REGULAR FUNCTIONS ON SOME SMALL NILPOTENT ORBITS AND UNIPOTENT REPRESENTATIONS

WAN-YU TSAI

Abstract. We aim at (1) decomposing the ring of regular functions on some small nilpotent orbits as a K-representation; (2) studying the unipotent representations attached to these nilpotent orbits and a certain infinitesimal character. In this manuscript, this goal is accomplished for the complex groups of type D. The technique will be used when it is generalized to the case of real groups (which will be mentioned more in the talk). This is joint work with Dan Barbasch.

1. Introduction

Let G be the real points of a complex linear reductive algebraic group with Lie algebra $\mathfrak{g}_0$ and maximal compact subgroup K. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ be the Cartan decomposition, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the complexification. Let $K_{\mathbb{C}}$ be the complexification of K. We are interested in studying unipotent representations associated to an orbit $\mathcal{O} = K_{\mathbb{C}} \cdot e \subset \mathfrak{s}$, where $e \in \mathfrak{s}$ is a nilpotent element. Let $C_{K_{\mathbb{C}}}(\mathcal{O}) := C_{K_{\mathbb{C}}}(e)$, the centralizer of e in $K_{\mathbb{C}}$, and $A(\mathcal{O}) := C_{K_{\mathbb{C}}}(e)/C_{K_{\mathbb{C}}}(e)^0$ be the component group.

Definition 1.1. *An irreducible $(\mathfrak{g}, K)$-module Ξ of G is called unipotent attached to $\mathcal{O}$ and with infinitesimal character $\lambda_{\mathcal{O}}$ if*

(a) *The annihilator $Ann_{U(\mathfrak{g})}\Xi$ is maximal subject to the condition that the infinitesimal character is $\lambda_{\mathcal{O}}$, and the associated variety (see* [V2]*) is $\mathcal{O}$.*

(b) *Ξ is unitary.*

Denote by $\mathcal{U}_G(\mathcal{O}, \lambda_{\mathcal{O}})$ the set of unipotent representations of G attached to $\mathcal{O}$ and $\lambda_{\mathcal{O}}$.

Assume that G is a complex group viewed as a real Lie group. In this case, $K_{\mathbb{C}} \cong G$ as complex groups, and K embeds in the complexification of G, which is isomorphic to $G \times G$, and $\mathfrak{s} \cong \mathfrak{g}_0$ with $K_{\mathbb{C}}$ acting by the adjoint action. We can view $(\mathfrak{g}, K)$-modules as $(\mathfrak{g}, K_{\mathbb{C}})$-modules.

Conjecture 1.2. *Assume that G is simply connected. There exists an infinitesimal character $\lambda_{\mathcal{O}}$ such that in addition to (a) and (b),*

(c) *there is a 1-1 correspondence $\psi \in \widehat{A(\mathcal{O})} \longleftrightarrow \Xi(\mathcal{O}, \psi) \in \mathcal{U}_G(\mathcal{O}, \lambda_{\mathcal{O}})$ satisfying the additional condition*

$$\Xi(\mathcal{O}, \psi) \mid_{K_{\mathbb{C}}} \cong R(\mathcal{O}, \psi).$$

Here

$$(1) \quad \begin{aligned} R(\mathcal{O}, \psi) &= \mathrm{Ind}_{C_{K_{\mathbb{C}}}(\mathcal{O})}^{K_{\mathbb{C}}}(\psi) \\ &= \{f : K_{\mathbb{C}} \to V_{\psi} \mid f(gx) = \psi(x)f(g) \ \forall g \in K_{\mathbb{C}},\ x \in C_{K_{\mathbb{C}}}(\mathcal{O})\} \end{aligned}$$

TABLE 1. D_n, $n = 2p$

$\mathcal{O}$	$[3\ 2^{n-2}\ 1]$	$[3\ 2^{2k}\ 1^{2n-4k-3}]$ $0 \le k \le p-1$	$[2^n]_I$	$[2^n]_{II}$	$[2^{2k}\ 1^{2n-4k}]$ $0 \le k \le p-1$
$\dim \mathcal{O}$	n^2	$4nk - 4k^2 + 4n - 8k - 4$	$n^2 - n$	$n^2 - n$	$4nk - 4k^2 - 2k$
$A_K(\mathcal{O})$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	1

TABLE 2. D_n, $n = 2p+1$

$\mathcal{O}$	$[3\ 2^{2k}\ 1^{2n-4k-3}]$ $0 \le k \le p-1$	$[2^{2k}\ 1^{2n-4k}]$ $0 \le k \le p$
$\dim \mathcal{O}$	$4nk - 4k^2 + 4n - 8k - 4$	$4nk - 4k^2 - 2k$
$A_K(\mathcal{O})$	$\mathbb{Z}_2$	1

is the ring of regular functions on $\mathcal{O}$ transforming according to ψ. Therefore, $R(\mathcal{O}, \psi)$ carries a $K_{\mathbb{C}}$-representation.

