STOKES PHENOMENON AND BRAIDS

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Abstract. In this note we shall present a definition of the spaces of deformation parameters of isomonodromic deformations of meromorphic connections with ramified irregular singularities on $\mathbb{P}^1$ and give a topological characterization of these spaces. Also we shall see an invariant of Stokes structure which is preserved under moving parameters of differential equation inside our space of deformation parameters.

1. Formal meromorphic connections on a disk

In this section we recall basic definitions and facts of formal meromorphic connections on a disk.

Definition 1.1. Let V be a finite dimensional vector space over $K(\!(x)\!)$. A *connection* on V is a K-linear map $\nabla\colon V \to V$ satisfying the Leibniz rule

$$\nabla(fv) = f\nabla(v) + \frac{df}{dx}\nabla(v)$$

for $f \in K(\!(x)\!)$ and $v \in V$. We call the pair (V, ∇) the $K(\!(x)\!)$-connection shortly.

The *rank* of (V, ∇) is the dimension of V as the $K(\!(x)\!)$-vector space. We say that (V, ∇) is *irreducible* if V has no proper nontrivial $K(\!(x)\!)$-subspace W such that $\nabla(W) \subset W$. Morphisms between connections (V_1, ∇_1) and (V_2, ∇_2) are $K(\!(x)\!)$-linear maps $\phi\colon V_1 \to V_2$ satisfying $\phi\nabla_1 = \nabla_2\phi$.

1.1. Indecomposable decompositions of connections.

Let us give a quick review of indecomposable decompositions of connections based on the works of Hukuhara, Turrittin, Levelt, Balser-Jurkat-Lutz, Babbitt- Varadarajan and so on, [10, 18, 13, 3, 1].

For a positive integer q and $f \in K(\!(x^{\frac{1}{q}})\!)$, let us define $E_{f,q} = (V, \nabla)$, a connection over $K(\!(x)\!)$, as follows. Regard $V = K(\!(x^{\frac{1}{q}})\!)$ as a $K(\!(x)\!)$-vector space and define $\nabla(v) = (\frac{d}{dx} + x^{-1}f)v$ for $v \in V$. The irreducibility and isomorphic class of $E_{f,q}$ are determined as follows. If $E_{f,q}$ and $E_{g,q}$ are isomorphic, then there exists an integer $0 \le r \le q-1$ such that

$$f(x^{\frac{1}{q}}) - g(\zeta_q^r x^{\frac{1}{q}}) \in R_q(x) = K(\!(x^{\frac{1}{q}})\!)/\left(x^{\frac{1}{q}}K[\![x^{\frac{1}{q}}]\!] + \frac{1}{q}\mathbb{Z}\right)$$

Also the converse is true. Let us define $R_q^o(x)$ as the set of $f \in R_q(x)$ that cannot be represented by elements of $K(\!(x^{\frac{1}{r}})\!)$ for any $0 < r < q$. Then the connection $E_{f,q}$ is irreducible if and only if the image of f in R_q^o.

Proposition 1.2 (Hukuhara-Turrittin-Levelt decomposition)**.** *Every* (V, ∇) *decomposes as*

$$(V, \nabla) \cong \bigoplus_i (E_{f_i,q_i} \otimes J_{m_i})$$

where $f_i \in R^o_{q_i}(x)$ *and* $J_m = (\mathbb{C}((x))^{\oplus m}, \frac{d}{dx} + x^{-1}N_m)$ *with the nilpotent Jordan block* N_m *of size* m*.*

For the above Hukuhara-Turrittin-Levelt decomposition, set $q := \mathrm{lcm}_i\{q_i\}$ and call *ramification index* of (V, ∇). Collecting indecomposable components satisfying $E_{f_i,q_i} \cong E_{f_j,q_j}$, we have the *coarse* Hukuhara-Turrittin-Levelt decomposition,

$$(V, \nabla) \cong \bigoplus_j E_{g_j,q_j} \otimes R_j \tag{1}$$

where $g_j \in x^{-\frac{1}{q_j}} K[x^{-\frac{1}{q_j}}]$ satisfying $g_j \neq g_{j'}$ $(j \neq j')$ and R_j are regular singular connections.

1.2. **Irregular type and dual Puiseux characteristic.**

Definition 1.3 (irregular type)**.** Let (V, ∇) be a connection with the coarse HTL decomposition as (1). The following subset of $x^{-\frac{1}{q}} K[x^{-\frac{1}{q}}]$ is called the *irregular type* of (V, ∇),

$$\sqcup_i \{g^i_j := \int_\infty^x \frac{g_i(\zeta^j_{q_i} x^{\frac{1}{q_i}})}{x}\, dx \mid j = 1, \ldots, q_i\}. \tag{2}$$

we say that an irregular type $\{p_1(x^{\frac{1}{q}}), \ldots, p_n(x^{\frac{1}{q}})\}$ is *reduced* if the following condition does not hold: $\deg_{x^{-\frac{1}{q}}}(p_1) = \cdots = \deg_{x^{-\frac{1}{q}}}(p_n)$ and coefficients of the leading terms of them are same.

