Symmetry breaking for representations of rank one orthogonal groups

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I am reporting on Joint work with T. Kobayashi

A representation π of a group G defines a representation of a subgroup G' by restriction. In general irreducibility is not preserved by the restriction. If G is compact then the restriction $\pi|_{G'}$ is isomorphic to a direct sum of irreducible representations π' of G' with multiplicities $m(\pi, \pi')$. These multiplicities are studied by using combinatorial techniques. If G' is not compact and the representation π is infinite-dimensional, then generically the restriction $\pi|_{G'}$ is not a direct sum of irreducible representations and we have to consider another notion of multiplicity.

For a continuous representation π of G on a complete, locally convex topological vector space H_π, the space H_π^∞ of C^∞-vectors of H_π is naturally endowed with a Fréchet topology, and (π, H_π) gives rise to a continuous representation π^∞ of G on H_π^∞. If $\mathcal{H}_\pi$ is a Banach space, then the Fréchet representation $(\pi^\infty, \mathcal{H}_\pi^\infty)$ depends only on the underlying $(\mathfrak{g}, K)$-module $(\mathcal{H}_\pi)_K$. Given another continuous representation π' of the subgroup G', we consider the space of continuous G'-intertwining operators (*symmetry breaking operators*)

$$\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, (\pi')^\infty).$$

The dimension $m(\pi, \pi')$ of this space yields important information of the restriction of π to G' and is called the *multiplicity* of π' occurring in the restriction $\pi|_{G'}$. Notice that the multiplicity $m(\pi, \pi')$ makes sense also for non-unitary representations π and π'. In general, $m(\pi, \pi')$ may be infinite. For detailed analysis on symmetry breaking operators, we concentrate on the case where $m(\pi, \pi')$ is finite. The criterion by T.Kobayashi and T.Oshima asserts that the multiplicity $m(\pi, \pi')$ is finite for all irreducible representations π of G and all irreducible representations π' of G' if and only if the minimal parabolic subgroup P' of G' has an open orbit on the real flag variety G/P, and that the multiplicity is uniformly bounded with respect to π and π' if and only if a Borel subgroup of $G'_{\mathbb{C}}$ has an open orbit on the complex flag variety of $G_{\mathbb{C}}$.

The classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$ for which the former case was recently accomplished by T. Kobayashi and T. Matsuki whereas the latter condition depends only on the complexified pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$, of which the classification was already known in 1970s by Krämer . In particular, the multiplicity $m(\pi, \pi')$ is uniformly bounded if the Lie algebras $(\mathfrak{g}, \mathfrak{g}')$ of (G, G') are real forms of $(\mathfrak{sl}(N+1, \mathbb{C}), \mathfrak{gl}(N, \mathbb{C}))$ or $(\mathfrak{o}(N+1, \mathbb{C}), \mathfrak{o}(N, \mathbb{C}))$. We confine ourselves to the case

$$(G, G') = (O(n+1, 1), O(n, 1)),$$

and study thoroughly symmetry breaking operators between principal series representations $I(V, \lambda)$ and $J(W, \nu)$ for the groups G and G'.

A classification of the"symmetry breaking operators" $T : I(V, \lambda)^\infty \to J(W, \nu)^\infty$ for any pair of principal series representations $I(V, \lambda)$ and $J(W, \nu)$, is carried out through an analysis of the distribution kernels K_T. We find explicit formulae of distribution kernels of the basis of

$$\mathrm{Hom}_{G'}(I(V, \lambda)^\infty, J(W, \nu)^\infty$$

The important properties of these symmetry breaking operators are the existence of the analytic continuation, and the functional equations that they satisfy with the Knapp–Stein intertwining operators of G and G'.

We state now our results more precisely for principal series representations. We realize $G = O(n+1, 1)$ as the automorphism group of a quadratic form

$$x_0^2 + x_1^2 + \cdots + x_n^2 - x_{n+1}^2$$

and the subgroup $G' = O(n,1)$ as the stabilizer of the basis vector $e_n = {}^t(0,\cdots,0,1,0)$.

A principal series representation $I(V,\lambda)$ of G is an (unnormalized) induced representation from an irreducible finite dimensional representation $V \otimes \chi_\lambda$ of a minimal parabolic subgroup P. Here χ_λ for $\lambda \in \mathbb{C}$ is a character of P and V is an irreducible representation of the orthogonal group O(n-1) . The representation is called spherical if V is the trivial representation. It is denoted by $I(\lambda)$. In what follows, we take the representation space of $I(V,\lambda)$ to be the space of C^∞-sections of the G-equivariant vector bundle $G \times_P (V \otimes \chi_\lambda, \mathbb{V}) \to G/P$, so that $I(V,\lambda)^\infty \simeq I(V,\lambda)$ is the Fréchet globalization having moderate growth in the sense of Casselman–Wallach . The parametrization is chosen so that the spherical representations $I(\lambda)$ is reducible if and only if $-\lambda \in \mathbb{N}$ or $\lambda - n \in \mathbb{N}$, and that $I(-i)$ $(i \in \mathbb{N})$ contains a finite-dimensional representation $F(i)$ as the unique subrepresentation, which is isomorphic to the representation of the identity component G_0 of G on the space $\mathcal{H}^i(\mathbb{R}^{n+1,1})$ of spherical harmonics of degree i, The irreducible Fréchet representation $I(-i)/F(i)$ of G is denoted by $T(i)$.