Such a parameter $\lambda_{\mathcal{O}}$ is given in detail in [B] for classical complex groups, along with results establishing the validity of this conjecture for large classes of nilpotent orbits in the classical complex groups. Conjectural parameters $\lambda_{\mathcal{O}}$ are available for the exceptional groups as well, [B] for F_4, and to appear elsewhere for type E.

This conjecture cannot be valid for all nilpotent orbits in the case of real groups; the intersection of a complex nilpotent orbit with $\mathfrak{s}$ consists of several components. $R(\mathcal{O}, \psi)$ can be the same for different components, whereas the representations with associated variety containing a given component have drastically different K-structures. Examples can be found in [V1]. As explained in [V1] Chapter 7 and [V2] Theorem 4.11, if the codimension of the orbit $\mathcal{O}$ is ≥ 2, then $\Xi\ |_{K_{\mathbb{C}}} = R(\mathcal{O}, \phi) - Y$ with ϕ an algebraic representation, and Y an $S(\mathfrak{g}/\mathfrak{k})$-module supported on orbits of strictly smaller dimension. The orbits $\mathcal{O}$ under consideration in this paper have codimension ≥ 2. Even when $codim\mathcal{O} \ge 2$, (e.g. the case of the minimal orbit in certain real forms of type D), many examples are known where there are no representations with associated variety $\mathcal{O}$ or any real form of its complexification.

In this lecture we investigate this conjecture for *small* orbits in the complex case, with a view towards generalizing it to the real case. We concentrate on the case of type D. For the condition of *small* we require that

$$[\mu : R(\mathcal{O}, \psi)] \le c_{\mathcal{O}}$$

ie that the multiplicity of any $\mu \in \widehat{K}$ be uniformly bounded. This puts a restriction on $\dim \mathcal{O}$:

$$\dim \mathcal{O} \le \operatorname{rank}(\mathfrak{k}) + |\Delta^+(\mathfrak{k}, \mathfrak{t})|, \tag{2}$$

where $\mathfrak{t} \subset \mathfrak{k}$ is a Cartan subalgebra, and $\Delta^+(\mathfrak{k}, \mathfrak{t})$ is a positive system.

2. THE K-STRUCTURE OF $R(\mathcal{O}, \psi)$ IN THE COMPLEX CASE

In this section, we calculate the K-structure of the regular functions of the nilpotent orbits of the complex group of type D_n. Thus $\mathfrak{g}_0 = \mathfrak{so}(2n)$, and $G = Spin(2n, \mathbb{C})$ while $K = Spin(2n)$. Since $K_{\mathbb{C}} \cong G$, we usually replace $K_{\mathbb{C}}$ with G.

Condition (2) is $\dim \mathcal{O} \leq n^2$. The orbits satisfying this condition are in Tables 1 and 2. They are parametrized by partitions of $2n$. Note that these are the orbits listed in [McG].

2.1. First of all, here is a summary of the component group $A(\mathcal{O}) := A_G(\mathcal{O})$ for each $\mathcal{O}$ in Table 1 and 2. The proof can be found in [CM].

Proposition 2.2. *Write $n = 2p$ or $2p + 1$.*

- *(a) Assume $n = 2p$, $\mathcal{O} = [3\ 2^{n-1}\ 1]$. Then $A(\mathcal{O}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*
- *(b) Assume $n = 2p$ or $2p + 1$, $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$ with $0 \leq k \leq p - 1$. Then $A(\mathcal{O}) \cong \mathbb{Z}_2$.*
- *(c) Assume $n = 2p$, $\mathcal{O} = [2^n]_{I,II}$. Then $A(\mathcal{O}) \cong \mathbb{Z}_2$.*
- *(d) Assume $n = 2p$ or $2p + 1$, $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ with $0 \leq k \leq p - 1$. Then $A(\mathcal{O}) \cong 1$.*

In these cases, $A(\mathcal{O})$ can be realized by elements in the center of G.

2.3. A representation of K (and also a finite dimensional representation of G), will be denoted by its highest weight, i.e.

$$\widehat{K} = \{V(\lambda) \mid \lambda \in P\} \text{ with}$$

$$P = \{(a_1, \ldots, a_n \mid a_1 \geq a_2 \geq \cdots \geq |a_n|,\ a_i\text{'s are all integers or all half-integers}\},$$

the weight lattice of type D_n.