Definition 1.4 (cyclic decomposition of an irregular type)**.** Let us take an irregular type of a meromorphic connection, $P = \{p_1(x^{\frac{1}{q}}), \ldots, p_n(x^{\frac{1}{q}})\}$.

Setting $x^{\frac{1}{q}} \mapsto \zeta_q x^{\frac{1}{q}}$, we obtain the permutation of P which is written as the disjoint permutation cycles,

$$\prod_k (i^k_1\, i^k_2\, \cdots\, i^k_{r_k})$$

where i^k_j are distinct elements in $\{1, 2, \ldots, n\}$. Along this decomposition, we decompose P as below,

$$P = \sqcup_k \{p_{i^k_1}, p_{i^k_2}, \ldots, p_{i^k_{r_k}}\}$$

and call *cyclic decomposition* of P.

For example, the decomposition (2) in Definition 1.3 is precisely cyclic decomposition.

As an analogy of the Puiseux characteristics of plane curve germs, we define the dual Puiseux characteristics of cyclic components. Let

$$\sqcup_i \{g^i_1, \ldots, g^i_{q_i}\}$$

be the cyclic decomposition of the irregular type as in Definition 1.3. Let us write $g_1^i(x^{\frac{1}{q_i}}) = a_n x^{\frac{n}{q_i}} + a_{n+1}x^{\frac{n+1}{q_i}} + \cdots$. Then define

$$-\beta_1^i := \min\{k \mid a_k \neq 0,\ q_i \not| k\} \qquad e_1^i := \gcd(q_i, \beta_1^i).$$

Also define

$$-\beta_l^i := \min\{k \mid a_k \neq 0,\ e_{l-1} \not| k\} \qquad e_l^i := \gcd(e_{l-1}, \beta_1^l).$$

inductively till we reach g_i with $e_{g_i}^i = 1$.

Definition 1.5 (dual Puiseux characteristic of cyclic component)**.** Let

$$\sqcup_i\{g_1^i, \dots, g_{q_i}^i\}$$

be the cyclic decomposition of the irregular type as above. Then the collection of the positive integers

$$(q_i; \beta_1^i, \dots, \beta_{g_i}^i)$$

defined as above is called the *dual Puiseux characteristic* of i-th cyclic component for each i.

1.3. **Irregularity.** For (V, ∇), let us fix a basis and identify $V \cong K(\!(x)\!)^{\oplus n}$. Then we can write $\nabla = \frac{d}{dx} + A(x)$ with $A(x) \in M(n, K(\!(x)\!))$. Moreover we can choose a suitable basis so that $A(x) \in M(n, K[x^{-1}])$, see [3] for example. We call $A(x) \in M(n, K[x^{-1}])$ the *normalized matrix* of (V, ∇).

Let us take K as the field of complex numbers $\mathbb{C}$ and $\mathbb{C}(\{x\})$ denote the field of meromorphic functions near 0. Let us fix a branch of q-th root $x^{\frac{1}{q}}$ of x.

The irregularity defined below measures the difference between formal and convergent solutions of $\frac{d}{dx} + A(x)$.

Definition 1.6 (see H. Komatsu [12] and B. Malgrange [14])**.** Let (V, ∇) be a $\mathbb{C}(\!(x)\!)$-connection. The *irregularity* of (V, ∇) is

$$\mathrm{Irr}(V, \nabla) = \chi\left(\frac{d}{dx} + A(x),\ \mathbb{C}(\!(x)\!)^{\oplus n}\right) - \chi\left(\frac{d}{dx} + A(x),\ \mathbb{C}(\{x\})^{\oplus n}\right).$$

Here $\chi(\Phi, V)$ is the *index* of the $\mathbb{C}$-linear map $\Phi\colon V \to V$, i.e.,

$$\chi(\Phi, V) = \dim_{\mathbb{C}} \mathrm{Ker}\,\Phi - \dim_{\mathbb{C}} \mathrm{Coker}\Phi.$$

It is known that $\mathrm{Irr}(V, \nabla)$ is independent from choices of normalized matrices $A(x)$. Moreover when we consider the coarse HTL decomposition

$$(V, \nabla) \cong \bigoplus_i E_{g_i, q_i} \otimes R_i,$$

the irregularity can be written by

$$\mathrm{Irr}((V, \nabla)) = \sum_i \deg_{x^{-\frac{1}{q_i}}}(g_i).$$

Thus not only for $\mathbb{C}$ but also general K, we can define the irregularity by the above formula.

The Komatsu-Malgrange irregularity can be computed from the dual Puiseux characteristics as follows.

