Analysis on non-Riemannian locally symmetric spaces
— an application of invariant theory

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Table of Contents

Locally homogeneous space $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

<u>Setting</u> $\underset{\text{discrete}}{\Gamma} \subset \underset{\text{Lie group}}{G_{\mathbb{R}}} \supset \underset{\text{closed subgroup}}{H_{\mathbb{R}}}$

$$
\begin{tikzcd}
 & G_{\mathbb{R}} \arrow[dl, "\text{covering}"'] \arrow[dr] & \\
\Gamma\backslash G_{\mathbb{R}} \arrow[dr] & & G_{\mathbb{R}}/H_{\mathbb{R}} = X_{\mathbb{R}} \arrow[dl, "p_\Gamma"'] \\
 & \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} = \Gamma\backslash X_{\mathbb{R}} &
\end{tikzcd}
$$

- $\Gamma\backslash G_{\mathbb{R}}$ and $X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$ are C^∞ manifolds.
- (locally homogeneous space) If Γ acts properly discontinuously and freely on $X_{\mathbb{R}}$, then the quotient $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$ becomes a C^∞ manifold with p_Γ being a covering.

<u>Definition</u> Say $\Gamma\backslash X_{\mathbb{R}}$ is a Clifford–Klein form of $X_{\mathbb{R}}$.

"Intrinsic" differential operators on $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

$\mathbb{D}(X_{\mathbb{R}})$: ring of differential operators on $X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$.

$\mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$: ring of $G_{\mathbb{R}}$-invariant differential operators on $X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$.

$\mathbb{D}(\Gamma\backslash X_{\mathbb{R}})$: ring of differential operators on $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$.

$$
\begin{tikzcd}
 & G_{\mathbb{R}} \arrow[dl, "\text{covering}"'] \arrow[dr] & \\
\Gamma\backslash G_{\mathbb{R}} \arrow[dr] & & G_{\mathbb{R}}/H_{\mathbb{R}} = X_{\mathbb{R}} \arrow[dl, "p_\Gamma"', "\text{covering}"] \\
 & \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} = \Gamma\backslash X_{\mathbb{R}} &
\end{tikzcd}
$$

Any element $D \in \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$ induces a differential operator D_Γ on $\Gamma\backslash X_{\mathbb{R}}$ if a subgroup Γ acts properly discontinuously and freely on $X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$.

ring homomorphism $(\mathbb{D}(X_{\mathbb{R}}) \supset)\ \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}}) \hookrightarrow \mathbb{D}(\Gamma\backslash X_{\mathbb{R}}),\ D \mapsto D_\Gamma$

$$D \circ p_\Gamma^* = p_\Gamma^* \circ D_\Gamma$$

<u>Example</u> Laplacian $\square_{X_{\mathbb{R}}} \in \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$, $\square_{\Gamma\backslash X_{\mathbb{R}}} \in \mathbb{D}(\Gamma\backslash X_{\mathbb{R}})$

Spectral analysis on $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

$$\begin{array}{rcccccccc} & X_{\mathbb{R}} & = & G_{\mathbb{R}}/H_{\mathbb{R}} & \qquad & \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}}) & \ni & D \\ \text{covering} & \downarrow & & & & \downarrow & & \downarrow \\ & \Gamma\backslash X_{\mathbb{R}} & = & \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} & & \mathbb{D}(\Gamma\backslash X_{\mathbb{R}}) & \ni & D_\Gamma \end{array}$$

General Problem Wish to understand joint eigenvalues and (to construct) joint eigenfunctions $f \in C^\infty(\Gamma\backslash X_{\mathbb{R}})$ (or $L^2(\Gamma\backslash X_{\mathbb{R}})$ etc):

$$D_\Gamma f = \lambda(D) f \quad \text{for all } D \in \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}}),$$

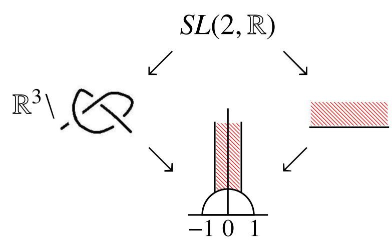
where $\lambda : \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}}) \to \mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism.

