Conformal Geometry and Branching Problems in Representation Theory

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共形幾何と表現論

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共形幾何学において自然に現れる表現論と部分群への分岐則に関して、最近活発に進展している話題から２つのテーマＡ，Ｂを取り上げて概説する。

A. 任意の擬リーマン多様体に対して、その共形変換群の表現を自然な形で構成し、その部分群である等長変換群への分岐則を手法として、大域解の空間を理解する。例えば、

- ユニタリ化 vs 微分方程式の保存量の存在
- 共形同相だ等長ではない幾何モデルの活用
- フーリエ変換の一般化と変形

などがこの話題に関連して自然な形で登場する。

B. 擬リーマン多様体 X とその部分多様体 Y が与えられたとき、２つの共形変換群が定義される。この組に関して自然に生じる分岐則の問題を紹介する。

Branching problems

$$\begin{tikzcd} G \arrow[r, "\pi", "\text{irreducible}"'] & GL(V) \\ \cup & \\ G' \arrow[ur, dashed, "\pi|_{G'}"'] & \end{tikzcd}$$

Branching problem

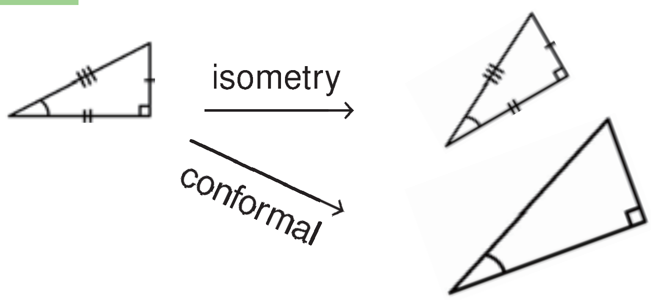
$\cdots$ wish to understand
how the restriction $\pi|_{G'}$
behaves as a G'-module.

Conformal vs Isometry

- $\mathbb{R}^2$ Euclidean space

$\mathrm{Isom}(\mathbb{R}^2) = O(2) \ltimes \mathbb{R}^2$ (Euclid motion gp.)

$\mathrm{Conf}(\mathbb{R}^2) = \mathbb{R}_{>0} \cdot \mathrm{Isom}(\mathbb{R}^2)$



- S^2 Sphere

$\mathrm{Isom}(S^2) = O(3)$

$\mathrm{Conf}(S^2) = O(3,1)/\{\pm I\}$

$$\begin{array}{ccc} O(3,1) & \curvearrowright & S^2 \\ \Vert & & \wr\Vert \\ SL(2,\mathbb{C}) & \curvearrowright & \mathbb{P}^1\mathbb{C} \end{array} \quad z \mapsto \frac{az+b}{cz+d}$$ (Möbius transform)

Hidden symmetry — conformal covariance

Harmonic function

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0 \quad \text{in } \mathbb{R}^2$$

Maxwell's equation

$$dw = 0 \text{ and } d^* w = 0 \quad \text{in } \mathcal{E}^2(\mathbb{R}^{1,3})$$

- Obviously, operators $D = \Delta_{\mathbb{R}^2}, d^*, \ldots$ are isometrically invariant.
  $D = \varphi^{-1} \circ D \circ \varphi$ if φ is isometric.
- Miracle: Solutions are conformally invariant!
  f is a solution $\Leftrightarrow$ $f \circ \varphi$ is a solution if φ is conformal.

$$\begin{tikzcd} \mathrm{Conf}(X) \arrow[r, dashed] & GL(\mathcal{S}ol) \\ \mathrm{Isom}(X) \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

Definition: $\mathrm{Conf}(X) \supset \mathrm{Isom}(X)$

(X, g) pseudo-Riemannian manifold
$\varphi \in \mathrm{Diffeo}(X)$

Def.

φ is isometry $\iff \varphi^* g = g$

φ is conformal $\iff$ ${}^\exists$positive function $C_\varphi \in C^\infty(X)$ s.t. $\varphi^* g = C_\varphi^2\, g$

C_φ : conformal factor

$$\mathrm{Diffeo}(X) \supset \underset{\text{Conformal group}}{\mathrm{Conf}(X)} \supset \underset{\text{isometry group}}{\mathrm{Isom}(X)}$$

Branching problems arising from conformal geometry

$\boxed{\text{Geometry } X}$

$\boxed{\text{Groups } G}$ $\curvearrowright$ rep $\curvearrowright$ $\updownarrow$

$\boxed{\text{Functions (solutions} \ldots \text{) on } X}$

- Branching problems $\cdots$ compare two objects.

$$G' \subset G \xrightarrow{\pi} GL_{\mathbb{C}}(V)$$

Topics for today

	Geometries	$\rightsquigarrow$	Two groups $G \supset G'$
A.	X	$\rightsquigarrow$	$\mathrm{Conf}(X) \supset \mathrm{Isom}(X)$
B.	$X \supset Y$	$\rightsquigarrow$	$\mathrm{Conf}(X) \supset \mathrm{Conf}(X;Y)$

General Reference on Topic A

Topic A: Branching problems for $\mathrm{Conf}(X) \supset \mathrm{Isom}(X)$

Reference — General Scheme/Open Problems

[1] T. Kobayashi. Special functions in minimal representations. Contemporary Mathematics, 610, (2014), pp. 253–266, Amer. Math. Soc.

