnothing, \ldots, \varnothing, (c^r, \lambda_{\mathrm{B}}^{(m)}(r)), \lambda^{(m+1)}, \ldots, \lambda^{(l)})$. We define $n_{\mathrm{T}} := |\lambda_{\mathrm{T}}|$ and $n_{\mathrm{B}} := |\lambda_{\mathrm{B}}|$.

Theorem 5.3 ([BS16]). *Let (λ, μ) be a pair of l-multipartitions of n such that $\lambda \trianglerighteq_\theta \mu$. If (λ, μ) admits a θ-diagonal cut at (r, c, m), then*

$$d_{\lambda\mu} = d_{\lambda_{\mathrm{T}}\mu_{\mathrm{T}}} \times d_{\lambda_{\mathrm{B}}\mu_{\mathrm{B}}},$$

where $d_{\lambda_{\mathrm{T}}\mu_{\mathrm{T}}}$ and $d_{\lambda_{\mathrm{B}}\mu_{\mathrm{B}}}$ are the corresponding graded decomposition numbers in the algebras $A(n_{\mathrm{T}}, \theta, \kappa)$ and $A(n_{\mathrm{B}}, \theta, \kappa)$ respectively.

Note that in the above result, the bottom piece of λ differs slightly from that in Theorem 3.10.

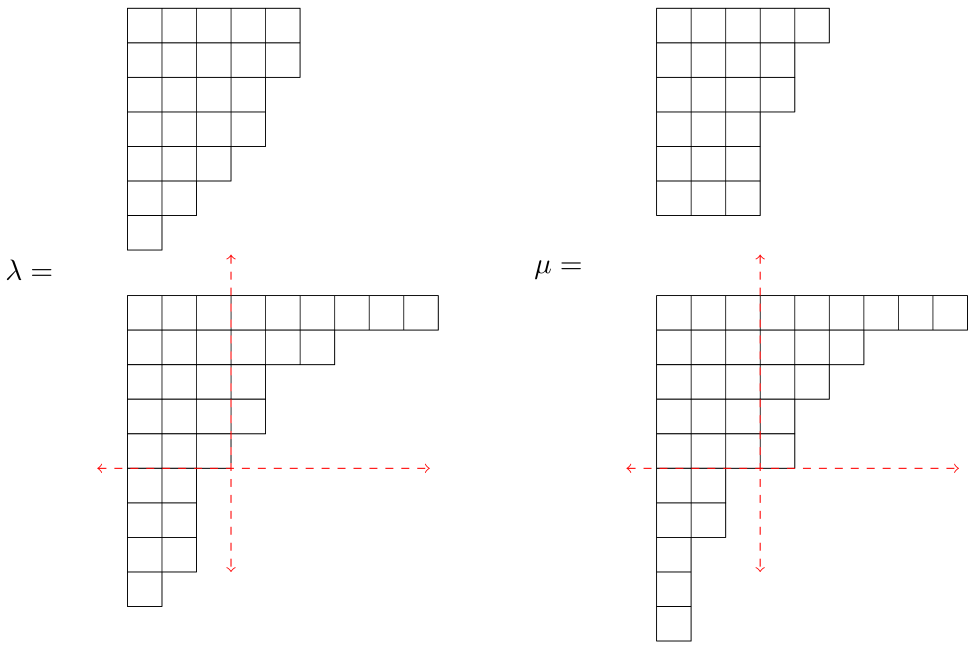
Remark. In [BS16], this result is given in more generality, for any weighting θ. If θ is not well-separated, then the result is phrased in terms of different combinatorics, where we must take a Russian convention for drawing Young diagrams, and make a diagonal cut which may intersect more than one component. Our notion of 'top' and 'bottom' pieces of the multipartitions must be adapted accordingly.

Example. Let $e = 3$, $\theta = (0)$, and $(\lambda, \mu) = ((5, 4, 3, 2, 1), (4^3, 1^3))$. We have that

$$d_{(5,4,3,2,1)(4^3,1^3)} = d_{(5,4,3)(4^3)} d_{(3^3,2,1)(3^3,1^3)} = v \times v = v^2.$$

regardless of the characteristic of the underlying field $\mathbb{F}$.

Example. Let $e = 3$, $\kappa = (0, 1)$ and $\theta = (0, 60)$ a well-separated weighting. The bipartitions $\lambda = ((5^2, 4^2, 3, 2, 1), (9, 6, 4^2, 3, 2^3, 1))$ and $\mu = ((5, 4^2, 3^3), (9, 6, 5, 4^2, 2^2, 1^3))$ of 57 admit a θ-diagonal cut at $(5, 3, 2)$.



This cut results in

$$\lambda_{\mathrm{T}} = ((5^2, 4^2, 3, 2, 1), (9, 6, 4^2, 3)), \qquad \lambda_{\mathrm{B}} = (\varnothing, (3^5, 2^3, 1)),$$
$$\mu_{\mathrm{T}} = ((5, 4^2, 3^3), (9, 6, 5, 4^2)), \qquad \mu_{\mathrm{B}} = (\varnothing, (3^5, 2^2, 1^3)).$$

This reduction yields multipartitions amenable to the techniques of [BS] (whereas λ and μ are not). The left-hand pieces should be compared with [BS, Example 2.6], which yields $d_{\lambda_{\mathrm{T}}\mu_{\mathrm{T}}} = v^{11} + 2v^9 + 2v^7 + v^5$. We may also apply [BS, Theorem 4.30] to calculate that $d_{\lambda_{\mathrm{B}}\mu_{\mathrm{B}}} = v$. Thus, we have $d_{\lambda\mu} = v^{12} + 2v^{10} + 2v^8 + v^6$.

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