$\lambda(D)$: joint eigenvalues for $D \in \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$
The above problem makes reasonable sense if $\mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$ is commutative ($\Longleftrightarrow X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical)

Spectral analysis on $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$ beyond Riemannian setting

Γ (discrete subgp) $\subset$ $G_{\mathbb{R}}$ (Lie group) $\supset$ $H_{\mathbb{R}}$ (subgroup) $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R}) \supset SO(2)$

$$\begin{array}{ccc} & G_{\mathbb{R}} & \\ \swarrow & & \searrow \\ \Gamma\backslash G_{\mathbb{R}} & & G_{\mathbb{R}}/H_{\mathbb{R}} \\ \searrow & & \swarrow \\ & \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} & \end{array}$$



$\boxed{\text{Special cases are already deep and rich.}}$

- $\Gamma = \{e\} \cdots$ non-commutative harmonic analysis on $L^2(G_{\mathbb{R}}/H_{\mathbb{R}})$
  Gelfand, Harish-Chandra, T. Oshima, Delorme, van den Ban, Schlichtkrull, . . .
- $H_{\mathbb{R}}$ compact $\cdots$ automorphic forms (local theory)
  Siegel, Selberg, Piateski-Shapiro, Langlands, Arthur, Sarnak, Müller, . . .
- $G_{\mathbb{R}} = \mathbb{R}^{p,q}$ (abelian, but non-Riemannian), $\Gamma = \mathbb{Z}^{p+q}$, $H_{\mathbb{R}} = \{e\}$
  Oppenheim conjecture, Dani–Margulis, Ratner, Eskin, Mozes, . . .

Spectral analysis on $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G_{\mathbb{R}}} \supset \underset{\text{subgroup}}{H_{\mathbb{R}}}$$

$$\begin{tikzcd} & G_{\mathbb{R}} \arrow[dl] \arrow[dr] & \\ \Gamma\backslash G_{\mathbb{R}} \arrow[dr] & & G_{\mathbb{R}}/H_{\mathbb{R}} \arrow[dl] \\ & \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} & \end{tikzcd}$$

New challenge: Initiate spectral analysis of $\mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$ on $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$ in a more general setting (non-abelian $G_{\mathbb{R}}$ and non-compact $H_{\mathbb{R}}$).

Further difficulties arise

- (geometry) existence of good geometry $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$? study of "local to global" beyond Riemannian setting
- (analysis) Laplacian is no more elliptic. No "Weyl's law."
- (representation theory) $\mathrm{vol}(\Gamma\backslash G_{\mathbb{R}}) = \infty$ even when $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$ is compact

Standard Clifford–Klein forms $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

$$\Gamma \subset G'_{\mathbb{R}} \subset G_{\mathbb{R}} \supset H_{\mathbb{R}}$$

Assume

$$G'_{\mathbb{R}} \text{ acts properly on } X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}} \qquad (*)$$

Then any torsion-less discrete subgp Γ of $G'_{\mathbb{R}}$ acts properly discontinuously and freely on $X_{\mathbb{R}}$.

Def. Let $G'_{\mathbb{R}} \subset G_{\mathbb{R}} \supset H_{\mathbb{R}}$ be three real reductive Lie groups satisfying $(*)$. Take $\Gamma \subset G'_{\mathbb{R}}$.
We say the locally homogeneous space $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$ is a standard Clifford–Klein form of $X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$

Remark

(1) Criterion of $(*)$ for reductive $G_{\mathbb{R}}$ is known (K– '89, '96; Benoist '96).

(2) If $G'_{\mathbb{R}}$ acts cocompactly on $X_{\mathbb{R}}$ (in particular, if $G'_{\mathbb{R}} \curvearrowright X_{\mathbb{R}}$ transitively),
$$\mathrm{vol}(\Gamma\backslash G'_{\mathbb{R}}) < \infty \iff \mathrm{vol}(\Gamma\backslash X_{\mathbb{R}}) < \infty.$$
$$\Gamma\backslash G'_{\mathbb{R}} \text{ compact} \iff \Gamma\backslash X_{\mathbb{R}} \text{ is compact}$$

(3) Existing examples of compact Clifford–Klein forms are obtained as either standard ones or their deformations.

Compact standard Clifford–Klein form $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

$G'_{\mathbb{R}} \subset G_{\mathbb{R}} \supset H_{\mathbb{R}}$ real linear reductive groups

Observation If $G'_{\mathbb{R}}$ acts transitively and properly on $X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$,
then any lattice Γ in $G'_{\mathbb{R}}$ yields $\Gamma\backslash X_{\mathbb{R}}$ of finite volume.