[2] T. Kobayashi. Geometric analysis on minimal representations. In J. Matsuzawa and S. Tsunoda, editors, Ninth Oka Symposium Lecture Notes, pages 27–61. (2011).

[3] T. Kobayashi. Algebraic analysis of minimal representations. Publ. RIMS, 47, (2011), pp.585–611. Special issue in commemoration of the golden jubilee of algebraic analysis.

[4] T. Kobayashi. Conformal geometry and global solutions to the Yamabe equations on classical pseudo-Riemannian manifolds. Rendiconti del Circolo Matematico di Palermo, Serie II, 71, (2003), pp.15–40.

Further generalization

- Geometric analysis of minimal representations
  — interaction of different fields of mathematics
  - PDEs, conservative quantities [4,13],
  - symplecitc geometry, geometric quantization [3,5,6],
  - real analysis, generalization/deformation of Fourier transform [3,7,9,12],
  - representation theory, construction of “small” reps [6], ...
- $\mathrm{Conf}(X) \supset \mathrm{Isom}(X)$
  (pseudo)-Riemannian manifolds $\rightsquigarrow$ Jordan algebra, etc. [5,6]

Geometric analysis on minimal reps — further readings

[5] Fock model and Segal–Bargmann transform for minimal reps ··· 77 pp. JFA 2012

[6] Minimal representations via Bessel operators ··· 70 pp. JMSJ 2014

[7] Laguerre semigroup and Dunkl operators ··· Compositio Math (2012)

[8] Classification of symmetric pairs with discretely decomposable restrictions ··· Crelles Journal (2015)

[9] Schrödinger model of minimal representations of $O(p,q)$ ··· Memoirs of Amer. Math. Soc. (2011), no.1000, 132 pp.

[10] Geometric analysis of small unitary reps of $GL(n,\mathbb{R})$ ··· JFA (2011).

[11] Special functions associated to a fourth order differential equation ··· Ramanujan J. Math (2011) I, II

[12] Inversion and holomorphic extension ··· R. Howe 60th birthday volume (2007), 65 pp.

[13] Analysis on minimal representations ··· Adv. Math. (2003) I, II, III,

Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers, B. Ørsted, Y. Oshima & M. Pevzner

Example — conformal group of sphere

X : Riemannian manifold of dimension $n \geq 3$.

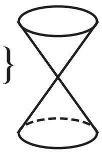
Fact (1) $\dim \mathrm{Conf}(X) \leq \frac{1}{2}(n+1)(n+2)$.
(2) "=" $\iff X \approx S^n$

Example $O(n+1,1) \curvearrowright S^n$ conformal

$$GL(n+2,\mathbb{R}) \overset{\text{linear}}{\curvearrowright} \mathbb{R}^{n+2}$$

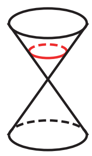
$$\cup \qquad\qquad \cup$$

$$O(n+1,1) \curvearrowright \Xi = \{x \in \mathbb{R}^{n+2} \setminus \{0\} : \sum_{j=1}^{n+1} {x_j}^2 = {x_{n+2}}^2\}$$



$$\cup \qquad \text{conformal} \curvearrowright \text{isometry} \qquad \downarrow$$

$$O(n+1) \curvearrowright S^n \simeq \Xi/\mathbb{R}^\times \simeq \{x \in \mathbb{R}^{n+2} \setminus \{0\} : \sum_{j=1}^{n+1} {x_j}^2 = 1\}$$



Upper estimate of $\dim \mathrm{Conf}(X)$

X : Riemannian manifold of dimension $n \geq 3$.

Fact (1) $\dim \mathrm{Conf}(X) \leq \frac{1}{2}(n+1)(n+2)$.
(2) "=" $\iff X \approx S^n$

Pseudo-Riemannian case.

