D-modules on partial flag varieties and intertwining functors

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1 Introduction

Beilinson and Bernstein proved the following.

Theorem 1. (Beilinson-Bernstein localization [1])
Let G be a reductive algebraic group over $\mathbb{C}$ and B be its Borel subgroup. Let $\mathcal{D}_{G/B}$ be the sheaf of differential operators on the full flag variety G/B.

Then the natural homomorphism $\mathcal{U}(\mathfrak{g}) \to \Gamma(G/B, \mathcal{D}_{G/B})$ induces an isomorphism $\mathrm{U} := \mathcal{U}(\mathfrak{g})/I(0) \cong \Gamma(G/B, \mathcal{D}_{G/B})$, where $I(0)$ is the minimal primitive ideal contained in the augmentation ideal, and the functors $\Gamma : \mathcal{D}_{G/B}\text{-mod} \to \mathrm{U}\text{-mod}$ and $\mathcal{D}_{G/B}\otimes_{\mathrm{U}} : \mathrm{U}\text{-mod} \to \mathcal{D}_{G/B}\text{-mod}$ are mutually inverse equivalences.

They proved the Kazhdan-Lusztig conjecture on the composition multiplicities of Verma modules using this localization theorem (for detail, see [7]). This theorem is applied to the proof of Jantzen's conjecture by Beilinson and Bernstein [3], and also to the study of $(\mathfrak{g},K)$-modules (see e.g., [13]), for example.

In another paper [2], Beilinson and Bernstein defined intertwining functors on $\mathcal{D}$-modules on full flag varieties (see Definition 6). They proved casselman's submodule theorem using intertwining functors.

In this article, we study $\mathcal{D}$-modules and intertwining functors on partial flag varieties. We show that intertwining functors induce equivalences between derived categories in certain cases (Theorem 8). We also show that intertwining functors to antidominant direction are compatible with the global section functors (Theorem 9).

2 Preliminary

2.1 Twisted differential operators

In this section we recall properties of sheaves of twisted differential operators. The original reference is [3]. We follow [8, 9]. Miličić [13] adopts a different definition.

Let Y be a smooth algebraic variety over $\mathbb{C}$. Denote by $\mathcal{O}_Y$ its structure sheaf and by Θ_Y its tangent sheaf. For an $\mathcal{O}_Y$-algebra $\mathcal{A}$, define a filtration by

$\mathcal{O}_Y$-submodules $(\mathrm{F}_i\mathcal{A})_{i\in\mathbb{N}}$ on $\mathcal{A}$ by $\mathrm{F}_0\mathcal{A} = \mathcal{O}_Y$ and $\mathrm{F}_i\mathcal{A} := \{a \in \mathcal{A} \mid [f,a] \in \mathrm{F}_{i-1}\mathcal{A}$ *for any* $f \in \mathcal{O}_Y\}$.

Definition 2. [3, Definition 2.1.1] An $\mathcal{O}_Y$-algebra $\mathcal{A}$ is called a sheaf of twisted differential operators (TDO) if the following conditions are satisfied.

1. The filtration $(\mathrm{F}_i\mathcal{A})_{i\in\mathbb{N}}$ is exhausting, i.e., $\mathcal{A} = \bigcup_{i\in\mathbb{N}} \mathrm{F}_i\mathcal{A}$
2. The first symbol map $\sigma^1 : \mathrm{F}_1\mathcal{A}/\mathrm{F}_0\mathcal{A} \to \Theta_Y, a \mapsto (f \mapsto [a,f])$ is an isomorphism.
3. The homomorphism $\mathrm{Sym}((\sigma^1)^{-1}) : \mathrm{Sym}_{\mathcal{O}_Y}(\Theta_Y) \to \mathrm{gr}^{\mathrm{F}}(\mathcal{A})$ is an isomorphism.

Remark 3. The last condition seems redundant.

By the condition 3, $\mathcal{A}$ is coherent. Let $\mathcal{M}$ be an coherent $\mathcal{A}$-module. The condition 3 of TDO allows one to associate to $\mathcal{M}$ a characteristic variety $\mathrm{Ch}(\mathcal{M}) \subset T^*Y$ in the same way as for $\mathcal{D}$-modules.

Definition 4. A coherent $\mathcal{A}$-module is holonomic if $\mathrm{Ch}(\mathcal{M})$ is a Lagrangian subvariety of T^*Y

$D^b_{hol}(\mathcal{A}\text{-mod})$ denotes the full subcategory of the derived category of the category of $\mathcal{A}$-modules consisting of complexes with holonomic cohomologies.