Proposition 1.7. *Let $E_{f,q}$ be an irreducible $K(\!(x)\!)$-connection with the dual Puiseux characteristic $(q, p; \beta_1, \ldots, \beta_g)$. Then we have*

$$\mathrm{Irr}(\mathrm{End}_{K(\!(x)\!)}(E_{f,q})) = \sum_{i=1}^{g} (e_{i-1} - e_i)\beta_i.$$

2. Representations of sequences of total orders and local moduli of differential equations

For a connection $(\widehat{V}, \widehat{\nabla})$ over $\mathbb{C}(\{x\})$, i.e., the pair of finite dimensional $\mathbb{C}(\{x\})$-vector space $\widehat{V}$ and the $\mathbb{C}$-linear connection $\widehat{\nabla}$, the *formalization* (V, ∇) is the connection over $\mathbb{C}(\!(x)\!)$ defined by $V = \mathbb{C}(\!(x)\!) \otimes_{\mathbb{C}(\{x\})} \widehat{V}$ and $\nabla(f \otimes \hat{v}) = \frac{d}{dx} f \otimes \hat{v} + f \otimes \widehat{\nabla}(\hat{v})$ for $f \in \mathbb{C}(\!(x)\!)$ and $\hat{v} \in \widehat{V}$. Let us fix a connection (V_0, ∇_0) over $\mathbb{C}(\!(x)\!)$ and consider a $\mathbb{C}(\{x\})$-connection $(\widehat{V}, \widehat{\nabla})$ whose formalization is isomorphic to (V_0, ∇_0). Let us fix an isomorphism $\xi\colon (V, \nabla) \to (V_0, \nabla_0)$ and call $((\widehat{V}, \widehat{\nabla}), \xi)$ a *marked pair* formally isomorphic to (V_0, ∇_0). We say that marked pairs $((\widehat{V}, \widehat{\nabla}), \xi)$ and $((\widehat{V}', \widehat{\nabla}'), \xi')$ are isomorphic if there exists an isomorphism $\hat{u}\colon (\widehat{V}, \widehat{\nabla}) \to (\widehat{V}', \widehat{\nabla}')$ as $\mathbb{C}(\{x\})$-connections such that $\xi = \xi' \circ u$ where u is the isomorphism between the formalizations of them induced by $\hat{u}$. The isomorphism class of marked pairs formally isomorphic to (V_0, ∇_0) is denoted by $\mathfrak{M}((V_0, \nabla_0))$. This local moduli space $\mathfrak{M}((V_0, \nabla_0))$ is studied by many authors (see for instance [2] and its references) and it is known that there exists a one to one correspondence from a space of certain unipotent matrices, so called Stokes matrices, to $\mathfrak{M}((V_0, \nabla_0))$ (see Theorem 2.4 for example).

2.1. Representations of sequences of total orders.

Let I be a finite set and $<_0, <_1, \ldots, <_h$ $(h \geq 1)$ a sequence of total orders of I. We shortly denote the pair of I and the sequence by

$$\mathcal{I} = (I, (<_i)_{i=0,\ldots,h}).$$

Let us define a representation of $\mathcal{I}$. For $\nu = 1, \ldots, h$, define subsets of $I \times I$ by

$$\rho_\nu = \{(j, k) \in I \times I \mid j \neq k, k <_{\nu-1} j, j <_\nu k\}.$$

Here we note that ρ_ν is *anti-symmetric*, i.e., $(j, k) \in \rho_\nu$ contradicts $(k, j) \in \rho_\nu$ and *transitive*, i.e., $(j, k) \in \rho_\nu$ and $(k, l) \in \rho_\nu$ implies $(j, l) \in \rho_\nu$. For each $k \in I$, take a finite dimensional $\mathbb{C}$-vector space V_k. Then *representations of* $\mathcal{I}$ are elements in

$$\mathrm{Rep}(\mathcal{I}, (V_k)_{k \in I}) = \bigoplus_{\nu=1}^{h} \bigoplus_{(j,k) \in \rho_\nu} \mathrm{Hom}_{\mathbb{C}}(V_k, V_j).$$

We call $(\dim_{\mathbb{C}}(V_k))_{k \in I} \in (\mathbb{Z}_{\geq 0})^I$ the *dimension vector* of $\mathrm{Rep}(\mathcal{I}, (V_k)_{k \in I})$. For a vector $\alpha = (\alpha_i) \in (\mathbb{Z}_{\geq 0})^I$, we write

$$\mathrm{Rep}(\mathcal{I}, \alpha) = \mathrm{Rep}(\mathcal{I}, (\mathbb{C}^{\alpha_k})_{k \in I}).$$

2.2. **Sequence of total orders and that of permutations.** Let us fix a sequence of total orders $\mathcal{I} = (I, (<_i)_{i=0,\dots,h})$. For each $i = 0, \dots, h$ let us arrange the elements in I,

$$t_1^{(i)} <_i t_2^{(i)} <_i \cdots <_i t_n^{(i)},$$

and define the bijection

$$\begin{array}{cccc} \phi_i\colon & I & \longrightarrow & \{1,\dots,n\} \\ & t_k^{(i)} & \longmapsto & k \end{array}.$$

Here n is the cardinality $\#I$ of I. Then we have a sequence of permutations of $\{1, \dots, n\}$,

$$r_\nu = \phi_\nu \circ \phi_{\nu-1}^{-1} \text{ for } \nu = 1, \dots, h.$$

Conversely if we fix a bijection ϕ_0 from I to $\{1, \dots, n\}$ and a sequence of permutations of $\{1, \dots, n\}$,

$$r_1, \dots, r_h,$$

then we can define a sequence of total orders as follows. Let us define bijections $\phi_\nu\colon I \to \{1, \dots, n\}$ by $\phi_\nu = r_\nu \circ \phi_{\nu-1}$ for $\nu = 1, \dots, h$. For each $i = 0, \dots, h$ define the total ordering $<_i$ of I as the pull back of the natural ordering of $\{1, \dots, n\}$ by ϕ_i. Thus we have the following.