A representation ψ of $A(\mathcal{O})$ is identified with a representation of $C_G(\mathcal{O})$ trivial on $C_G(\mathcal{O})^0$. Recall $R(\widetilde{\mathcal{O}})$, the regular functions on the universal cover of $\mathcal{O}$, and $R(\mathcal{O}, \psi)$ from (1). Then

$$R(\widetilde{\mathcal{O}}) = Ind^G_{C_G(\mathcal{O})^0}(Triv) = \sum_{\psi \in \widehat{A(\mathcal{O})}} R(\mathcal{O}, \psi). \tag{3}$$

We will decompose $R(\mathcal{O}, \psi)$ explicitly as a representation of K for every $\mathcal{O}$ in the tables.

Theorem 2.1. *The regular functions on the universal cover of $\mathcal{O}$ can be decomposed as*

$$R(\widetilde{\mathcal{O}}) = \sum_{\lambda \in S} V(\lambda),$$

where S is a subset of the weight lattice, which varies for each $\mathcal{O}$ as follows.

- (a) *$n = 2p$, $\mathcal{O} = [3\ 2^{n-1}\ 1]$: $S = P$, i.e. every K-type occurs exactly once.*
- (b) *$n = 2p$ or $2p + 1$, $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$ with $0 \leq k \leq p - 1$,*
  $S = \{\lambda = (a_1, \ldots, a_{k+2}, 0, \ldots, 0) \mid \lambda \in P\}$.
- (c) *$n = 2p$, $\mathcal{O} = [2^n]_{I,II}$:*
  $S = \{(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, a_{n-1}) \in P\}$ or $\{(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, -a_{n-1}) \in P\}$ (depending on the choice of I or II).
- (d) *$n = 2p$ or $2p + 1$, $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ with $0 \leq k \leq p - 1$:*
  $S = \{\lambda = (a_1, a_1, a_3, a_3, \ldots, a_{2k-1}, a_{2k-1}, 0, \ldots, 0) \mid \lambda \in P\}$.

Proof. This is a proved by a case by case calculation. We just give a very short idea of the proof here.

Let $\mathcal{O}$ be the G-orbit given by the nilpotent element e, and let $\{e, h, f\}$ be the corresponding Lie triple. Let

- $C_{\mathfrak{k}}(h)_i$ be the i-eigenspace of $ad(h)$ in $\mathfrak{k}$,

- $C_{\mathfrak{k}}(e)_i$ be the i-eigenspace of $ad(h)$ in the centralizer of e in $\mathfrak{k}$,
- $C_{\mathfrak{k}}(h)^+ := \sum_{i>0} C_{\mathfrak{k}}(h)_i$, and $C_{\mathfrak{k}}(e)^+ := \sum_{i>0} C_{\mathfrak{k}}(e)_i$.

Let χ denote the trivial character of $C_{\mathfrak{k}}(e)_0$. We start with $\mathrm{Hom}_{C_G(\mathcal{O})^0}[V,\psi] = \mathrm{Hom}_{C_{\mathfrak{k}}(e)}[V,\psi]$, since $\exp_G C_{\mathfrak{k}}(e) = C_G(\mathcal{O})^0$. This equals $\Big[V^*/(C_{\mathfrak{k}}(e)^+V^*)\Big]^{\chi}$. The results in this theorem are the conditions on a_i's so that the space $\Big[V^*/(C_{\mathfrak{k}}(e)^+V^*)\Big]^{\chi}$ is nontrivial.

□

By Theorem 2.1,

$$Ind^G_{C_G(\mathcal{O})^0}[Triv] = \sum_{\lambda\in S} V(\lambda).$$

On the other hand,

$$\mathrm{Ind}^G_{C_G(\mathcal{O})^0}(Triv) = \mathrm{Ind}^G_{C_G(\mathcal{O})}\left[\mathrm{Ind}^{C_G(\mathcal{O})}_{C_G(\mathcal{O})^0}(Triv)\right] = \sum_{\psi\in\widehat{A(\mathcal{O})}} R(\mathcal{O},\psi). \tag{4}$$

We first decompose $\mathrm{Ind}\,^G_{C_G(\mathcal{O})^0}(Triv)$ as a sum of characters in $\widehat{A(\mathcal{O})}$.

Lemma 2.4.