Example $X = S^p \times S^q$

pseudo-Riemannian metric: $\underbrace{+\cdots+}_{p}\ \underbrace{-\cdots-}_{q}$

$$\mathrm{Isom}(X) \simeq O(p+1) \times O(q+1)$$

$$\cap$$

$$\mathrm{Conf}(X) \simeq O(p+1, q+1)$$

In particular, $\dim \mathrm{Conf}(X) = \frac{1}{2}(n+1)(n+2)$ if $n := p+q$

Twisted reps of conformal groups

$G \curvearrowright X$ pseudo-Riemannian metric g, $\quad h \mapsto (x \mapsto L_h x)$

This action is conformal if $^\exists \Omega \in C^\infty(G \times X)_{>0}$ such that

$$h^* g_{L_h x} = \Omega(h, x)^2 g_x \quad ({}^\forall h \in G,\ \forall x \in X)$$

Ω satisfies a cocycle condition. Therefore

<u>Def</u> ([KØ 2003, Part I]) $^\exists$natural family of reps ϖ_λ of the conformal group G on $C^\infty(X)$ with parameter $\lambda \in \mathbb{C}$ given by

$\varpi_\lambda(h) f := \Omega(h^{-1}, \cdot)^\lambda L^*_{h^{-1}} f$ for $h \in G$

In other words, $^\exists$natural $\mathrm{Conf}(X)$-equivariant line bundle $\mathcal{L}_\lambda \to X$.

<u>Remark</u> $\varpi_\lambda|_{\mathrm{Isom}(X)}$ is independent of λ.

$$\begin{array}{ccc} \mathrm{Conf}(X) & \xrightarrow{\varpi_\lambda} & GL_{\mathbb{C}}(C^\infty(X)) \\ \cup & \nearrow & \\ \mathrm{Isom}(X) & & \end{array}$$

Space of solutions of conformally covariant ops

Suppose $D : C^\infty(X) \to C^\infty(X)$ satisfies

$$\varpi_\nu(h) \circ D = D \circ \varpi_\lambda(h) \quad \text{for all} \quad h \in \mathrm{Conf}(X). \qquad (*)$$

$\mathcal{S}ol(D) := \mathrm{Ker}(D)$ is a $\mathrm{Conf}(X)$-submodule via ϖ_λ.

$$\begin{array}{ccc} \mathrm{Conf}(X) & \xrightarrow{\varpi_\lambda} & GL_{\mathbb{C}}(\mathcal{S}ol(D)) \\ \cup & \nearrow & \\ \mathrm{Isom}(X) & & \end{array}$$

<u>Remark</u> $(*)$ means that D is $\mathrm{Isom}(X)$-invariant.

I raised in [1, Prob 1.8]:

<u>Program A.</u> Understand $\mathcal{S}ol(D)$ via branching "$\mathrm{Conf}(X) \downarrow \mathrm{Isom}(X)$".

(1) (existence of global solution) When is $\mathcal{S}ol(D) \neq \{0\}$?

(2) (irreducibility) Is $(\varpi_\lambda, \mathcal{S}ol(D))$ irreducible as a $\mathrm{Conf}(X)$-module?

(3) (unitarization) Find a $\mathrm{Conf}(X)$-invariant inner product on $\mathcal{S}ol(D)$ (if exists)

(4) (branching law) Decompose $\mathcal{S}ol(D)$ into irreducibles of $\mathrm{Isom}(X)$.

Branching law $(\mathrm{Conf}(X) \downarrow \mathrm{Isom}(X))$ as a clue

$$\mathcal{S}ol(D) = \{f \in C^\infty(X) : D f = 0\}$$

$$\begin{array}{ccc} \circlearrowright \text{invariant} & & \circlearrowright \text{invariant} \\ \mathrm{Conf}(X) & \supset & \mathrm{Isom}(X) \end{array}$$

(4) Branching law for $\mathrm{Conf}(X) \downarrow \mathrm{Isom}(X)$ (clue)

$\swarrow \quad \downarrow \quad \searrow$

(1) Do global solution exist?

(2) Understand $\mathcal{S}ol(D)$ as a $\mathrm{Conf}(X)$-module (irreducibility)

(3) Do conservative quantities exist? (unitarization)

Point $X \sim X_1 \sim X_2 \sim \cdots$ conformally equivalent

$\rightsquigarrow$ $\mathrm{Isom}(X) \neq \mathrm{Isom}(X_1) \neq \mathrm{Isom}(X_2) \neq \cdots$ in general

$$\begin{array}{ccccc} \cap & \cap & \cap & \cdots \\ & \mathrm{Conf}(X) & & \\ \circlearrowleft & \circlearrowleft & \circlearrowright & \cdots \\ \mathcal{S}ol(X) \simeq & \mathcal{S}ol(X_1) \simeq & \mathcal{S}ol(X_2) \simeq & \cdots \end{array}$$

Example — Yamabe operator

X : pseudo-Riemannian manifold of dimension n.

$$\underset{\text{Yamabe operator}}{\widetilde{\Delta}_X} := \underset{\text{Laplacian}}{\Delta_X} + \frac{n-2}{4(n-1)} \underset{\text{scalar curvature}}{\kappa}$$

$$\begin{array}{ccc} \widetilde{\Delta}_X : C^\infty(X) & \to & C^\infty(X) \\ \varpi_{\frac{n-2}{2}} \circlearrowleft & & \circlearrowright \varpi_{\frac{n+2}{2}} \\ & \mathrm{Conf}(X) & \end{array}$$