Proposition 2.1. *Let I be a finite set of the cardinality n. Then there exists a one to one correspondence between sequences of total orders of I and the pairs of a bijection $\phi_0\colon I \to \{1, \dots, n\}$ and a sequence of elements in $\mathfrak{S}_n$.*

The identity element $\mathrm{id} \in \mathfrak{S}_n$ may be included in the sequence of permutations $r_1, \dots, r_h$ corresponding to $(I, (<_i)_{i=0,\dots,h})$. It is equivalent to the existence of $i \in \{1, \dots, h\}$ such that $<_i$ and $<_{i+1}$ define the same order. Thus we may omit $\mathrm{id} \in \mathfrak{S}_n$ in the sequence of permutations and call the consequent sequence $r'_1, \dots, r'_{h'}$ without $\mathrm{id} \in \mathfrak{S}_n$ the *reduced sequence* of permutations.

2.3. **Local moduli space and representations of sequences of total orders.** We shall construct a sequence of total orders from the Stokes structure of connections.

Let us consider a $\mathbb{C}(\!(x)\!)$-connection (V, ∇) with a normalized matrix $A(x) \in M(n, \mathbb{C}[x^{-1}])$. Then it is known that there exists $F \in \mathrm{GL}(n, \mathbb{C}(\!(x^{\frac{1}{q}})\!))$ with $q \in \mathbb{Z}_{>0}$ such that

$$F^{-1}A(x)F + \left(\frac{d}{dx}F^{-1}\right)F =$$

$$\begin{pmatrix} \tilde{p}_1 I_{m_1} & & & \\ & \tilde{p}_2 I_{m_2} & & \\ & & \ddots & \\ & & & \tilde{p}_s I_{m_s} \end{pmatrix} x^{-1} + \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_s \end{pmatrix} x^{-1}$$

where $p_i \in x^{-\frac{1}{q}}\mathbb{C}[x^{-\frac{1}{q}}]$ ($p_i \neq p_j$ if $i \neq j$) and $L_i \in M(m_i, \mathbb{C})$. Then we can see that the differential equation

$$\frac{d}{dx}Y = A(x)Y$$

has the following formal fundamental solution,

$$H := Fx^L e^{P_A}$$

where $L = \mathrm{diag}(L_1, L_2, \dots, L_s)$ and $P_A = \mathrm{diag}(p_1 I_{m_1}, p_2 I_{m_2}, \dots, p_s I_{m_s})$ with $p_i := \int_\infty^x \frac{\tilde{p_i}}{x}\, dx$ for $i = 1, 2, \dots, s$. We denote the irregular type of this connection by

$$P = \{p_1, p_2, \dots, p_s\},$$

Then define a sequence of total orders as follows. For $d \in \mathbb{R}$, we write

$$j <_d k \quad \text{if} \quad \mathrm{Re}(a_0 e^{-\sqrt{-1} l_0 d}) > 0$$

where $p_j - p_k = a_0 x^{-l_0} + a_1 x^{-l_1} + \cdots + a_t x^{-l_t}$ with $l_0 > l_1 > \cdots > l_t$, $a_0 \neq 0$ and say d is a *Stokes direction* if there exist two distinct integers $1 \leq j, k \leq s$ such that these are incomparable by $<_d$. Thus we note that if d is not a Stokes direction, $<_d$ defines a total order on P_A.

Let $0 \leq d_1 < d_2 < \cdots < d_h < 2\pi$ be the collection of all Stokes directions in $[0, 2\pi)$, so called *basic Stokes directions* (see [4]). Let us choose $\varepsilon > 0$ so that $\tilde{d_i} = d_i + \varepsilon < d_{i+1}$ and for $i = 0, \dots, h$, where d_0 is the maximum of Stokes directions $d < 0$ and we formally set $d_{h+1} = 2\pi$. Then we have the sequence of total orders

$$\mathcal{I}_P = (P, (<_{\tilde{d}_i})_{i=0,\dots,h})$$

and call it *Stokes sequence of total orders* of (V, ∇).

Remark 2.2. *In the above setting, we see only the basic Stokes directions d_i because there exists $\sigma \in \mathfrak{S}_s$ such that*

$$p_{\sigma(i)}(e^{2\pi\sqrt{-1}} x) = p_i(x)$$

for all $i = 1, \dots, s$ and we have

$$j <_d k \text{ if and only if } \sigma(j) <_{d+2\pi} \sigma(k)$$

for $d \in \mathbb{R}$.