Proposition 5. [3, Lemma 2.16], [9, Theorem 2.6.1] *There is a natural bijection between the set of isomorphism classes of TDO on* Y *and* $\mathbb{R}^2\Gamma(Y, \overset{\bullet}{0} \to \Omega^1_Y \to \Omega^2_Y \to \Omega^3_Y \to \cdots) =: \mathrm{H}^2(Y, \sigma^{\geq 1}\Omega^\bullet_Y)$.

Let $f : X \to Y$ be a morphism of smooth algebraic varieties. To each TDO $\mathcal{A}$ on Y, a TDO $f^{\#}\mathcal{A}$ (pullback of $\mathcal{A}$) is defined. As in the case of $\mathcal{D}$-modules, functors $f_+ : D^b(f^{\#}\mathcal{A}\text{-mod}) \to D^b(\mathcal{A}\text{-mod})$ and $f^! : D^b(\mathcal{A}\text{-mod}) \to D^b(f^{\#}\mathcal{A}\text{-mod})$ are defined [8, §2.9, §2.11]. For holonomic modules, functors $f_! : D^b_{hol}(f^{\#}\mathcal{A}\text{-mod}) \to D^b_{hol}(\mathcal{A}\text{-mod})$ and $f^+ : D^b_{hol}(\mathcal{A}\text{-mod}) \to D^b_{hol}(f^{\#}\mathcal{A}\text{-mod})$ are defined using duality [9, §1.2].

Let G be an algerbaic group and X be a homogeneous G-variety. In this setting, a method of construction of TDO on X with G-equivariant structure is known [8, §4.9]. Let $\mathfrak{g}_X := \mathfrak{g} \otimes \mathcal{O}_X$ be the Lie algebroid (for the definition of Lie algebroids see [3, §1.2]) with a natural homomorphism $\mathfrak{g} \to \Theta_X$ induced by G-action on X. Denote by $\mathcal{I}_X$ be the kernel of this Lie algebroid homomorphism. Let $\mathcal{U}_X$ be the enveloping algebra $\mathcal{U}(\mathfrak{g}_X)$ of Lie algebroid $\mathfrak{g}_X$. The quotient of $\mathcal{U}_X$ by two-sided ideal generated by $\mathcal{I}_X$ id isomorphic to $\mathcal{D}_X$. Let $\tilde{\lambda}$ be an Lie algebroid homomorphism $\mathcal{I}_X \to \mathcal{O}_X$. We define $\mathcal{D}^{\tilde{\lambda}}_X$ by $\mathcal{U}_X/\langle X - \tilde{\lambda}(X) \mid X \in \mathcal{I}_X\rangle$. This is a TDO on X. We call this homogeneous TDO. There is a parametrization of homogeneous TDO using a stabilizer of G at a point $x \in X$. Fix a point $x \in X$. Let G_x be the stabilizer subgroup of G at x and $\mathfrak{g}_x$ be its Lie algebra. Assume that G_x is connected We see that $\mathcal{I}_X$ is isomorphic to associated Lie algebroid $G \times^{G_x} \mathfrak{g}_x$. We have a natural bijection between $\{\tilde{\lambda} : \mathcal{I}_X \to \mathcal{O}_X\}$ and $\lambda \in (\mathfrak{g}_x/[\mathfrak{g}_x, \mathfrak{g}_x])^*$. Thus we have a homogeneous TDO $\mathcal{D}^\lambda_X$ associated to each character $\lambda \in (\mathfrak{g}_x/[\mathfrak{g}_x, \mathfrak{g}_x])^*$ of $\mathfrak{g}_x$.

2.2 Flag varieties

Let G be a semisimple algebraic group over $\mathbb{C}$, B be its Borel subgroup, H be a Cartan subgroup contained in B and $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$ be their Lie algebras. Let Δ be the set of roots associated to these data, Π be the set of simple roots. To each subset $I \subset \Pi$, one associates parabolic subgroup P_I of G in the way that $P_\emptyset = B$ holds and its Levi subgroup $\mathfrak{l}_I$.

We have the following bijection describing G-orbits of product of partial flag varieties.

$$G\backslash(G/P_J \times G/P_I) \cong P_J\backslash G/P_I \cong W_J\backslash W/W_I$$

Under this bijection $w \in W$ corresponds to the orbit $G(w,1) =: \mathbb{O}_w$. Let p_1 and p_2 be the first and second projection restricted to $\mathbb{O}_w$. The map p_1 and p_2 are smooth surjective morphisms.