Theorem $\mathrm{Conf}(X)$ acts on

$\mathcal{S}ol(\widetilde{\Delta}_X) := \{f \in C^\infty(X) : \widetilde{\Delta}_X f = 0\}$

by $f \mapsto C_\varphi^{-\frac{n-2}{2}} f \circ \varphi$ for $\varphi \in \mathrm{Conf}(X)$

Example $(n = 2)$ If $n = 2$, then $\widetilde{\Delta}_X = \Delta_X$ and f is harmonic $\Leftrightarrow$ $f \circ \varphi$ is harmonic.

Yamabe harmonic function

General construction of representation of $\mathrm{Conf}(X)$.

(X, g) $\rightsquigarrow$ $\mathrm{Conf}(X) \overset{\varphi}{\curvearrowright} \mathcal{S}ol(\widetilde{\Delta}_X)$

pseudo-Riemannian mfd. representation of conformal group

Problem A.1 Is $\mathcal{S}ol(\widetilde{\Delta}_X) \neq \{0\}$?

Example

$\mathbb{R}^{p,q} := (\mathbb{R}^{p+q}, dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2)$

$S^{p,q}$:= space form of pseudo-Riemannian mfd. with sectional curvature $\equiv 1$

($S^{n,0} = S^n$, $S^{0,n}$ = hyperbolic space, $S^{1,n-1}$ = anti de Sitter space)

X	$\mathrm{Isom}(X)$	Is $\mathcal{S}ol(\widetilde{\Delta}_X)$ nonzero?
$\mathbb{R}^{p,q}$	$O(p,q) \ltimes \mathbb{R}^{p+q}$	Yes
$S^{p,q}$	$O(p+1,q)$	Yes
$S^p \times S^q$	$O(p+1) \times O(q+1)$	No if $p+q$ is odd.

$\mathcal{S}ol(\widetilde{\Delta}_x)$ as a $\mathrm{Conf}(X)$-module

Problem A.2 Does $\mathrm{Conf}(X)$ act irreducibly on $\mathcal{S}ol(\widetilde{\Delta}_X)$?

Conformal compactification $\mathbb{R}^{p,q} \hookrightarrow \underset{+\cdots+}{S^p} \times \underset{-\cdots-}{S^q} \hookleftarrow S^{p,q}$

Theorem [13] Let $X = S^p \times S^q$. Then $\mathrm{Conf}(X) \simeq O(p+1, q+1)$ acts irreducibly on $\mathcal{S}ol(\widetilde{\Delta}_X)$, and for $p+q > 4$, it is a minimal rep. $\varpi^{p+1,q+1}$ of $O(p+1, q+1)$.

Theorem [4] Let $X = S^{p,q}$. The $\mathfrak{conf}(X) \simeq \mathfrak{o}(p+1, q+1)$-module structure on $\mathcal{S}ol(\widetilde{\Delta}_X)_K$ is described as

$$0 \to d\varpi^{p+1,q+1} \to \mathcal{S}ol(\widetilde{\Delta}_X)_K \to d\varpi^{p+2,q} \to 0.$$

Conservative quantities vs unitarizability

Problem 3 Find conservative quantities in $\mathcal{S}ol(\widetilde{\Delta}_X)$ if exist.

Eg. (wave equation)
"Energy" is a conservative quantity w.r.t. time translation.

Definition (unitarizability) A continuous representation of a group G on a topological vector space V is unitarizable if $^\exists$ G-invariant dense subsp. $W \subset V$ and $^\exists$ G-invariant inner product $(\, , \,)$.

- The Hilbert completion of $(W, (\, , \,))$ gives a unitary rep of G.
- $G \curvearrowright V$ irreducible $\Rightarrow$ $(\, , \,)$ is unique upto scalar.
- Algebraic rep. theory sometimes proves "unitarizability" without giving an explicit inner product $(\, , \,)$.

Construction of invariant inner product — conservative quantities

- Construct a $\mathrm{Conf}(X)$-invariant inner product on $\mathrm{Ker}(\widetilde{\Delta}_X)$

when $\mathrm{Conf}(X)$ (or $\mathfrak{conf}(X)$) attains its maximum dimension.