Example 1 (anti de Sitter manifold, $\mathrm{rank}\, X_{\mathbb{R}} = 1$)
$X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}} = O(2m, 2)/O(2m, 1)$
Take $G'_{\mathbb{R}} = U(m, 1)$

Switching the role of $H_{\mathbb{R}}$ and $G'_{\mathbb{R}}$

Example 2 (indefinite Kähler manifold)
$X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}} = O(2m, 2)/U(m, 1)$
Take $G'_{\mathbb{R}} = O(2m, 1)$

Example 3 (15-diml'l pseudo-Riemannian mfd of signature $(8, 7)$)
$X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}} = O(8, 8)/O(8, 7)$
Take $G'_{\mathbb{R}} = Spin(1, 8)$ (acting on $\mathbb{R}^{16}$ via spin representation)

Analysis on standard Clifford–Klein form $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

Setting $\Gamma \subset G'_{\mathbb{R}} \subset G_{\mathbb{R}} \supset H_{\mathbb{R}}$
$\rightsquigarrow$ standard Clifford–Klein forms $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

Problem (today) Understand joint eigenvalues and
eigenfunctions $f \in C^{\infty}(\Gamma\backslash X_{\mathbb{R}})$ for pseudo-Riemannian
standard Clifford–Klein form $\Gamma\backslash X_{\mathbb{R}}$
$D_{\Gamma} f = \lambda(D) f$ for all $D \in \mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$
by using the branching law $G_{\mathbb{R}} \downarrow G'_{\mathbb{R}}$.

Idea: "Invariant theory" for $(G'_{\mathbb{C}}, G_{\mathbb{C}}, H_{\mathbb{C}})$ (cf. Howe–Umeda) (§2)
$\Rightarrow$ Branching of representations $G_{\mathbb{R}} \downarrow G'_{\mathbb{R}}$ (§3)
$\Rightarrow$ Relate $\Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$ with Riemannian case (§4)

Another approach $\cdots$ not for today ([Kassel–K '15], Poincaré series. . .)
Construction of some eigenfunctions f built on
Flensted–Jensen's discrete series for $G_{\mathbb{R}}/H_{\mathbb{R}}$
$\Longleftarrow$ quantitative estimate for properly discontinuous actions

Table of contents

Spherical varieties — revisited

$G_{\mathbb{C}}$: complex reductive Lie group
$H_{\mathbb{C}}$: algebraic reductive subgp (for simplicity)

Definition–Theorem We say $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if the following 5 equivalent conditions (i)–(v) on $(G_{\mathbb{C}}, H_{\mathbb{C}})$ hold:

- Geometry (flag variety)
  (i) $X_{\mathbb{C}}$ admits an open orbit of a Borel subgp of $G_{\mathbb{C}}$.
- Ring $\mathbb{D}_G(X)$ of invariant differential operators
  (ii) $\mathbb{D}_G(X)$ is commutative.
  (iii) $\mathbb{D}_G(X)$ is a polynomial ring.
- Representation theory ((BM) = bounded multiplicity property)
  (iv) (BM) holds for some real form $(G_{\mathbb{R}}, H_{\mathbb{R}})$.
  (v) (BM) holds for all real forms $(G_{\mathbb{R}}, H_{\mathbb{R}})$.

(BM) $^{\exists}C > 0$ s.t. for all irreducible admissible rep π of $G_{\mathbb{R}}$,

$$\dim \mathrm{Hom}_{G_{\mathbb{R}}}(\pi, C^{\infty}(G_{\mathbb{R}}/H_{\mathbb{R}})) \leq C.$$

Overgroup

$$\begin{array}{ccccc} G_{\mathbb{R}} & & & \supset & H_{\mathbb{R}} \\ \cup & & & & \cup \\ G'_{\mathbb{R}} & \supset & K'_{\mathbb{R}} & \supset & H'_{\mathbb{R}} := H_{\mathbb{R}} \cap G'_{\mathbb{R}} \end{array}$$

If $G'_{\mathbb{R}}$ acts on $X_{\mathbb{R}}$ transitively, then

$$X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}} \xleftarrow{\sim} G'_{\mathbb{R}}/H'_{\mathbb{R}}.$$

<u>Definition</u> Say $G_{\mathbb{R}}$ is an <u>overgroup</u> of $G'_{\mathbb{R}}/H'_{\mathbb{R}}$.