(a) *Assume that* $n = 2p$, *and* $\mathcal{O} = [3\ 2^{n-1}\ 1]$. *Let* μ_i, $1 \le i \le 4$, *be the following* K*-types parametrized by their highest weights:*

$$\mu_1 = (0,\dots,0), \mu_2 = (1,0,\dots,0),$$
$$\mu_3 = (\tfrac{1}{2},\dots,\tfrac{1}{2}), \mu_4 = (\tfrac{1}{2},\dots,\tfrac{1}{2},-\tfrac{1}{2}).$$

Let ψ_i *be the restriction of the highest weight of* μ_i *to* $C_G(\mathcal{O})$, *respectively. Then*

$$Ind^{C_G(\mathcal{O})}_{C_G(\mathcal{O})^0}(Triv) = \sum_{i=1}^{4} \psi_i.$$

(b) *Assume that* $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$, *with* $1 \le k \le p-1$. *Let* μ_1, μ_2 *be the following* K*-types parametrized by their highest weights:*

$$\mu_1 = (0,\dots,0), \mu_2 = (1,0,\dots,0).$$

Let ψ_i *be the restriction of the highest weight of* μ_i *to* $C_G(\mathcal{O})$, *respectively. Then*

$$Ind^{C_G(\mathcal{O})}_{C_G(\mathcal{O})^0}(Triv) = \psi_1 + \psi_2.$$

(c) *Assume that* $n = 2p$, *and* $\mathcal{O}_I = [2^n]_I, \mathcal{O}_{II} = [2^n]_{II}$. *Let* μ_1, μ_2, ν_1, ν_2, *be:*

$$\mu_1 = (1,\dots,1), \mu_2 = (\tfrac{1}{2},\dots\tfrac{1}{2}),$$
$$\nu_1 = (1,\dots,1,-1), \nu_2 = (\tfrac{1}{2},\dots,\tfrac{1}{2},-\tfrac{1}{2}).$$

Let ψ_i *be the restriction of the highest weight of* μ_i *to* $C_G(e)$, *and* ϕ_i *be the restriction of the highest weight of* ν_i, *respectively. Then*

$$\begin{aligned} Ind^{C_G(\mathcal{O}_I)}_{C_G(\mathcal{O}_I)^0}(Triv) &= \psi_1 + \psi_2, \\ Ind^{C_G(\mathcal{O}_{II})}_{C_G(\mathcal{O}_{II})^0}(Triv) &= \phi_1 + \phi_2. \end{aligned}$$

The ψ_i, ϕ_i *are viewed as representations of* $\widehat{A}(\mathcal{O}_{I,II})$*, and* ψ_1 *and* ϕ_1 *are Triv,* ψ_2, ϕ_2 *are Sgn.*

(d) *Assume that* $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ *with* $0 \leq k \leq p$*. Then*

$$Ind_{C_G(\mathcal{O})^0}^{C_G(\mathcal{O})}(Triv) = Triv.$$

Then we are able to split up $R(\widetilde{\mathcal{O}})$ as a sum of $R(\mathcal{O}, \psi)$ as in (3).

Proposition 2.5. *Let* ψ_i*,* ϕ_j *be as in Lemma 2.4 (with respect to each case). We have the following:*

(a) *Assume that* $n = 2p$ *and* $\mathcal{O} = [3\ 2^{n-1}\ 1]$*. Then the decomposition in (3) is*

$$\begin{aligned}
V_1 &= R(\mathcal{O}, \psi_1) = Ind_{C_G(\mathcal{O})}^{G}(\psi_1) = \sum_{a \in S_1} V(\lambda),\\
V_2 &= R(\mathcal{O}, \psi_2) = Ind_{C_G(\mathcal{O})}^{G}(\psi_2) = \sum_{a \in S_2} V(\lambda),\\
V_3 &= R(\mathcal{O}, \psi_3) = Ind_{C_G(\mathcal{O})}^{G}(\psi_3) = \sum_{a \in S_3} V(\lambda),\\
V_4 &= R(\mathcal{O}, \psi_4) = Ind_{C_G(\mathcal{O})}^{G}(\psi_4) = \sum_{a \in S_4} V(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid\ a_1 \geq \cdots \geq |a_n|, \sum a_i \in 2\mathbb{Z}\},\\
S_2 &= \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid\ a_1 \geq \cdots \geq |a_n|, \sum a_i \in 2\mathbb{Z}+1\},\\
S_3 &= \{(a_1, \ldots, a_n) \in (\mathbb{Z}+\frac{1}{2})^n \mid\ a_1 \geq \cdots \geq |a_n|, \sum a_i \in 2\mathbb{Z}+p\},\\
S_4 &= \{(a_1, \ldots, a_n) \in (\mathbb{Z}+\frac{1}{2})^n \mid\ a_1 \geq \cdots \geq |a_n|, \sum a_i \in 2\mathbb{Z}+p+1\}.
\end{aligned}$$