Let us associate the representations of $\mathcal{I}_P$ with the space of certain unipotent matrices, i.e., so called Stokes matrices. For each $\nu = 1, \dots, h$, define

$$\mathrm{Sto}_{d_\nu}(P) =$$

$$\left\{ (X_{i,j})_{1 \leq i,j \leq s} \in \bigoplus_{1 \leq i,j, \leq s} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_j}, \mathbb{C}^{m_i}) \;\middle|\; X_{i,j} = \begin{cases} \mathrm{id}_{\mathbb{C}^{m_i}} & \text{if } i = j \\ 0 & \text{if } (i,j) \notin \rho_\nu \end{cases} \right\}.$$

Then we have the isomorphism

$$\mathrm{Rep}(\mathcal{I}_P, (m_i)_{i=1,\dots,s}) \cong \bigoplus_{\nu=1}^{h} \mathrm{Sto}_{d_\nu}(P)$$

as $\mathbb{C}$-vector spaces.

The actual solution of

$$\frac{d}{dx} Y = A(x) Y$$

defines an element in $\mathrm{Rep}(\mathcal{I}_P, (m_i)_{i=1,\dots,s}) \cong \bigoplus_{\nu=1}^h \mathrm{Sto}_{d_\nu}(P)$ as follows. For $\alpha, \beta, r \in \mathbb{R}$ with the ordering $\alpha < \beta$, define the *sectorial region*

$$S(\alpha, \beta; r) := \{ z \in \mathbb{C} \mid 0 < |z| < r,\ \alpha < \arg(z) < \beta \}.$$

Let us take $\rho \in \mathbb{R}$ so that there is no singular point of $\frac{d}{dx}Y = A(x)Y$ in $\{z \in \mathbb{C} \mid 0 < |z| < \rho\}$. Then for each Stokes direction d_ν, we define the *normal sector* by $S_\nu := S(d_{\nu-1}, d_{\nu+1}; \rho)$. The theory of Hukuhara and Turrittin assures that there exists $G_\nu \in \mathrm{GL}(n, \mathcal{O}(S_\nu))$ such that $G_\nu \cong F$ in S_ν, namely,

$$\lim_{r\to 0} \sup_{x\in S(\alpha,\beta;r)} |(G_\nu - \sum_{i=-k}^{n} F_i x^i)x^{-n}| = 0$$

for any $d_{\nu-1} < \alpha < \beta < d_{\nu+1}$ and $n \in \mathbb{Z}_{>0}$. Here $F = \sum_{i=-k}^{\infty} F_i x^i$. In the nonempty intersection $S_{\nu-1} \cap S_\nu$, there exists a constant invertible matrix C_ν and we have

$$G_{\nu-1} x^L e^{P_A} = G_\nu x^L e^{P_A} C_\nu$$

for each $\nu = 1, \ldots, h$. Since $G_{\nu-1}$ and G_ν have the same asymptotic expansion, we have

$$x^L e^{P_A} C_\nu e^{-P_A} x^{-L} = G_\nu^{-1} G_{\nu-1} \cong I_n$$

in $S_{\nu-1} \cap S_\nu$. Let us denote block components of C_ν by $C_{i,j} \in M(m_i, m_j; \mathbb{C})$ for $1 \leq i, j \leq s$ along the block decompositions of L and P_A. Then the above observation implies that

$$C_{i,j} = \begin{cases} I_{m_i} & \text{if } i = j, \\ \text{arbitrary} & \text{if } i <_d j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore it is known the following normalization theorem of Stokes matrices (see [4] for the proof).

Theorem 2.3. *The above actual solutions G_ν can be chosen so that $C_\nu \in \mathrm{Sto}_{d_\nu}(P)$.*

By this theorem, we can associate an element in $\mathrm{Rep}(\mathcal{I}_P, (m_i)_{i=1,\ldots,s})$ to an element in $\mathfrak{M}((V, \nabla))$. Moreover it is known that this correspondence is a well-defined map and bijection. The following is the direct consequence of Theorem VII and its Remark 2 of [4].

Theorem 2.4. *We have a one to one correspondence*

$$\mathrm{Rep}(\mathcal{I}_P, (m_i)_{i=1,\ldots,s}) \cong \bigoplus_{\nu=1}^{h} \mathrm{Sto}_{d_\nu}(P) \cong \mathfrak{M}((V, \nabla)).$$

3. Main theorem

3.1. **Braids and Plane curve germs.** Let us recall the well known theorem by Brauner that irreducible plane curve germs describe iterated torus knots around the singular point . The detail can be found in standard references ([9] for instance). Let $C(x, y) \in \mathbb{C}\{x, y\}$ be an irreducible plane curve germ with the Puiseux characteristic $(m; \beta_1, \ldots, \beta_g)$. Exchanging x and y if necessary, we may assume that f has a good parametrization $x = t^m$, $y = \sum_{i\geq n} a_i t^i \in \mathbb{C}\{t\}$, $(a_n \neq 0)$, with $n \geq m$. If we let x run around a sufficiently small circle ,then

$$K = \left\{ (x, y) \;\middle|\; x \in S_\eta,\ y = \sum_{i\geq n} a_i x^{\frac{i}{m}} \right\}$$

describes a knot in a solid torus

$$S_\eta \times D_\delta = \left\{ (\eta e^{\sqrt{-1}s}, \epsilon e^{\sqrt{-1}t}) \mid s,t\in\mathbb{R},\, 0\le \epsilon\le\delta \right\}$$

with a suitable $\delta > 0$.