In this parametrization, $I(\lambda)$ is unitarizable if $\lambda \in \frac{n}{2} + \sqrt{-1}\mathbb{R}$ (*unitary principal series representation*) or $0 < \lambda < n$ (*complementary series representations*, Spherical principal series representations of the subgroup G' are denoted by $J(\nu)$ and are parametrized so that the finite-dimensional representations $F(j)$ is a subrepresentation of $J(-j)$. The irreducible Fréchet representation $J(-j)/F(j)$ of G' is denoted by $T(j)$.

Consider pairs of nonpositive integers and define

$$L_{\text{even}} := \{(-i,-j) : j \leq i \text{ and } i \equiv j \mod 2\}\,,$$
$$L_{\text{odd}} := \{(-i,-j) : j \leq i \text{ and } i \equiv j+1 \mod 2\}\,.$$

We prove:

Theorem 0.1 (multiplicities for spherical principal series,)**.** *We have*

$$m(I(\lambda), J(\nu)) = \begin{cases} 1 & \text{if } (\lambda,\nu) \in \mathbb{C}^2 - L_{\text{even}}, \\ 2 & \text{if } (\lambda,\nu) \in L_{\text{even}}. \end{cases}$$

From the viewpoint of geometry, G is the conformal group of the standard sphere S^n, and conformally equivariant line bundles $\mathcal{L}_\lambda$ over S^n are parametrized by $\lambda \in \mathbb{C}$ (we normalize $\mathcal{L}_\lambda$ such that $\mathcal{L}_0$ is the trivial line bundle and $\mathcal{L}_n$ is the bundle of volume densities). The subgroup G' is the conformal group of the 'great circle' S^{n-1} in S^n, and conformally equivariant line bundles $\mathcal{L}_\nu$ over S^{n-1} are parametrized by $\nu \in \mathbb{C}$. Then the Theorem determines the dimension of conformally covariant linear maps (*i.e.*, G'-equivariant operators) from $C^\infty(S^n, \mathcal{L}_\lambda)$ to $C^\infty(S^{n-1}, \mathcal{L}_\nu)$.

More generally we proved for all principal series representations

Theorem 0.2 (multiplicities for general principal series representations)**.** *We have*

$$m(I(V, \lambda), J(W, \nu)) \leq 2$$

From representation theoretic viewpoint, it was proved recently in Sun and Zhu that $m(\pi, \pi') \leq 1$ for all irreducible admissible representations π of G and π' of G'. However, it is more involved to tell whether $m(\pi, \pi') = 0$ or 1 for given irreducible representations π and π'.

The following theorem determines $m(\pi, \pi')$ for irreducible subquotients of spherical principal series representations at reducible points.

Theorem 0.3 (multiplicities for composition factors of spherical principal series)**.** *Let* $i, j \in \mathbb{N}$.

(1) *Suppose that* $i \geq j$.

(1-a) *Assume* $i \equiv j \mod 2$, *namely,* $(-i, -j) \in L_{\mathrm{even}}$. *Then*

$$m(T(i), T(j)) = 1, \qquad m(T(i), F(j)) = 0, \qquad m(F(i), F(j)) = 1.$$

(1-b) *Assume* $i \equiv j + 1 \mod 2$, *namely,* $(-i, -j) \in L_{\mathrm{odd}}$. *Then*

$$m(T(i), T(j)) = 0, \qquad m(T(i), F(j)) = 1, \qquad m(F(i), F(j)) = 0.$$

(2) *Suppose that* $i < j$. *Then*

$$m(T(i), T(j)) = 0, \quad m(T(i), F(j)) = 1, \quad m(F(i), F(j)) = 0.$$

In the special case $\nu = 0$, our results on symmetry breaking operators for spherical principal series representations are closely related to the analysis on the indefinite hyperbolic space

$$X(n+1,1) := \{\xi \in \mathbb{R}^{n+2} : \xi_0^2 + \cdots + \xi_n^2 - \xi_{n+1}^2 = 1\} \simeq G/G'.$$

As a hypersurface of the Minkowski space

$$\mathbb{R}^{n+1,1} \equiv (\mathbb{R}^{n+2}, d\xi_0^2 + \cdots + d\xi_n^2 - d\xi_{n+1}^2),$$

$X(n+1,1)$ carries a Lorentz metric for which the sectional curvature is constant -1, and thus is a model space of anti-de Sitter manifolds. The Laplacian Δ of the Lorentz manifold $X(n+1,1)$ is a hyperbolic operator, and for $\lambda \in \mathbb{C}$, we consider its eigenspace:

$$\mathcal{S}ol(G/G';\lambda) := \{f \in \mathcal{C}^\infty(G/G') : \Delta f = -\lambda(\lambda - n)f\}.$$