Take a maximal subgroup $K'_{\mathbb{R}}$. We have a fibration

$$\begin{array}{ccccc} F_{\mathbb{R}} & \to & X_{\mathbb{R}} & \to & Y_{\mathbb{R}} \\ \vdots\!\vdots & & \parallel & & \vdots\!\vdots \\ K'_{\mathbb{R}}/H'_{\mathbb{R}} & & G'_{\mathbb{R}}/H'_{\mathbb{R}} & & G'_{\mathbb{R}}/K'_{\mathbb{R}} \\ & & \parallel & & \\ & & G_{\mathbb{R}}/H_{\mathbb{R}} & & \end{array}$$

Invariant differential operators for overgroups

Suppose $X = G'/H'$ has an overgroup G, namely,

$$\begin{array}{ccc} G & \supset & H \\ \cup & & \cup \\ G' & \supset & H' = G' \cap H \end{array} \quad \text{s.t.} \quad G/H \simeq G'/H' = X.$$

<u>Problem</u> Understand two rings $\mathbb{D}_G(X)$ and $\mathbb{D}_{G'}(X)$.

In this section, we assume G is compact

$X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$		$U(2p+1,2q)/Sp(p,q)$
$\downarrow$	complexification	$\downarrow$
$X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$		$GL(2n+1,\mathbb{C})/Sp(n,\mathbb{C}) \quad (n = p+q)$
$\uparrow$	complexification	$\uparrow$
$X = G/H$ (compact)		$U(2n+1)/Sp(n)$

$\mathbb{D}_{G_{\mathbb{R}}}(X_{\mathbb{R}})$: $G_{\mathbb{R}}$-inv. differential ops on $X_{\mathbb{R}}$
$\wr$
$\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$: $G_{\mathbb{C}}$-inv. holomorphic differential ops on $X_{\mathbb{C}}$
$\wr$
$\mathbb{D}_G(X)$: G-inv. differential ops

Invariant differential operators on G/H (review)

$G \supset H$: compact connected Lie groups, $X = G/H$

$\mathbb{D}_G(X)$: ring of G-invariant differential operators on $X = G/H$

$$\boxed{\mathbb{D}_G(X) \text{ is commutative} \iff X_{\mathbb{C}} \text{ is } G_{\mathbb{C}}\text{-spherical}}$$

$$dl \otimes dr : U(\mathfrak{g}_{\mathbb{C}}) \otimes U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{D}(G)$$

$$\begin{array}{rccccl} & Z(\mathfrak{g}_{\mathbb{C}}) \otimes \mathbb{C} & \to & \mathbb{D}_G(X) & (1) \\ & \cap & & \cap & \\ dl \otimes dr : & U(\mathfrak{g}_{\mathbb{C}}) \otimes U(\mathfrak{g}_{\mathbb{C}})^H & \to & \mathbb{D}(X) & \\ & \cup & & \cup & \\ & \mathbb{C} \otimes U(\mathfrak{g}_{\mathbb{C}})^H & \to & \mathbb{D}_G(X) & (2) \end{array}$$

- $Z(\mathfrak{g}_{\mathbb{C}})$ is understood by Harish-Chandra isomorphism, but (1) is not always surjective ("abstract Capelli problem" à la Howe–Umeda may have a negative answer) when $X = G/H$ is non-symmetric.
- (2) is surjective, but the ring $U(\mathfrak{g}_{\mathbb{C}})^H$ is hard to treat in general.

Invariant differential operator for overgroups

$F = K'/H' \hookrightarrow X = G'/H' = G/H$ (compact) $\to Y = G'/K'$ (fibration)

Three ~~Two~~ subalgebras of $\mathbb{D}_{G'}(X)$

$$\boxed{\mathcal{P} := \mathbb{D}_G(X), \quad \mathcal{Q} := \iota(\mathbb{D}_{K'}(F)), \quad \mathcal{R} := dl(Z(\mathfrak{g}'_{\mathbb{C}}))}$$

Theorem A Assume $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

(1) The commutative algebra $\mathbb{D}_{G'}(X)$ is generated by $\mathcal{P}$ and $\mathcal{R}$.
Let K' be a maximal subgroup of G' containing H'.

(2) $\mathbb{D}_{G'}(X)$ is generated by $\mathcal{P}$ and $\mathcal{Q}$, too.