Theorem 3.1 (K. Brauner [7]). *The above K is an iterated torus knot of order g and type $(m/e_1, \beta_1/e_1), (e_1/e_2, \beta_2/e_2), \ldots, (e_{g-1}/e_g, \beta_g/e_g)$.*

Now let us recall the construction the iterated torus knot from the good parametrization. First we decompose $y(x)$ as $y(x) = \sum_{k=1}^{g} a_{\beta_k} x^{\frac{\beta_k}{m}} + r(x)$ where $r(x)$ is the term of small oscillations which may be ignored. Thus we focus only on $\tilde{y}(x) = \sum_{k=1}^{g} a_{\beta_k} x^{\frac{\beta_k}{m}}$. Let us first look at $\tilde{y}^{(1)} = a_{\beta_1} x^{\frac{\beta_1}{m}}$. Then

$$K_1 = \{ (x, \tilde{y}^{(1)}(x)) \mid x\in S_\eta \}$$

is the torus knot of type $(m/e_1, \beta_1/e_1)$ which can be seen as the closed braid of the geometric braid B_1 with the m/e_1 strings

$$\tilde{y}^{(1)}_l(t) = a_{\beta_1}\eta^{\frac{\beta_1}{m}} e^{\sqrt{-1}\frac{\beta_1}{m}(t+l)} \quad (0\le t\le 2\pi),$$

for $l = 1, \ldots, m/e_1$. Here we note that there exists a permutation $\tau_1 \in \mathfrak{S}_{m/e_1}$ such that

$$\tilde{y}^{(1)}_l(t+2\pi) = \tilde{y}^{(1)}_{\tau_1(l)}(t)$$

for $l = 1, \ldots, m/e_1$. Here $\mathfrak{S}_n$ denotes the symmetric group of n symbols.

Then next, $\tilde{y}^{(2)} = a_{\beta_1} x^{\frac{\beta_1}{m}} + a_{\beta_2} x^{\frac{\beta_2}{m}}$ improves the approximation and

$$K_2 = \{ (x, \tilde{y}^{(2)}(x)) \mid x\in S_\eta \}$$

is the iterated torus knot of order 2 and type $(m/e_1, \beta_1/e_1), (e_1/e_2, \beta_2/e_2)$. Indeed, for each $l_1 = 1, \ldots, m/e_1$, one has e_1/e_2 points

$$\tilde{y}^{(2)}_{l_1,l_2}(t) = a_{\beta_1}\eta^{\frac{\beta_1}{m}} e^{\sqrt{-1}\frac{\beta_1}{m}(t+l_1)} + a_{\beta_2}\eta^{\frac{\beta_2}{m}} e^{\sqrt{-1}\frac{\beta_2}{m}(t+l_2)} \quad (l_2 = 1,\ldots, e_1/e_2)$$

in the circle of radius $|a_{\beta_2}|\eta^{\frac{\beta_2}{m}}$ around the point $\tilde{y}^{(1)}_{l_1}(t)$. Thus for each l_1, we have the set $\widehat{B}_{l_1}$ of the strings $\tilde{y}^{(2)}_{l_1,l_2}(t)$ for $l_2 = 1, \ldots, e_1/e_2$. As we noted above, we can identify $\widehat{B}_{l_1}$ and $\widehat{B}_{\tau_1(l_1)}$ by substituting $t + 2\pi$ for t. Thus it suffices to see $\widehat{B}_{l_1}$ for one $l_1 \in \{1, \ldots, n/e_1\}$. Then $\widehat{B}_{l_1}$ defines a geometric braid B_2 if t runs in the interval $[0, (m/e_1)2\pi]$ and we have the torus knot of type $(e_1/e_2, \beta_2/e_2)$ as the closed braid of B_2.

Then one can repeat this process to refine the approximation and obtain the iterated torus knot of the plane curve $C(x, y)$.