(b) *Assume that* $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$*, with* $1 \leq k \leq p-1$*. Then the decomposition in (4) is*

$$\begin{aligned}
V_1 &= R(\mathcal{O}, \psi_1) = Ind_{C_G(\mathcal{O})}^{G}(\psi_1) = \sum_{a \in S_1} V(\lambda),\\
V_2 &= R(\mathcal{O}, \psi_2) = Ind_{C_G(\mathcal{O})}^{G}(\psi_2) = \sum_{a \in S_2} V(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \{(a_1, \ldots, a_{2k+2}, 0, \ldots, 0) \in \mathbb{Z}^n \mid\ a_1 \geq \cdots \geq a_{2k+2} \geq 0, \sum a_i \in 2\mathbb{Z}\},\\
S_2 &= \{(a_1, \ldots, a_{2k+2}, 0, \ldots, 0) \in \mathbb{Z}^n \mid\ a_1 \geq \cdots \geq a_{2k+2} \geq 0, \sum a_i \in 2\mathbb{Z}+1\}.
\end{aligned}$$

(c) *Assume that* $n = 2p$, *and* $\mathcal{O}_I = [2^n]_I, \mathcal{O}_{II} = [2^n]_{II}$. *Then the decomposition in (4) is*

$$\begin{aligned} V_1 &= R(\mathcal{O}_I, \psi_1) = Ind^G_{C_G(\mathcal{O}_I)}(\psi_1) = \sum_{a \in S_1} V(\lambda), \\ V_1' &= R(\mathcal{O}_I, \psi_2) = Ind^G_{C_G(\mathcal{O}_I)}(\psi_2) = \sum_{a \in S_2} V(\lambda), \\ V_2 &= R(\mathcal{O}_{II}, \phi_1) = Ind^G_{C_G(\mathcal{O}_{II})}(\phi_1) = \sum_{a \in S_1'} V(\lambda), \\ V_2' &= R(\mathcal{O}_{II}, \phi_2) = Ind^G_{C_G(\mathcal{O}_{II})}(\phi_2) = \sum_{a \in S_2'} V(\lambda), \end{aligned}$$

where

$$\begin{aligned} S_1 &= \{(a_1, a_1, a_2, a_2, \dots, a_p, a_p) \in \mathbb{Z}^n \mid \ a_1 \geq \cdots \geq |a_p|, a_i \in \mathbb{Z}\}, \\ S_2 &= \{(a_1, a_1, a_2, a_2, \dots, a_p, a_p) \in \mathbb{Z}^n \mid \ a_1 \geq \cdots \geq |a_p|, a_i \in \mathbb{Z} + \frac{1}{2}\}, \\ S_1' &= \{(a_1, a_1, a_2, a_2, \dots, a_p, -a_p) \in \mathbb{Z}^n \mid \ a_1 \geq \cdots \geq |a_p|, a_i \in \mathbb{Z}\}, \\ S_2' &= \{(a_1, a_1, a_2, a_2, \dots, a_p, -a_p) \in \mathbb{Z}^n \mid \ a_1 \geq \cdots \geq |a_p|, a_i \in \mathbb{Z} + \frac{1}{2}\}. \end{aligned}$$

(d) *Assume that* $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ *with* $0 \leq k \leq p-1$. *Then*

$$V = R(\mathcal{O}, Triv) = R(\widetilde{O}) = \sum_{\lambda \in S} V(\lambda),$$

where S *is from Theorem 2.1 (d).*

Proof. We just look at the proof of (a). The rest ones are the same. For $V(\lambda) \in \widehat{K}$ with $\lambda \in S_1$, $V(\lambda)$ and μ_1 (the notation in Lemma 2.4) have the same central character and hence $V(\lambda)|_{C_G(\mathcal{O})} = \mu_1|_{C_G(\mathcal{O})} = \psi_1$. Thus, $V(\lambda)$ occurs in $\mathrm{Ind}^G_{C_G(\mathcal{O})}(\psi_1)$. Applying the same argument to the other cases, we get the decompositions of all V_i's. ☐

3. REPRESENTATIONS WITH SMALL SUPPORT, COMPLEX GROUPS

3.1. **Langlands Classification.** Let G be a complex linear algebraic reductive group viewed as a real Lie group. Let θ be a Cartan involution with fixed points K. Let $G \supset B = HN \supset H = TA$ be a Borel subgroup containing a fixed θ-stable Cartan subalgebra H, with

$$\begin{aligned} T = \{h \in H \ \mid \ \theta(h) = h\}, \\ A = \{h \in H \ \mid \ \theta(h) = h^{-1}\}. \end{aligned}$$

The Langlands classification is as follows. Let $\chi \in \widehat{H}$. Denote by

$$X(\chi) := Ind^G_B[\chi \otimes \mathbb{1}]_{K\text{-finite}}$$

the corresponding admissible standard module (Harish-Chandra induction). Let (μ, ν) be the differentials of $\chi\ |_T$ and $\chi\ |_A$ respectively. Let $\lambda_L = (\mu + \nu)/2$ and $\lambda_R = (\mu - \nu)/2$. We write $X(\mu, \nu) = X(\lambda_L, \lambda_R) = X(\chi)$.