$\underset{\text{space form}}{S^{p,q}} \subset S^p \times S^q \supset \mathbb{R}^{p,q}$ conformal compactification

(a) $X = S^p \times S^q$ $(p+q$ even$)$

$$\begin{array}{ccccc} \mathrm{Conf}(X) \simeq O(p+1, q+1) & \supset & \mathrm{Isom}(X) \simeq O(p+1) \times O(q+1) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Ker}(\widetilde{\Delta}_X) & \underset{\text{dense}}{\supset} & \bigoplus_{a,b \in \mathbb{N}, a-b = \frac{p-q}{2}} \mathcal{H}^a(S^p) \boxtimes \mathcal{H}^b(S^q) \end{array}$$

Parseval–Plancherel type thm will give the invariant inner product on $\mathrm{Ker}(\widetilde{\Delta}_X)$. (ref. Binegar–Zierau, K–Ø [13], Part II)

(b) $X = S^{p,q}$

$$\begin{array}{ccccc} V := \mathrm{Ker}(\widetilde{\Delta}_X) & \supset & \text{"half" of } V & \supset & \text{"}\tfrac{1}{4}\text{ of } V\text{"} \\ \circlearrowleft & & \circlearrowleft & & \cap \\ \mathrm{Isom}(X) = O(p+1, q) & \subset & \underset{\text{unitarizable}}{\boxed{\mathrm{Conf}(X) = O(p+1, q+1)}} & & L^2(X) \end{array}$$

$\cdots$ asymptotic behaviour at infinity of $S^{p,q}$ (ref. K- 2003)

Construction of invariant inner product—conservative quantities

(c) $X = \mathbb{R}^{p,q}$

(c-1) Use "Green function" to give the inner product explicitly.

(c-2) (an analogue of "energy")

Integration of some "data" of Yamabe harmonic functions over a hyperplane give the inner product explicitly (independence of the choice of hyperplanes is a part of unitarizability)

(d) $X = (\mathbb{R}^{p,q})^\vee$ (dual space)

The Fourier transform of the model (c-2) gives the "Schrödinger model" of the minimal representation.

(K–Mano [9], Memoirs of AMS, no. 1000, 2011)

Branching problems arising from conformal geometry

Geometry X

Groups G rep $\updownarrow$

Functions (solutions . . .) on X

- Branching problems $\cdots$ compare two objects.

$$G' \subset G \xrightarrow{\pi} GL_{\mathbb{C}}(V)$$

Topics for today

	Geometries		Two groups $G \supset G'$
A.	X	$\rightsquigarrow$	$\mathrm{Conf}(X) \supset \mathrm{Isom}(X)$
B.	$X \supset Y$	$\rightsquigarrow$	$\mathrm{Conf}(X) \supset \mathrm{Conf}(X; Y)$

Topic (B): Branching for $\mathrm{Conf}(X) \supset \mathrm{Conf}(X;Y)$ for $X \supset Y$

X: manifold of dimension
$\mathcal{E}^i(X) = \{$differential i-forms on $X\}$ $(0 \le i \le n)$
$d : \mathcal{E}^i(X) \to \mathcal{E}^{i+1}(X)$ exterior derivative

- $\varphi^* \circ d = d \circ \varphi^*$ for all $\varphi \in \mathrm{Diffeo}(X)$

Conversely, if $T : \mathcal{E}^i(X) \to \mathcal{E}^{i+1}(X)$ is a linear differential op. s.t.
$\varphi^* \circ T = T \circ \varphi^*$ for all $\varphi \in \mathrm{Diffeo}(X)$,
then T is a scalar multiple of d.

How about replacing $\mathrm{Diffeo}(X)$ with $\mathrm{Conf}(X)$ or $\mathrm{Isom}(X)$?

Topic (B): Conformal reps on differential forms

X : Riemannian manifold of dimension n
$G = \mathrm{Conf}(X)$:= {conformal diffeos of X} $\supset$ $\mathrm{Isom}(X)$

<u>Example</u> $X = S^n$ (standard sphere)
$\Rightarrow$ $G = \mathrm{Conf}(S^n) \simeq O(n+1,1)/\{\pm I_{n+2}\} \supset \mathrm{Isom}(S^n) \simeq O(n+1)$.

Form a family of $\mathrm{Conf}(X)$-equivariant line bundles: for $u \in \mathbb{C}$ and $\delta \in \{\pm\}$,

$$\mathcal{L}_{u,\delta} \to X.$$

$\Rightarrow$ A family of reps $\Pi^i_{u,\delta}$ of $G = \mathrm{Conf}(X)$ for $u \in \mathbb{C}$ and $\delta \in \{\pm\}$ on

$$\mathcal{E}^i(X, \mathcal{L}_{u,\delta}) \simeq \mathcal{E}^i(X) \text{ for } 0 \le i \le n$$

<u>Remark</u> The restriction $\Pi^i_{u,\delta}|_{\mathrm{Isom}(X)}$ is independent of $u \in \mathbb{C}$.