3.2. **Space of local deformation parameters and links.** Let us take a connection with the coarse HTL decomposition $(V, \nabla) = \bigoplus_{j=1}^{r} E_{p_i,q_i} \otimes R_i$ which has the reduced irregular type with the cyclic decomposition,

$$P := \sqcup_{j=1}^{r} \{ p^{(j)}_1, \ldots, p^{(j)}_{q_j} \}$$

where $p^{(j)}_k := \int_\infty^x \frac{p_j(\zeta_{q_j}^k x^{\frac{1}{q_j}})}{x}\,dx$. Set

$$\begin{aligned} n_{j,k} :&= \mathrm{Irr}(\mathrm{Hom}_{\mathbb{C}((x))}(E_{p_j,q_j}, E_{p_k,q_k})) \\ &= -\mathrm{ord}_x \prod_{s=1}^{q_j}\prod_{t=1}^{q_k}(p^{(j)}_s - p^{(k)}_t). \end{aligned}$$

Here for $f = a_1x^{r_1} + a_2x^{r_2} + \cdots$ with $a_1 \neq 0$ and rational numbers $r_1 < r_2 < \cdots$, we define $\mathrm{ord}(f) := r_1$. For another reduced irregular type with the cyclic decomposition

$$P' := \sqcup_{j=1}^{r'}\{p'^{(j)}_1, \ldots, p'^{(j)}_{q'_j}\},$$

we say that P and P' have the same *cyclic type* if $r' = r$ and the length of each cyclic components is same, i.e., $q_j = q'_j$ for $j = 1, \ldots, r$.

Definition 3.2 (admissible deformation)**.** Let P and P' be reduced irregular types defined as above. We say that P' is an *admissible deformation* of P if the followings are satisfied.

(1) P' and P have the same cyclic type.
(2) For $n_{j,k}$, $1 \leq j, k \leq r$ defined above, we have

$$n_{j,k} = -\mathrm{ord}_x \prod_{s=1}^{q_j}\prod_{t=1}^{q_k}(p'^{(j)}_s - p'^{(k)}_t).$$

(3) For $i = 1, \ldots .r$, each i-th cyclic components of P and P' have the same dual Puiseux characteristic.

Definition 3.3 (space of local deformation parameters)**.** Let (V, ∇) be as above. Then the *space of local deformation parameters* $\mathbb{T}_{(V,\nabla)}$ is the set of all admissible deformations of P.

Remark 3.4. Let us recall integrable (isomonodromic) deformations of meromorphic connections in a very rough way (see [16] for the detail). Let $\mathbb{T}$ be a complex manifold and V a finite dimensional $\mathbb{C}$-vector space. Then the family $(\nabla_t)_{t\in\mathbb{T}}$, $\nabla_t = d_{\mathbb{P}^1} - A(s)$ of meromorphic connections on $\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} V$ is called *integrable* if there exists a flat meromorphic connection ∇ on $\mathcal{O}_{\mathbb{P}^1\times\mathbb{T}} \otimes_{\mathbb{C}} V$ such that $\nabla|_{\mathbb{P}^1\times\{t\}} = \nabla_t$. It is known that elements in an integrable family, roughly speaking, have the equivalent monodromies and Stokes structures, thus it is called isomonodromic deformation.

In the theory of isomonodromic deformations, the complex manifold $\mathbb{T}$ is called the space of deformation parameters and plays a role of the space of time parameters of time dependent Hamiltonians which describe the isonomodromic deformations. For instance, $\mathbb{T}$ can be taken as the configuration space of ordered points of $\mathbb{P}^1$ in the case of Fuchsian isomonodromic deformations. Our deformation parameters defined in Definition 3.3 can be seen as a generalization to the irregular singular isonomodromic deformations following preceding studies of irregular isomonodromic deformations, for example in [11],[6], [8],[5].

In the Fuchsian isomonodromy, the space of deformation parameters can be seen as the configuration space of ordered points and its topological structures, fundamental group for example, play important roles for the study of isomonodromic deformations. Thus we are interested in a topological structure of our space of local deformation parameters for irregular singular equations.

For this purpose, we shall give a topological characterization of our space of local deformation parameters by looking at the links associated to the irregular types as follows.

Definition 3.5 (link of an irregular type)**.** Let $P = \{p_1(x^{\frac{1}{q}}), \ldots, p_n(x^{\frac{1}{q}})\}$ be a reduced irregular type of the ramification index q. Then for a sufficiently small $\epsilon > 0$,

$$L_P := \bigcup_{i=1}^{n} \{(x, p_i(x^{\frac{1}{q}}) \mid |x| = \epsilon\}$$

defines a link in a solid $S_\epsilon \times D_R$ with a suitable $R > 0$, called the *link of the irregular type* P.

The following is our main theorem which gives a topological characterization of the spaces of local deformation parameters.

Theorem 3.6. *Let us take two reduced irregular types P and P' of the same cyclic type. Then P' is an admissible deformation of P if and only if the corresponding links L_P and $L_{P'}$ are isotopic in a solid torus.*

Remark 3.7. It is known that two algebraic plane curve germs are *equisingular* (see [9] for the definition) if and only if the corresponding links are isotopic. The above theorem can be seen as an analogy of this fact for the differential equations.

3.3. **Invariants of connections and that of links.** Theorem 3.6 tells us that there will be similarities between singular points of differential equations and links. Thus we shall see some relationships between invariants of differential equations and that of links.