By the Bruhat decomposition of G/P_I, we have $\mathrm{H}^2(G/P_I;\mathbb{C}) \cong \mathrm{H}^2(G/P_I, \sigma^{\geq 1}\Omega^\bullet_{G/P_I})$. We have also isomorphisms $\mathrm{H}^2(G/P_I;\mathbb{C}) \cong (\mathfrak{h}/\mathfrak{h}_I)^* \cong (\mathfrak{p}_I/[\mathfrak{p}_I,\mathfrak{p}_I])^*$, where $\mathfrak{h}_I$ is the Lie subalgebra of $\mathfrak{h}$ generated by $\check{\alpha}$, $\alpha \in I$. From this isomorphism and the construction of a homogeneous TDO in §2.1, we see that every TDO on G/P_I is homogeneous TDO. If $\lambda \in (\mathfrak{h}/\mathfrak{h}_I)^*$ is integral, then $\mathcal{L}(\lambda) := G \times^{P_I} \lambda$ is $\mathcal{D}^\lambda_{G/P_I}$-module.

3 Intertwining functors

Intertwining functors are integral transforms for TDO modules associated to orbits $\mathbb{O}_w$. For full flag varieties they were introduced by Beilinson and Bernstein [2, §11].

Definition 6. To each $w \in W$, we associate an integral transform $R^w_? : D^b(\mathcal{D}^\lambda_{G/P_I}\text{-mod}) \to D^b(\mathcal{D}^{w^{-1}*\lambda}_{G/P_J}\text{-mod})$ defined by

$$R^w_?(M) := p_{1?}(\det(\Theta_{p_1}) \otimes p_2^!(M)), \quad ? =!, *$$

where $w * \lambda := w(\lambda-\rho)+\rho$, Θ_{p_1} is the relative tangent sheaf of the projection p_1 and det means the determinant line bundle associated to a vector bundle.

We call this functor an intertwining functor or a Radon transform.

Remark 7. The tensor product with the determinant line bundle of the relative tangent bundle Θ_{p_1} is not necessary in Theorem 8. This twist is crucial when we consider global sections (§3.2).

3.1 Equivalences

In the following, we always consider $w \in W$ satisfying $I = wJ$. In this case p_1 and p_2 are affine space fibrations. In this case we have following equivalences of categories.

Theorem 8.
$R_+^w : D^b_{hol}(\mathcal{D}^\lambda_{G/P_I}\text{-mod}) \to D^b_{hol}(\mathcal{D}^{w^{-1}*\lambda}_{G/P_J}\text{-mod})$ and $R_!^{w^{-1}} : D^b_{hol}(\mathcal{D}^{w^{-1}*\lambda}_{G/P_J}\text{-mod}) \to D^b_{hol}(\mathcal{D}^\lambda_{G/P_I}\text{-mod})$ are mutually inverse equivalences of categories.

This theorem is proved for $\mathcal{D}$-modules in the case of Grassmann varieties by Marastoni [10, Theorem 1] (for projective spaces this seems to be previously known but the author does not know a reference) and for any semisimple group when P_J is conjugate to opposite of P_I [11, Theorem 1.1].

The proof of Theorem 8 relies on the following observation due to Kashiwara (see [10, §2.1]): If the following three conditions are satisfied, then $R_?$ ($? =!, *$) is an equivalence.

1. $G/P_I =: X$, $G/P_J =: Y$ are simply connected.
2. For each $x_1, x_2 \in X$ (resp. $y_1, y_2 \in Y$) we have
$$\mathbb{R}\Gamma(Y_{x_1}, \mathbb{C}_{Y_{x_2}}) \cong \begin{cases} 0 & if\ x_1 \neq x_2 \\ \mathbb{C} & if\ x_1 = x_2 \end{cases} \text{ and } \mathbb{R}\Gamma(X_{y_1}, \mathbb{C}_{X_{y_2}}) \cong \begin{cases} 0 & if\ y_1 \neq y_2 \\ \mathbb{C} & if\ y_1 = y_2. \end{cases}$$
3. Follwing noncharacteristic conditions are satisfied.
$$\mathrm{Ch}(j_{w+}(\mathcal{O}_{\mathbb{O}_w})) \cap (T^*_Y Y \times T^*X) \subset T^*_{Y\times X} Y \times X$$
$$\mathrm{Ch}(j_{w+}(\mathcal{O}_{\mathbb{O}_w})) \cap (T^*Y \times T^*_X X) \subset T^*_{Y\times X} Y \times X$$

Here $j_w : \mathbb{O}_w \to G/P_J \times G/P_I$ is the canonical inclusion and for $x \in X$ and $y \in Y$, we define $Y_x := p_2(p_1^{-1}(x))$, $X_y := p_1(p_2^{-1}(y))$.