Theorem 3.1.

(1) $X(\mu,\nu)$ *has a unique irreducible subquotient denoted* $\overline{X}(\mu,\nu)$ *which contains the* K*-type with extremal weight* μ *occurring with multiplicity one in* $X(\mu,\nu)$.

(2) $\overline{X}(\mu,\nu)$ *is the unique irreducible quotient when* $\langle Re\nu,\alpha\rangle > 0$ *for all* $\alpha \in \Delta(\mathfrak{n},\mathfrak{h})$, *and the unique irreducible submodule when* $\langle Re\nu,\alpha\rangle < 0$.

(3) $\overline{X}(\mu,\nu) \cong \overline{X}(\mu',\nu')$ *if and only if there is* $w \in W$ *such that* $w\mu = \mu', w\nu = \nu'$. *Similarly for* (λ_L,λ_R).

Assume λ_L, λ_R are both dominant integral. Then $\overline{X}(\lambda_L,-\lambda_R)$ is the finite dimensional representation $F(\lambda_L)\otimes F(-w_0\lambda_R)$ where $w_0 \in W$ is the long Weyl group element. The lowest K-type has extremal weight $\lambda_L-\lambda_R$. Weyl's character formula implies

$$\overline{X}(\lambda_L,-\lambda_R) = \sum_{w\in W} \epsilon(w) X(\lambda_L,-w\lambda_R).$$

In our case, we let $G = Spin(2n,\mathbb{C})$, $K = Spin(2n)$ be the maximal compact subgroup of G.

From [B], we can associate to each $\mathcal{O}$ from Table 1 and 2 an infinitesimal character (λ_L,λ_R). The fact is that $\mathcal{O}$ is the minimal orbit which can be the associated variety of a $(\mathfrak{g},K)$-module with infinitesimal character (λ_L,λ_R). They are listed as follows.

(a) $n=2p$, $\mathcal{O}=[3\ 2^{n-1}\ 1]$: λ_L and λ_R are both conjugate to

$$(p-\frac{1}{2},\dots,\frac{3}{2},\frac{1}{2} \mid p-1,\dots,1,0).$$

(b) $n=2p$ or $2p+1$, $\mathcal{O}=[3\ 2^{2k}\ 1^{2n-4k-3}]$, $0\le k\le p-1$: λ_L and λ_R are both conjugate to

$$(k+\frac{1}{2},\dots,\frac{3}{2},\frac{1}{2} \mid n-k-2,\dots,1,0).$$

(c) $n=2p$, $\mathcal{O}=[2^n]_{I,II}$: λ_L and λ_R are both conjugate to

$$\left(p-\frac{1}{2},p-\frac{3}{2},\dots,-p+\frac{3}{2},\pm(p-\frac{1}{2})\right),$$

or

$$\left(\frac{n-1}{4},\frac{n-5}{4},\dots,\frac{-(n-7)}{4},\frac{\pm(n-3)}{4}\right).$$

(d) $n=2p$ or $2p+1$, $\mathcal{O}=[2^{2k}\ 1^{2n-4k}]$, $0\le k\le p-1$: λ_L and λ_R are both conjugate to

$$(k,k-1,\dots,1;n-k-1,\dots,1,0)$$

Notice that the infinitesimal characters in (a) and (b) are nonintegral. For instance, in (a), $\lambda_L=\lambda_R=\rho/2$, where ρ is half sum of the positive roots of type D_{2p}. The integral system is of type $D_p\times D_p$. The notation $\mid$ is to separate the two D_p.