(3) $\mathbb{D}_{G'}(X)$ is generated by $\mathcal{Q}$ and $\mathcal{R}$, too, if G is simple.

Invariant differential operator for overgroups

$F = K'/H' \hookrightarrow X = G'/H' = G/H$ (compact) $\to Y = G'/K'$ (fibration)

Three ~~Two~~ subalgebras of $\mathbb{D}_{G'}(X)$

$$\boxed{\mathcal{P} := \mathbb{D}_G(X), \quad Q := \iota(\mathbb{D}_{K'}(F)), \quad \mathcal{R} := dl(Z(\mathfrak{g}'_{\mathbb{C}}))}$$

$$\begin{array}{ccccc} \mathbb{C}[R_1,\dots,R_l] \simeq Z(\mathfrak{g}'_{\mathbb{C}}) & \longrightarrow & \mathbb{D}_{G'}(X) & \hookleftarrow & \mathbb{D}_G(X) \simeq \mathbb{C}[P_1,\dots,P_m] \\ & & \uparrow \iota & & \\ & & \mathbb{D}_{K'}(F) \simeq \mathbb{C}[Q_1,\dots,Q_n] & & \end{array}$$

For all the triples (G, G', H) (classified),

Theorem B We find explicit relations among generations P_j, $\iota(Q_j)$, and $dl(R_j)$ of $\mathcal{P}$, Q, and $\mathcal{R}$ in $\mathbb{D}_{G'}(X)$.

$G'_{\mathbb{C}}/H'_{\mathbb{C}}$ spherical with simple overgroup $G_{\mathbb{C}}$

	$G_{\mathbb{C}}$	$H_{\mathbb{C}}$	$G'_{\mathbb{C}}$	$H'_{\mathbb{C}}$	$K'_{\mathbb{C}}$
i)	$SO(2n+2)$	$SO(2n+1)$	$GL(n+1)$	$GL(n)$	$GL(n)\times GL(1)$
ii)	$SO(2n+2)$	$GL(n+1)$	$SO(2n+1)$	$GL(n)$	$SO(2n)$
iii)	$SL(2n+2)$	$GL(2n+1)$	$Sp(n+1)$	$Sp(n)\times GL(1)$	$Sp(n)\times Sp(1)$
iv)	$SL(2n+2)$	$Sp(n+1)$	$GL(2n+1)$	$Sp(n)\times GL(1)$	$GL(2n)\times GL(1)$
v)	$SO(4n+4)$	$SO(4n+3)$	$Sp(n+1)\times Sp(1)$	$Sp(n)\times\Delta(Sp(1))$	$Sp(n)\times Sp(1)\times Sp(1)$
vi)	$SO(16)$	$SO(15)$	$Spin(9)$	$Spin(7)$	$Spin(8)$
vii)	$SO(8)$	$Spin(7)$	$SO(5)\times SO(3)$	$SL(2)\times\Delta(SO(3))$	$SO(4)\times SO(3)$
viii)	$SO(7)$	G_2	$SO(5)\times SO(2)$	$SL(2)\times\Delta(SO(2))$	$SO(4)\times SO(2)$
ix)	$SO(7)$	G_2	$SO(6)$	$SL(3)$	$GL(3)$
x)	$SO(7)$	$SO(6)$	G_2	$SL(3)$	—
xi)	$SO(8)$	$Spin(7)$	$SO(7)$	G_2	—
xii)	$SO(8)$	$SO(7)$	$Spin(7)$	G_2	—
xiii)	$SO(8)$	$Spin(7)$	$SO(6)\times SO(2)$	$SL(3)\times\Delta(SO(2))$	$GL(3)\times SO(2)$
xiv)	$SO(8)$	$SO(6)\times SO(2)$	$Spin(7)$	$SL(3)\times\Delta(SO(2))$	$Spin(6)$

$$F_{\mathbb{C}} = K'_{\mathbb{C}}/H'_{\mathbb{C}} \hookrightarrow X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}} \simeq G'_{\mathbb{C}}/H'_{\mathbb{C}} \to Y_{\mathbb{C}} = G'_{\mathbb{C}}/K'_{\mathbb{C}}$$

Take maximal reductive subgp $K'_{\mathbb{C}}$ in $G'_{\mathbb{C}}$ containing $H'_{\mathbb{C}}$.