Branching problems arising from conformal geometry

Geometry X

Groups G ⟳ rep ⟳ $\updownarrow$

Functions (solutions . . .) on X

- Branching problems $\cdots$ compare two objects.

$$G' \subset G \xrightarrow{\pi} GL_{\mathbb{C}}(V)$$

Setting for today

	Geometries		Two groups $G \supset G'$
A.	X	$\rightsquigarrow$	$\mathrm{Conf}(X) \supset \mathrm{Isom}(X)$
B.	$X \supset Y$	$\rightsquigarrow$	$\mathrm{Conf}(X) \supset \mathrm{Conf}(X; Y)$

Books for topic B

[15] A. Juhl, Families of Conformally Covariant Differential Operators, Q-Curvature and Holography, Prog. Math., 275.

[16] T. Kobayashi and B. Speh. Symmetry Breaking for Representations of Rank One Orthogonal Groups, Memoirs of American Mathematical Society 238, 2015. 118 pp. ISBN: 978-1-4704-1922-6.

[17] T. Kobayashi, T. Kubo, and M. Pevzner, Conformal symmetry breaking operators for differential forms on spheres, Lecture Notes in Mathematics, vol. 2170, 2016. viii+192 pp. ISBN: 978-981-10-2656-0.

General references on Topic B

Reference — General Scheme/Classification Theory/Open Problems

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[21] T. Kobayashi and T. Oshima. Finite multiplicity theorems for induction and restriction. Advances in Mathematics, 248, (2013), pp. 921–944.

F-method

[22] T. Kobayashi. F-method for constructing equivariant differential operators. Contemporary Mathematics, 598, pp. 141–148. Amer. Math. Soc., 2013. (Special volume for Helgason's 85th birthday.)

[23] T. Kobayashi. F-method for symmetry breaking operators. Differential Geometry and its Applications, 33, (2014), pp. 272–289, (Special issue for Michael Eastwood's 60th birthday).

[24] T. Kobayashi, B. Ørsted, P. Somberg, and V. Souček. Branching laws for Verma modules and applications in parabolic geometry. I. Advances in Mathematics, 285, (2015), pp. 1796–1852.

[25] T. Kobayashi and M. Pevzner. Differential symmetry breaking operators. I. General theory and F-method. Selecta Mathematica, 22, (2016), pp. 801–845,

[26] II. Rankin-Cohen operators for symmetric pairs, Selecta Mathematica 22, (2016), pp. 847–911.

Problem B: Conformally covariant SBOs

Setting $X \supset Y$ Riemannian manifolds

$G := \mathrm{Conf}(X) \supset G' := \mathrm{Conf}(X;Y) \equiv \{\varphi \in G \mid \varphi(Y) = Y\}$

$\circlearrowleft \Pi^i_{u,\delta} \quad (u \in \mathbb{C},\ \delta \in \{\pm\}) \qquad \circlearrowright \pi^j_{v,\varepsilon} \quad (v \in \mathbb{C},\ \varepsilon \in \{\pm\})$

$\mathcal{E}^i(X, \mathcal{L}_{u,\delta}) = \mathcal{E}^i(X) \overset{D}{\dashrightarrow} \mathcal{E}^j(Y) = \mathcal{E}^j(Y, \mathcal{L}_{v,\varepsilon})$

General Problem B • For which $(i, j, u, v, \delta, \varepsilon)$, does there exist a differential operator $D : \mathcal{E}^i(X, \mathcal{L}_{u,\delta}) \to \mathcal{E}^j(Y, \mathcal{L}_{v,\varepsilon})$ which respects $\mathrm{Conf}(X;Y)$?
• Construct "natural" D explicitly.

Def Say D is a differential symmetry breaking op (SBO)
$\mathrm{Diff}_{\mathrm{Conf}(X;Y)}(\mathcal{E}^i(X, \mathcal{L}_{u,\delta}), \mathcal{E}^j(Y, \mathcal{L}_{v,\varepsilon})) = \{$differential SBOs$\}$

Background

General Problem B • For which $(i, j, u, v, \delta, \varepsilon)$, does there exist a differential operator $D : \mathcal{E}^i(X, \mathcal{L}_{u,\delta}) \to \mathcal{E}^j(Y, \mathcal{L}_{v,\varepsilon})$ which intertwines $\mathrm{Conf}(X;Y)$ and $\pi^i_{v,\varepsilon}$?
• Construct "natural" D explicitly.

1. $(X = Y)$ Yamabe op, Paneitz op, and GJMS ops are for $i = j = 0$.
2. $(X = Y)$ Exterior derivative d $(j = i+1)$, codifferential d^* $(j = i-1)$, and Hodge star op $(j = n - i)$ also have conformal covariance for some $(u, v, \delta, \varepsilon)$.
3. "natural" operators $D \cdots$ not interesting for individual pairs (X, Y) such as $\mathrm{Conf}(X;Y) = \{e\}$.
4. The larger $\mathrm{Conf}(X;Y)$ is, the more constraints on D.