We write λ_L and λ_R to be the infinitesimal characters as above for case (a), (b), (d). For case (c), write

$$\begin{aligned}
\lambda_L^I=\lambda_R^I &= \left(p-\frac{1}{2},p-\frac{3}{2},\dots,-p+\frac{3}{2},-p+\frac{1}{2}\right),\\
\lambda_L^{I'}=\lambda_R^{I'} &= \left(\frac{n-1}{4},\frac{n-5}{4},\dots,\frac{-(n-7)}{4},\frac{-(n-3)}{4}\right),\\
\lambda_L^{II}=\lambda_R^{II} &= \left(p-\frac{1}{2},p-\frac{3}{2},\dots,-p+\frac{3}{2},p-\frac{1}{2}\right),\\
\lambda_L^{II'}=\lambda_R^{II'} &= \left(\frac{n-1}{4},\frac{n-5}{4},\dots,\frac{-(n-7)}{4},\frac{(n-3)}{4}\right).
\end{aligned}$$

Then define the following irreducible modules in terms of Langlands classification:

(a) $\underline{n=2p,\ \mathcal{O}=[3\ 2^{n-1}\ 1]}$:
  (i) $\Xi_1=\overline{X}(\lambda_L,-\lambda_R)$;
  (ii) $\Xi_2=\overline{X}(\lambda_L,-w_1\lambda_R)$, where $w_1\lambda_R=(p-\frac{1}{2},\dots,\frac{3}{2},-\frac{1}{2}\mid p-1,\dots,1,0)$;
  (iii) $\Xi_3=\overline{X}(\lambda_L,-w_2\lambda_R)$, where $w_2\lambda_R=(p-1,\dots,1,0\mid p-\frac{1}{2},\dots,\frac{3}{2},\frac{1}{2})$;
  (iv) $\Xi_4=\overline{X}(\lambda_L,-w_3\lambda_R)$, where $w_3\lambda_R=(p-1,\dots,1,0\mid p-\frac{1}{2},\dots,\frac{3}{2},-\frac{1}{2})$.

(b) $\underline{n=2p \text{ or } 2p+1,\ \mathcal{O}=[3\ 2^{2k}\ 1^{2n-4k-3}],\ 0\le k\le p-1}$:
  (i) $\Xi_1=\overline{X}(\lambda_L,-\lambda_R)$;
  (ii) $\Xi_2=\overline{X}(\lambda_L,-w\lambda_R)$, $w\lambda_R=(k+\frac{1}{2},\dots,\frac{3}{2},\frac{1}{2}\mid n-k-2,\dots,1,0)$.

(c) $\underline{n=2p,\ \mathcal{O}=[2^n]_{I,II}}$:
  (i) $\Xi_I=\overline{X}(\lambda_L^I,-\lambda_R^I)$;
  (i′) $\Xi_{I'}=\overline{X}(\lambda_L^{I'},-\lambda_R^{I'})$;
  (ii) $\Xi_{II}=\overline{X}(\lambda_L^{II},-\lambda_R^{II})$;
  (ii′) $\Xi_{II'}=\overline{X}(\lambda_L^{II'},-\lambda_R^{II'})$.

(d) $\underline{n=2p \text{ or } 2p+1,\ \mathcal{O}=[2^{2k}\ 1^{2n-4k}],\ 0\le k\le p-1}$:
  (i) $\Xi=\overline{X}(\lambda_L,-\lambda_R)$.

The representations introduced above form the set $\mathcal{U}_G(\mathcal{O},\lambda_{\mathcal{O}})$.

Theorem 3.2. *Attain the notation above. Let $G=Spin(2n,\mathbb{C})$ be viewed as a real group. The K-structure of each representations in $\mathcal{U}_G(\mathcal{O},\lambda_{\mathcal{O}})$ is calculated explicitly and matches the K-structure of the $R(\mathcal{O},\psi)$ in Proposition 2.5. That is, there is a 1-1 correspondence $\psi\in\widehat{A_K(\mathcal{O})}\longleftrightarrow\Xi(\mathcal{O},\psi)\in\mathcal{U}_G(\mathcal{O},\lambda_{\mathcal{O}})$ satisfying*

$$\Xi(\mathcal{O},\psi)\mid_{K_{\mathbb{C}}}\cong R(\mathcal{O},\psi).$$

Proof. Case (c) and (d) are well known. The cases $[2^n]_{I,II}$ follow by Helgason's theorem since (D_n,A_{n-1}) is a symmetric pair (for the real form $SO^*(2n)$). They also follow by the method outlined below for the other cases.

For $2k<n$, the methods outlined in [BP] combined with [B] give the answer; the representations are Θ-lifts of the trivial representation of $Sp(2k,\mathbb{C})$. More precisely $\overline{X}(\lambda_R,-\lambda_L)$ is $\Omega/[\mathfrak{sp}(2k,\mathbb{C})\Omega]$ where Ω is the oscillator representation for the pair $O(2n,\mathbb{C})\times Sp(2k,\mathbb{C})$. The K-structure can then be computed using seesaw pairs, namely Ω is also the oscillator representation for the pair $O(2n)\otimes Sp(4k,\mathbb{R})$.