Condition 1 is well-known. Condition 2 follows from the fact that X_{y_1} and $X_{y_1} \setminus X_{y_2}$ are contractible. Condition 3 is easily deduced from the fact that X and Y are homogeneous.

3.2 Global sections

In this section we study how global sections of TDO modules behave under intertwining functors.

Taking global sections gives a functor $\Gamma : \mathcal{D}^\lambda_{G/P_I}\text{-mod} \to \Gamma(\mathcal{D}^\lambda_{G/P_I})\text{-mod}$. The algebras $\Gamma(\mathcal{D}^\lambda_{G/P_I})$ and $\Gamma(\mathcal{D}^{w^{-1}*\lambda}_{G/P_J})$ are a priori not comparable. By the structure of homogeneous TDO, we have a homomorphsim of algebras $\mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{D}^\lambda_{G/P_I})$. The kernel of this hohomorphism is known to be the annihilator $I_{\mathfrak{p}_I}(\lambda)$ of the generalized Verma module $M_{\mathfrak{p}_I}(-\lambda - 2\rho + 2\rho_I) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_I)} \mathbb{C}_{-\lambda-2\rho+2\rho_I}$ [5, Remark 3.9]. Here ρ is the half sum of positive roots and ρ_I is ρ for $\mathfrak{l}_I$ Using this homomorphism we consider global sections as a functor $\Gamma : \mathcal{D}^\lambda_{G/P_I}\text{-mod} \to \mathcal{U}(\mathfrak{g})/I_{\mathfrak{p}_I}(\lambda)\text{-mod}$.

From the definition of intertwining functor, we have morphism of functors $\mathbb{R}\Gamma \to \mathbb{R}\Gamma \circ R_+^w$. By Theorem 8, we also have a morphism $\mathbb{R}\Gamma \circ R_!^w \to \mathbb{R}\Gamma$ (for right modules, see [9, §1.6])

To each $I \subset \Pi$ and $a \in \Pi \setminus I$, we associate an element in Weyl group W by $v[a, I] := w_0(I \cup \{a\})w_0(I)$ where $w_0(I)$ is the longest element in W_I. We use the following result of Brink and Howlett [6, Proposition 2.3]. Let $w \in W$, $I, J \subset \Pi$ satisfy $I = wJ$. Then there exist $a_i \in \Pi$ and $J_i \subset \Pi$ satisfying the following conditions.

(1) $w = v[a_1, J_1] \cdots v[a_n, J_n]$

(2) $J_n = J$

(3) $a_i \in \Pi \setminus J_i$

(4) $\ell(w) = \Sigma_{1 \leq i \leq n} \ell(v[a_i, J_i])$

In such a case, we have $R_?^w \cong R_?^{v[a_n, J_n]} \circ \cdots \circ R_?^{v[a_1, J_1]}$ for $? =!, *$.

Theorem 9. *Let $\lambda \in (\mathfrak{h}/\mathfrak{h}_I)^*$ such that $\lambda + \rho_I$ is regular. If all generalized Verma modules $M_{\mathfrak{l}_{J_i \cup \{a_i\}} \cap \mathfrak{p}_{J_i}}(-\langle \lambda, \check{a}_i \rangle \varpi_i)$ over $\mathfrak{l}_{J_i \cup \{a_i\}}$ are irreducible, then the morphism of functors $\mathbb{R}\Gamma \to \mathbb{R}\Gamma \circ R_+^w$ is an isomorphism.*

This theorem is proved in the case of full flag varieties by Miličić [13, Theorem L.4.23], Kashiwara and Tanisaki [9, Corollary 1.6.2].

This theorem is deduced from the following lemma (due to Hisayosi Matumoto) and reduction to maximal parabolic cases.

Lemma 10. *If $\lambda + \rho_I$ is regular, the morphism $\mathcal{U}(\mathfrak{g})/I_{\mathfrak{p}_I}(\lambda) \to \Gamma(\mathcal{D}_{G/P_I}^\lambda)$ is an isomorphism.*

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