Fact (upper estimate of conformal group of n-dim'l Riemannian mfd X)
$\dim \mathrm{Conf}(X) \leq \frac{1}{2}(n+1)(n+2)$.
"=" $\iff (\mathrm{Conf}(X), X) \approx (O(n+1,1), S^n)$.

Solution to Problem B for the model space

Model space: $(X, Y) = (S^n, S^{n-1})$.
$(\mathrm{Conf}(X), \mathrm{Conf}(X; Y)) \approx (O(n+1, 1), O(n, 1))$

Theorem B (K–Kubo–Pevzner, Lecture Notes in Math, 2016)
Suppose $n \geq 3$, $0 \leq i \leq n$, $0 \leq j \leq n-1$, $u, v \in \mathbb{C}$, and $\delta, \varepsilon \in \{\pm\}$. Then the following three conditions on 6-tuple $(i, j, u, v, \delta, \varepsilon)$ are equivalent:

(i) $\mathrm{Diff}_{\mathrm{Conf}(X;Y)}(\mathcal{E}^i(X, \mathcal{L}_{u,\delta}), \mathcal{E}^j(Y, \mathcal{L}_{v,\varepsilon})) \neq \{0\}$.

(ii) $\dim_{\mathbb{C}} \mathrm{Diff}_{\mathrm{Conf}(X;Y)}(\mathcal{E}^i(X, \mathcal{L}_{u,\delta}), \mathcal{E}^j(Y, \mathcal{L}_{v,\varepsilon})) = 1$.

(iii) $\{j, n-j-1\} \cap \{i-2, i-1, i, i+1\} \neq \emptyset$
+ some explicit positivity/parity cond. on $(u, v, \delta, \varepsilon)$.

Example

- Case $j = i$. $v \in \mathbb{C}$, $v - u \in \{0, 1, 2, \ldots\}$, $\delta \equiv \varepsilon \equiv v - u \mod 2$.
- Case $j = i - 2$ $(2 \leq i \leq n-1)$.
$(u, v) = (n - 2i, n - 2i + 3)$, $\delta \equiv \varepsilon \equiv 1 \mod 2$.

Constructing SBOs for differential forms

Question
Construct explicitly a family of $\mathrm{Conf}(X; Y)$-covariant differential ops

$$\mathcal{E}^i(X, \mathcal{L}_{u,\delta}) \equiv \mathcal{E}^i(X) \xrightarrow{D^{i\to j}} \mathcal{E}^j(Y) \equiv \mathcal{E}^j(Y, \mathcal{L}_{v,\varepsilon})$$

$$\mathrm{Conf}(X) \supset \mathrm{Conf}(X; Y)$$

for the model space $(X, Y) = (S^n, S^{n-1})$

Via the conformal compactification $\mathbb{R}^n \hookrightarrow S^n$, we shall give an explicit formula in the flat coordinates:

$$\begin{array}{ccc} \mathcal{E}^i(S^n) & \xrightarrow{D^{i\to j}} & \mathcal{E}^j(S^{n-1}) \\ \downarrow & & \downarrow \\ \mathcal{E}^i(\mathbb{R}^n) & \overset{\mathcal{D}^{i\to j}}{\dashrightarrow} & \mathcal{E}^j(\mathbb{R}^{n-1}) \end{array}$$

Construction of matrix-valued SBOs

Scalar-valued SBOs For $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, we set

$$\mathcal{D}_k^\lambda := \sum_i a_i(\lambda)(-\Delta_{\mathbb{R}^{n-1}})^i \left(\frac{\partial}{\partial x_n}\right)^{k-2i} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

where $\{a_i(\lambda)\}$ are coefficients of the Gegenbauer polynomial

$$C_k^\lambda(t) = \sum_{0 \le i \le [\frac{k}{2}]} a_i(\lambda) t^{k-2i}.$$

Matrix-valued SBOs

We introduce a family of differential operators of order k

$$\mathcal{D}_{u,k}^{i \to j} : \mathcal{E}^i(\mathbb{R}^n) \to \mathcal{E}^j(\mathbb{R}^{n-1})$$

- Case $j = i$.

$$\mathcal{D}_{u,k}^{i \to i} := \mathrm{Rest}_{x_n=0} \circ \left(\mathcal{D}_{k-2}^{\lambda+1} dd^* + a\mathcal{D}_{k-1}^\lambda d\iota_{\frac{\partial}{\partial x_n}} + b\mathcal{D}_k^\lambda\right)$$

where

$$a := \begin{cases} 1 & (k: \text{ odd}) \\ u + i - \frac{n}{2} + k & (k: \text{ even}) \end{cases}, \quad b := \frac{u+k}{2}, \quad \lambda := u + i - \frac{n-1}{2}.$$

Classification of differential SBOs on forms

$$\mathcal{D}_{u,k}^{i \to i} := \mathrm{Rest}_{x_n=0} \circ \left(\mathcal{D}_{k-2}^{\lambda+1} dd^* + a\mathcal{D}_{k-1}^\lambda d\iota_{\frac{\partial}{\partial x_n}} + b\mathcal{D}_k^\lambda\right) : \mathcal{E}^i(\mathbb{R}^n) \to \mathcal{E}^i(\mathbb{R}^{n-1})$$

Theorem B2 ~~($j = i$ case)~~

Suppose $\nu \in \mathbb{C}$, $k := \nu - u \in \mathbb{N}$, and $\delta \equiv \varepsilon \equiv k \mod 2$.