Table of contents

Three applications of Theorems A and B

$F = K'/H' \hookrightarrow X = G/H = G'/H' \twoheadrightarrow Y = G'/K'$ (fibration)

Three subalgebras of $\mathbb{D}_{G'}(X)$:

$\mathcal{P} = \mathbb{D}_G(X)$, $\mathcal{Q} = \iota(\mathbb{D}_{K'}(F))$, $\mathcal{R} = dl(Z(\mathfrak{g}'_{\mathbb{C}}))$.

$\boxed{\text{Theorem A} \quad \mathbb{D}_{G'}(X) = \langle \mathcal{P}, \mathcal{Q} \rangle = \langle \mathcal{Q}, \mathcal{R} \rangle = \langle \mathcal{R}, \mathcal{P} \rangle \text{ if } G_{\mathbb{C}} \text{ is simple and } X_{\mathbb{C}} \text{ is } G'_{\mathbb{C}}\text{-spherical.}}$

Three applications

(1) Explicit description of the polynomial ring $\mathbb{D}_{G'}(X)$

(2) ($\mathcal{R} \subset \langle \mathcal{P}, \mathcal{Q} \rangle$) Analysis on $G_{\mathbb{R}}/H_{\mathbb{R}}$ and branching law $G_{\mathbb{R}} \downarrow G'_{\mathbb{R}}$

(3) ($\mathcal{P} \subset \langle \mathcal{Q}, \mathcal{R} \rangle$) Analysis on $\Gamma \backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

Take any real forms $G'_{\mathbb{R}} \subset G_{\mathbb{R}} \supset H_{\mathbb{R}}$ of $G'_{\mathbb{C}} \subset G_{\mathbb{C}} \supset H_{\mathbb{C}}$.

Corollary C ($G_{\mathbb{R}} \downarrow G'_{\mathbb{R}}$) Assume $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical and $F_{\mathbb{R}}$ is compact. Let π be any irreducible unitary representation of $G_{\mathbb{R}}$ occurring in $\mathcal{D}'(G_{\mathbb{R}}/H_{\mathbb{R}})$ as subquotients.
Then the restriction $\pi|_{G'_{\mathbb{R}}}$ is discretely decomposable.

Table of contents

$Z(\mathfrak{g}_{\mathbb{C}})$-actions on $\Gamma\backslash X_{\mathbb{R}}$ v.s. $Z(\mathfrak{g}'_{\mathbb{C}})$-actions on Y_Γ

Let $\Gamma\backslash X_{\mathbb{R}}$ be a standard Clifford–Klein form of $X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}}$ given by a transitive and proper $G'_{\mathbb{R}}$-action on $X_{\mathbb{R}}$.

$$\begin{array}{ccccccccl} \Gamma & \subset & G_{\mathbb{R}} & & & \supset & H_{\mathbb{R}} & & \\ & \searrow & \cup & & & & \cup & & \\ & & G'_{\mathbb{R}} & \supset & K'_{\mathbb{R}} & \supset & H'_{\mathbb{R}} & = & G'_{\mathbb{R}} \cap H_{\mathbb{R}} \text{ (compact)} \end{array}$$

$$\begin{array}{lllll} X_{\mathbb{R}} = G_{\mathbb{R}}/H_{\mathbb{R}} \simeq & G'_{\mathbb{R}}/H'_{\mathbb{R}} & \to & \Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} & \text{non-Riemannian} \\ \downarrow & \downarrow & & \downarrow & \\ Y_{\mathbb{R}} \quad = & G'_{\mathbb{R}}/K'_{\mathbb{R}} & \to & \Gamma\backslash Y_{\mathbb{R}} = \Gamma\backslash G'_{\mathbb{R}}/K'_{\mathbb{R}} & \underline{\text{Riemannian}} \end{array}$$

$$\begin{array}{ccc} Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{D}_G(X) & \curvearrowright & C^\infty(\Gamma\backslash X_{\mathbb{R}}) \\ & & \cup \\ Z(\mathfrak{g}'_{\mathbb{C}}) & \curvearrowright & C^\infty(\Gamma\backslash Y_{\mathbb{R}}) \end{array}$$

- In general, $Z(\mathfrak{g}_{\mathbb{C}})$-actions on $C^\infty(\Gamma\backslash X_{\mathbb{R}})$ are NOT controlled by $Z(\mathfrak{g}'_{\mathbb{C}})$-actions on $C^\infty(\Gamma\backslash Y_{\mathbb{R}})$
- However, Theorems A and B assure an explicit relationship if $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical

Summary of notation

$$\begin{array}{ccccc} & & & & \mathcal{W}_\tau = \Gamma\backslash G'_{\mathbb{R}} \times_{K'_{\mathbb{R}}} \tau \\ & & & & \downarrow \\ F_{\mathbb{R}} = K'_{\mathbb{R}}/H'_{\mathbb{R}} & \hookrightarrow & \Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} & \twoheadrightarrow & \Gamma\backslash Y_{\mathbb{R}} = \Gamma\backslash G'_{\mathbb{R}}/K'_{\mathbb{R}} \\ & & C^\infty(\Gamma\backslash X_{\mathbb{R}}) & \overset{\iota_\tau}{\hookleftarrow} & C^\infty(\Gamma\backslash Y_{\mathbb{R}}, \mathcal{W}_\tau) \end{array}$$

For $\lambda : \mathbb{D}_G(X) \to \mathbb{C}$
$C^\infty(\Gamma\backslash X_{\mathbb{R}}; \mathcal{M}_\lambda) := \{f \in C^\infty(\Gamma\backslash X_{\mathbb{R}}) : D_\Gamma f = \lambda(D) f \ \ ({}^\forall D \in \mathbb{D}_G(X))\}$

For $\nu : Z(\mathfrak{g}'_{\mathbb{C}}) \to \mathbb{C}$
$C^\infty(\Gamma\backslash Y_{\mathbb{R}}; \mathcal{W}_\tau; \mathcal{N}_\nu) := \{\varphi \in C^\infty(\Gamma\backslash Y_{\mathbb{R}}, \mathcal{W}_\tau) : z\varphi = \nu(z)\varphi \ \ ({}^\forall z \in Z(\mathfrak{g}'_{\mathbb{C}}))\}$

Construction of joint eigenfunctions on $\Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}}$

$$\begin{array}{ccccc} & & & & \mathcal{W}_\tau = \Gamma\backslash G'_{\mathbb{R}} \times_{K'_{\mathbb{R}}} \tau \\ & & & & \downarrow \\ F_{\mathbb{R}} = K'_{\mathbb{R}}/H'_{\mathbb{R}} & \hookrightarrow & \Gamma\backslash X_{\mathbb{R}} = \Gamma\backslash G_{\mathbb{R}}/H_{\mathbb{R}} & \twoheadrightarrow & \Gamma\backslash Y_{\mathbb{R}} = \Gamma\backslash G'_{\mathbb{R}}/K'_{\mathbb{R}} \\ & & C^\infty(\Gamma\backslash X_{\mathbb{R}}) & \overset{\iota_\tau}{\hookleftarrow} & C^\infty(\Gamma\backslash Y_{\mathbb{R}}, \mathcal{W}_\tau) \end{array}$$

<u>Thm D′</u> (L^2 case) Assume $\Gamma\backslash X_{\mathbb{R}}$ is standard and $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.
(1) For every $\tau \in \widehat{F}\ (\subset \widehat{K'_{\mathbb{R}}})$, there is an (explicitly described) map
$\nu_\tau : \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}'_{\mathbb{C}}), \mathbb{C}),\ \lambda \mapsto \nu_\tau(\lambda)$
s.t. for any λ, the following cond. on $\varphi \in C^\infty(Y_\Gamma, \mathcal{W}_\tau)$ are equivalent:
(i) $i_\tau(\varphi) \in L^2(\Gamma\backslash X_{\mathbb{R}}; \mathcal{M}_\lambda)$,
(ii) $\varphi \in L^2(\Gamma\backslash Y_{\mathbb{R}}, \mathcal{W}_\tau; \mathcal{N}_{\nu_\tau(\lambda)})$.
(2) For every $\lambda \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C})$, $L^2(\Gamma\backslash X_{\mathbb{R}}; \mathcal{M}_\lambda)$ contains

$$\bigoplus_{\tau \in \widehat{F}} i_\tau\big((L^2 \cap C^\infty)(Y_\Gamma, \mathcal{W}_\tau; \mathcal{N}_{\nu_\tau(\lambda)}) \big)$$

as a dense subspace.

<u>Remark</u> We allow the case $\mathrm{vol}(\Gamma\backslash X_{\mathbb{R}}) = \infty$.