The underlying $(\mathfrak{g},K)$-module $\mathcal{S}ol(G/G';\lambda)_K$ is isomorphic to the underlying $(\mathfrak{g},K)$-module of a principal series representation. For $\lambda = -i \in -\mathbb{N}$ there are two inequivalent reducible principal series representations $I(-i)_K$ and $I(n+i)_K$, and our results on the symmetry breaking operators $\widetilde{\mathbb{A}}_{\lambda,0}$ for $(\lambda,0) \in L_{\mathrm{even}}$ give another proof of the following $(\mathfrak{g},K)$-isomorphism:

$$\mathcal{S}ol(G/G';\lambda)_K \simeq \begin{cases} I(-i)_K & \text{if } \lambda = -i \in -2\mathbb{N}, \\ I(n+i)_K & \text{if } \lambda = -i \in -2\mathbb{N}-1. \end{cases}$$

More generally, we apply our results on symmetry breaking operators for $\nu \in -\mathbb{N}$ to the analysis on vector bundles. We note that harmonic analysis on (general) semisimple symmetric spaces has been studied actively by many people during the last fifty years, however, not much has been known for vector bundle sections. Using our results on symmetry breaking operators for spherical principal series representations we construct some irreducible subrepresentations in the space of sections of the G-equivariant vector bundles $G \times_{G'} F(j) \to G/G'$.

We have complete results about multiplicities for another family of representations. Let V_i be the representation of O(n-1) on $\bigwedge^i \mathbb{R}^{n-1}$. Let Π_i be irreducible subrepresentation of the representation $I(V_i,i)$. Our parametrization implies that Π_0 is the trivial representation. If n=2m and if n = 2m+1 then Π_m is tempered. All these representations Π_i are unitary and their underlying $(\mathfrak{g}, K)$-module $T(i)_K$ is isomorphic to $A_{\mathfrak{q}}(\lambda)$-modules where $\mathfrak{q}$ is a certain maximal θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In particular they are the representations with nontrivial $(\mathfrak{g}, K)$–cohomology.

We denote the corresponding representations of the smaller group G' by the small letter π_j.

The following tables summarize our results about symmetry breaking $\Pi_i \to \pi_j$ The results are represented graphically in the following tables. The first row are representations of G, the second row are representations of G'. Symmetry breaking operators are represented by black arrows .

Table 1: Symmetry breaking for O(2m+1,1), O(2m.1)

$$\begin{array}{cccccccccc}
\Pi_0 & & \Pi_1 & & \dots & \dots & \dots & \Pi_{m-1} & & \Pi_m \\
\downarrow & \swarrow & \downarrow & \swarrow & \dots & & & \downarrow & \swarrow & \downarrow \\
\pi_0 & & \pi_1 & & \dots & \dots & \dots & \pi_{m-1} & & \pi_m
\end{array}$$

default

Table 2: Symmetry breaking for O(2m+2,1), O(2m+1.1)

$$\begin{array}{ccccccccccc}
\Pi_0 & & \Pi_1 & & \dots & \dots & \dots & \Pi_{m-1} & & \Pi_m & \Pi_{m+1} \\
\downarrow & \swarrow & \downarrow & \swarrow & \dots & & & \downarrow & \swarrow & \downarrow & \swarrow \\
\pi_0 & & \pi_1 & & \dots & \dots & \dots & \pi_{m-1} & & \pi_m &
\end{array}$$

default

About 20 years ago Gross and Prasad formulated a conjecture about the restriction of an irreducible admissible tempered representation of an inner form G of the group $O(n)$ over a local field to a subgroup G" which is an inner form $G' = O(n-1)$. The conjecture by Gross and Prasad relates the existence of nontrivial homomorphisms to the value of an L-function at $1/2$ and the value of the epsilon factor. We expect that our theorem proves a special case of this conjecture and also the equivalent conjecture in the non tempered case.

Let H be a subgroup of G and π_H a representation of H. A representation $\Pi_G \otimes \pi_H$ of $G \times H$ has a nontrivial (H,ψ),-period if

$$\mathrm{Hom}_H(\pi_G \otimes \pi_H, 1_\psi) \neq 0$$

for a one dimensional representation 1_ψ of H.
We say that the period is nontrivial on a vector v ,if v is not in the kernel of a symmetry breaking operator in $\mathrm{Hom}_H(\pi_H \times \pi_H, I_\psi)$. and up to a sign the absolute value of the image of a unit vector v is referred to as the value of the period of v. If π_H is the trivial representation we say that the representation π_G has an H period.

We can reformulate our

Corollary 0.4. *The representation*

$$\Pi_i \otimes \pi_j$$

has for j= i or j=i-1 a nontrivial $O(n,1)$ -period. The period is nontrivial on the minimal K-type

Theorem 0.5. *.*

1. *The maximal orthogonal period of Π_i is O(n-i,1).*
2. *The maximal period is nontrivial on a vector v in the minimal K-type.*
3. *The absolute value of the maximal orthogonal period on v is*

$$\frac{2^i \pi^\gamma}{\prod_{j=1}^{i} \Gamma(j+1)}$$

The constant γ and sign depend on n.