(1) $\mathcal{D}_{u,k}^{i \to j}$ extends to a conformally covariant differential operator

$\mathcal{E}^i(S^n, \mathcal{L}_{u,\delta}) \to \mathcal{E}^j(S^{n-1}, \mathcal{L}_{\nu,\varepsilon})$.

(2) Conversely, any $O(n,1)$-invariant differential operator from

$\mathcal{E}^i(S^n, \mathcal{L}_{u,\delta})$ to $\mathcal{E}^j(S^{n-1}, \mathcal{L}_{\nu,\varepsilon})$ is proportional to $\mathcal{D}_{u,k}^{i \to i}$.

- Classified in all the cases $\{j, n-j-1\} \cap \{i-2, i-1, i, i+1\} \neq \emptyset$ in K–Kubo–Pevzner (Lecture Notes in Math. 2016)
- $j = i = 0$: scalar-valued case
  Juhl (Progr. Math. 2009), Fefferman–Graham (2013), Kobayashi–Ørsted–Souček–Somberg (Adv. Math. 2015)
- new approach $\cdots$ "F-method" based on "algebraic Fourier transform" of generalized Verma module (K– , 2013)

Classification of Symmetry Breaking Operators (SBOs)

$$\begin{array}{ccccc} G & \supset & G' & \text{real reductive} & \mathcal{V} = G \times_P V & & \mathcal{W} = G' \times_{P'} W \\ \cup & & \cup & & \downarrow & & \downarrow \\ P & \supset & P' & \text{parabolic} & X = G/P & \leftarrow & Y = G'/P' \\ \sigma \downarrow & & \downarrow \tau & & & & \\ GL(V) & & GL(W) & & & & \end{array}$$

<u>General Problem</u> Determine all SBOs,

$$\begin{array}{ccccc} T: & C^\infty\text{-}\mathrm{Ind}_P^G(\sigma) & \to & C^\infty\text{-}\mathrm{Ind}_{P'}^{G'}(\tau) & \text{for "good" } G \supset G'. \\ & \| & & \| & \\ & \mathcal{E}(X, \mathcal{V}) & & \mathcal{E}(Y, \mathcal{W}) & \end{array}$$

<u>Rem</u> 1. "good" $G \supset G' \iff \#(P' \backslash G/P) < \infty$

2. Classified all symmetric pairs (G, G') with $\#(P' \backslash G/P) < \infty$ (K–Matsuki, Dynkin's 90th issue (2014)).

Strategy of classifying all SBOs

<u>General Problem</u> Determine all SBOs,

$$T: \quad C^\infty\text{-}\mathrm{Ind}_P^G(\sigma) \to C^\infty\text{-}\mathrm{Ind}_{P'}^{G'}(\tau) \quad \text{for "good" } G \supset G'.$$

<u>Strategy</u>

- Try to find the distribution kernel K_T of T.
- $\mathrm{Supp}\, K_T$ is a closed subset in $\mathrm{Diag}(G') \backslash (G \times G')/(P \times P') \simeq P' \backslash G/P$.
- Starting point of classification: $\mathrm{Supp}\, K_T$ is the smallest closet subset in $P' \backslash G/P$ $\iff$ T is a "differential operator" (classified in Theorem B)
- Classify K_T modulo smaller closed subsets in $P' \backslash G/P$, inductively.

Strategy of classifying all SBOs

General Problem Determine all SBOs,

$T: \quad C^\infty\text{-Ind}_P^G(\sigma) \quad \to \quad C^\infty\text{-Ind}_{P'}^{G'}(\tau)$ for "good" $G \supset G'$.

Following this strategy, we find:

Solution $\cdots$ Test case: $(G, G') = (O(n+1,1), O(n,1))$.

$\dim \sigma = \dim \tau = 1$ (K–Speh, Memoirs AMS, 2015).

$\sigma \simeq \wedge^i(\mathbb{C}^n), \ \tau \simeq \wedge^j(\mathbb{C}^{n-1})$ (work in progress).

$(G, G') = O(p+1, q+1), \ O(p, q+1))$

(K–Leontiev, work in progress)

Thank you very much!

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