Example for $\nu_\tau : \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}'_{G_\mathbb{C}}), \mathbb{C})$

$$\begin{array}{ccccc} F_\mathbb{R} = K'_\mathbb{R}/H'_\mathbb{R} & \hookrightarrow & X_\mathbb{R} = G_\mathbb{R}/H_\mathbb{R} & \twoheadrightarrow & G'_\mathbb{R}/K'_\mathbb{R} \\ \| & & \| & & \| \\ U(2n)/Sp(n) & & SU(2,2n)/Sp(1,n) & & SU(1,2n)/U(2n) \end{array}$$

$$\begin{aligned} \widehat{F} &= \{\tau \in \widehat{U(2n)} : \tau \text{ has } Sp(n)\text{-fixed vector}\} \\ &\simeq \{k = (k_1, k_1, k_2, k_2, \dots, k_n, k_n) \in \mathbb{Z}^{2n} : k_1 \ge k_2 \ge \cdots \ge k_n\}. \end{aligned}$$

Fix such $\tau \equiv \tau(k) \in \widehat{F}$.

$$\begin{array}{c} \nu_\tau : \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}'_{G_\mathbb{C}}), \mathbb{C}), \ \lambda \mapsto \nu_\tau(\lambda) \\ \qquad \| \qquad\qquad\qquad\qquad\qquad \| \\ \mathbb{C}[P_1, \dots, P_{n+1}] \qquad\qquad \mathbb{C}[R_1, \dots, R_{2n+1}] \end{array}$$

is given by

$$\boxed{\nu_\tau(\lambda)(R_l) = \sum_{j=1}^{n+1} \left(\frac{1}{2}\lambda(P_j)\right)^l + \sum_{i=1}^{n} (k_i + n - 2i + 1)^l. \quad ({}^\forall l).}$$

Self-adjoint extension of $\square_{\Gamma\backslash X_\mathbb{R}}$ on $\Gamma\backslash X_\mathbb{R} = \Gamma\backslash G_\mathbb{R}/H_\mathbb{R}$

<u>Corollary E</u> Let $X_\mathbb{R} = G_\mathbb{R}/H_\mathbb{R}$ be a reductive homogeneous space.
Assume that a reductive subgroup $G'_\mathbb{R}$ acts properly on $X_\mathbb{R}$, and $G'_\mathbb{C}$ acts spherically on $X_\mathbb{C}$.
Then for any torsion-free discrete subgroup Γ of $G'_\mathbb{R}$,

(1) the Laplacian $\square_{\Gamma\backslash X_\mathbb{R}}$ extends to a self-adjoint operator on $L^2(\Gamma\backslash X_\mathbb{R})$;

(2) there exists countably many L^2-eigenvalues of $\square_{\Gamma\backslash X_\mathbb{R}}$ on $\Gamma\backslash X_\mathbb{R}$ if $\Gamma\backslash X_\mathbb{R}$ is compact or if Γ is an arithmetic in $G'_\mathbb{R}$.

Eigenvalues of Laplacian on pseudo-Riemannian space forms

Example 1 M : standard anti de Sitter manifold. $\Gamma\backslash O(m,2)/O(m,1)$
(Lorentz mfd with sectional curvature $\equiv -1$)

(1) $^{\exists}\lambda_j$: L^2-eigenvalues of the hyperbolic Laplacian $\square_M$ s.t. $\lim_{j\to\infty} \lambda_j = +\infty$.

(2) Suppose $m \in 2\mathbb{N}$. If M is compact or if Γ is arithmetic in $U(m/2,1)$, then there also exist L^2-eigenvalues λ'_j of $\square_M$ s.t. $\lim_{j\to\infty} \lambda'_j = -\infty$.

Example 2 $M = \Gamma\backslash X_{\mathbb{R}}$: 15 dim'l standard Clifford–Klein form of $X_{\mathbb{R}} = O(8,8)/O(8,7)$. (standard $\Longleftrightarrow \Gamma \subset Spin(1,8)$)
(pseudo-Riemannian mfd of signature $(8,7)$ with sectional curvature $\equiv -1$)
The same conclusions (1) and (2) hold.

Use $\mathbb{D}_{Spin(9)}(S^{15}) = \mathbb{C}[\Delta_{S^{15}}, \iota(\Delta_{S^7})]$ with $C_G = 4dl(C_{G'}) - 3\,\iota(C_K)$
for $S^7 \hookrightarrow S^{15} \to S^8$ and the branching $O(8,8) \downarrow Spin(1,8)$.

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