We sketch the proof for case (a), and the proof for case (b) is similar. The arguments are refinements of those in [McG].

Let $K'=Spin(n)\times Spin(n)$ and $\overline{K}=K'/\{\pm(I,I)\}$. Irreducible representations of $\overline{K}$ can be viewed as K'-representations such that $\pm(I,I)$ acts trivially.

Cases (i) and (ii) factor to representations of $SO(2n,\mathbb{C})$, (iii) and (iv) are genuine for $Spin(2n,\mathbb{C})$.

The Kazhdan-Lusztig conjectures for nonintegral infinitesimal character together with Weyl's formula for the character of a finite dimensional module, imply that

$$\overline{X}(\rho/2,-w_i\rho/2)=\sum_{w\in W(D_p\times D_p)}\epsilon(w)X(\rho/2,-ww_i\rho/2). \tag{5}$$

Restricting (5) to K and using Frobenius reciprocity, we get

$$\overline{X}(\rho/2,-w_i\rho/2)\mid_K= Ind_{K'}^{K}[F_1(\rho/2)\otimes F_2(-w_i\rho/2)], \tag{6}$$

where $F_{1,2}$ are finite dimensional representations of the two factors $Spin(2p,\mathbb{C})\times Spin(2p,\mathbb{C})$, so tensor products of representations themselves. The terms $[F_1(\rho/2)\otimes F_2(-w_i\rho/2)]$ are

(i): $F(1/2,\dots,1/2)\otimes F(1/2,\dots,1/2)\boxtimes F(0,\dots,0)\otimes F(0,\dots,0)$,
(ii): $F(1/2,\dots,-1/2)\otimes F(1/2,\dots,1/2)\boxtimes F(0,\dots,0)\otimes F(0,\dots,0)$,
(iii): $F(1/2,\dots,1/2)\otimes F(0,\dots,0)\boxtimes F(0,\dots,0)\otimes (1/2,\dots,1/2)$,
(iv): $F(1/2,\dots,1/2)\otimes F(0,\dots,0)\boxtimes F(0,\dots,0)\otimes F(1/2,\dots,-1/2)$

as $K'=Spin(n)\times Spin(n)$-representations.

Furthermore, we have

(7)

$$\begin{aligned}
&(i)\ \overline{X}(\rho/2,-\rho/2)\mid_K &&=\mathrm{Ind}_{K'}^{K}\left[\sum_{0\le k\le[\frac{p}{2}]}F(\underbrace{1,\dots,1}_{2k},0,\dots,0)\boxtimes F(0,\dots,0)\right]\\
&(ii)\ \overline{X}(\rho/2,-w_1\rho/2)\mid_K &&=\mathrm{Ind}_{K'}^{K}\left[\sum_{0\le k\le[\frac{p-1}{2}]}F(\underbrace{1,\dots,1}_{2k+1},0,\dots,0)\boxtimes F(0,\dots,0)\right]\\
&(iii)\ \overline{X}(\rho/2,-w_2\rho/2)\mid_K &&=\mathrm{Ind}_{K'}^{K}\left[F(1/2,\dots,1/2)\boxtimes F(1/2,\dots,1/2)\right]\\
&(iv)\ \overline{X}(\rho/2,-w_3\rho/2)\mid_K &&=\mathrm{Ind}_{K'}^{K}\left[F(1/2,\dots,1/2)\boxtimes F(1/2,\dots,-1/2)\right].
\end{aligned}$$

In the first two cases we can substitute $\big(G^{split},K^{split}\big):=\big(SO(2p,2p),S[O(2p)\times O(2p)]\big)$ for $\big(K,\overline{K}\big)$, and $\big(Spin(2p,2p),Spin(2p)\times Spin(2p)/\{\pm(I,I)\}\big)$ for the last two cases . The problem of computing the K-structure of $\overline{X}$ reduces to finding the finite dimensional representations of $\widetilde{G}^{split}$ which contain factors of $F(\rho/2)\otimes F(-w_i\rho/2)$. Any finite dimensional representation of $\widetilde{G}^{split}$ is a Langlands quotient of a principal series. Principal series have fine lowest K-types (see [V]). Let MA be a split Cartan subgroup of $\widetilde{G}^{split}$. A principal series is parametrized by a $(\delta,\nu)\in\widehat{MA}$. The δ are called fine, and each fine K^{split}-type μ is a direct sum of a Weyl group orbit of a $fine$ δ. This implies that the multiplicities in (7) are all one, and all the finite dimensional representations occur in $(i),(ii),(iii),(iv)$. We obtain the four formulas correspond to the various orbits of the δ as shown in Proposition 2.5 (a).

□

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