Suppose $(V, \nabla) = E_{p,q}$, an irreducible connection with $p \in x^{-\frac{1}{q}}\mathbb{C}[x^{-\frac{1}{q}}]$ and the irregular type $P = \{p_1, \ldots, p_q\}$ where $p_i := \int_\infty^x \frac{p(\zeta_q^i x^{\frac{1}{q}})}{x}\, dx$ for $i = 1, \ldots, q$. Then we can see that the corresponding link L_P becomes a knot.

The following is an analogue of Brauner's theorem.

Theorem 3.8 (dual Puiseux characteristics and iterated torus knot)**.** *Let $(q; \beta_1, \beta_2, \ldots, \beta_g)$ be the dual Puiseux characteristic of $E_{p,q}$. Then L_P is the iterated torus knot of order g and type*

$$(q/e_1, \beta_1/e_1), (e_1/e_2, \beta_2/e_2), \ldots, (e_{g-1}/e_g, \beta_g/e_g).$$

Let us take another irreducible connection $E_{p',q'}$ with the irregular type P'. Then linking number $\mathrm{lk}(L_P, L_{P'})$ of the knots L_P and $L_{P'}$ relates the Komatsu-Malgrange irregularity as follows.

Theorem 3.9 (linking number and irregularity)**.** *We have*

$$\mathrm{lk}(L_P, L_{P'}) = \mathrm{Irr}(\mathrm{Hom}_{\mathbb{C}((x))}(E_{p,q}, E_{p',q'})).$$

Let us denote the Alexander polynomial of L_P by Δ_P.

Theorem 3.10 (degree of Alexander polynomial and irregularity)**.** *We have*

$$\deg \Delta_P = \mathrm{Irr}(\mathrm{End}_{\mathbb{C}((x))}(E_{p,q})) - (q - 1).$$

Moreover the following says that Alexander polynomials of corresponding knots are completely determined by dual Puiseux characteristics.

Theorem 3.11 (Alexander polynomial and dual Puiseux characteristic)**.** *Suppose $q = q'$. Then the dual Puiseux characteristics of $E_{p,q}$ and $E_{p',q'}$ are same if and only if the Alexander polynomials of L_P and $L_{P'}$ are same.*

Since general irregular type is union of the above kind of irregular types as cyclic components, Theorem 3.11 indicates a direction of the implication of our main theorem.

3.4. **Braids and Stokes sequence of total orders.** Let (V, ∇) be a meromorphic connection with the irregular type $P = \{p_1(x^{\frac{1}{q}}), \dots, p_n(x^{\frac{1}{q}})\}$. Then

$$b_P := \bigcup_{i=1}^{n} \{(t, p_i(\eta e^{\sqrt{-1}\frac{t}{q}})) \mid 0 \leq t \leq 2\pi\}$$

defines a geometric braid whose closure is L_P. Then the projection

$$[0, 2\pi] \times \mathbb{C} \in (t, y) \mapsto (t, \mathrm{Re}\,(y)) \in [0, 2\pi] \times \mathbb{R}$$

gives the *braid diagram* of b_P. Let $0 \leq d_1 < d_2 < \cdots < d_h < 2\pi$ be the basic Stokes directions of (V, ∇) and define $\tilde{d}_i$ for $i = 0, \dots, h$ as before. Then restricting the braid diagram on the segments $[\tilde{d}_{i-1}, \tilde{d}_i]$ for $i = 1, \dots, h$, we obtain $b_i \in B_n$, elements in the braid group B_n such that

$$b_h b_{h-1} \cdots b_1 = b_P \in B_n.$$

Finally taking the natural projection $\tau_n \colon B_n \to S_n$ from braid group to symmetric group, we obtain the sequence of permutations

$$r_1 := \tau_n(b_1), r_2 := \tau_n(b_2), \dots, r_h := \tau_n(b_h)$$

which is nothing but the Stokes sequence of permutations of (V, ∇). Thus we call the $b_1, b_2, \dots, b_h$ the *lift* of the Stokes sequence of permutations of (V, ∇).

If we move the parameters of irregular types, Stokes directions and Stokes matrices are changed in general. However to consider "isomonodromic" deformations, some structure of Stokes phenomena should be preserved even when we move these parameters. The following says that a structure of Stokes phenomena are preserved under the move of parameters inside the space of deformation parameters.

Theorem 3.12 (Stokes sequences and conjugacy problem of braid groups)**.** *Take two Stokes sequences of permutations $\{r_1, \dots, r_h\}$ and $\{r'_1, \dots, r'_h\}$ of reduced irregular types P and P' of the same cyclic type respectively. Then P' is admissible deformation of P if and only if the lifts $\{b_1, \dots, b_h\}$ and $\{b'_1, \dots, b'_{h'}\}$ of these Stokes sequences are conjugate, namely, $b := b_h b_{h-1} \cdots b_1$ and $b' := b'_{h'} b'_{h'-1} \cdots b'_1$ are conjugate in the braid group B_n where $n = \#P$.*

This theorem relates Stokes sequences of irregular types in a space of local deformation parameters to the work "like" problem or conjugacy problem of the braid group.

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