Totally geodesic surfaces in Riemannian symmetric spaces and nilpotent orbits

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26 November, 2014
(Symposium on Representation Theory 2014)

1 Introduction

Let M be a connected Riemannian (global) symmetric space of non-compact type and denote by $\mathrm{Isom}(M)$ the group of isometries on M (then $\mathrm{Isom}(M)$ is a non-compact semisimple Lie group). A connected submanifold S of M is said to be *totally geodesic* if for each point $p \in S$, any geodesic in M which is tangent to S at p is a curve in S.

One of fundamental problems of totally geodesic submaniflds in M is to classify these. It is well-known that maximal flat totally geodesic submaniflds in M can be constructed easily and unique up to $\mathrm{Isom}(M)$-congugate. Thus, we are interested in classifications of non-flat totally geodesic submanifolds in M. Note that by taking the compact dual M' of the Riemannian symmetric space M of non-compact type, the classification of totally geodesic submanifolds in M gives that of in M', and the converse is also true.

The full classifications of totally geodesic submanifolds have been known for irreducible Riemannian symmetric spaces of rank one (cf. Wolf [13, 14]) and two (done by Klein [3, 4, 6] recently). However, a general classification problem remains widely open. In [5], one can found a survey of the his-

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tory to study classifications of totally geodesic submanifolds in Riemannian symmetric spaces.

In this talk, we study classifications of totally geodesic surfaces (i.e. 2-dimensional submanifolds) in Riemannian symmetric spaces M of non-compact type. We also refer to the recent work by Mashimo [8] and Fujimaru–Kubo–Tamaru [2], who study totally geodesic surfaces in symmetric spaces of classical type.

In the study on totally geodesic submanifolds in Riemannian symmetric spaces M, reductive subalgebras of the Lie algebra of $\mathrm{Isom}(M)$ play fundamental roles. In particular, $\mathrm{Isom}(M)$-conjugate classes of non-flat totally geodesic surfaces in M corresponds to the $\mathrm{Isom}(M)$-conjugacy classes of reductive subalgebras of the Lie algebra of $\mathrm{Isom}(M)$ which is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Therefore, by the Jacobson–Morozov theorem and Kostant's conjugacy theorem, we have a surjection from the set of nilpotent adjoint orbits in the semisimple Lie algebra of $\mathrm{Isom}(M)$ to the set of $\mathrm{Isom}(M)$-conjugacy classes of totally geodesic surfaces in M.

The goal of this talk is to study the fibers of such surjective map and give an algorithm to classify $\mathrm{Isom}_0(M)$-conjugacy classes of totally geodesic surfaces in M in terms of Djokovic's classifications of nilpotent adjoint orbits in real semisimple Lie algebras, where $\mathrm{Isom}_0(M)$ is the connected component of $\mathrm{Isom}(M)$.

Acknowledgements.

The authors would like to give heartfelt thanks to Atsumu Sasaki, whose suggestions were of inestimable value for this research.

2 Main theorem

Let G be a semisimple linear Lie group (which may not be connected) and K a maximal compact subgroup of G. In this talk, submanifolds S and S' in the Riemannian symmetric space G/K are said to be *G-conjugate* if there exists $g \in G$ such that $gS = S'$. The purpose of this talk is to study the set of G-conjugacy classes of non-flat totally geodesic surfaces (i.e. 2-dimensional submanifolds) in G/K.

In order to describe our results, let us set up our notation. We denote by

$\mathfrak{g}$ the Lie algebra of G and set $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g})$ to the automorphism group and the inner-automorphism group of $\mathfrak{g}$, respectively. The adjoint action of G on $\mathfrak{g}$ will be denoted by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. An $\mathrm{Ad}(G)$-orbit and an $\mathrm{Int}(\mathfrak{g})$-orbit in $\mathfrak{g}$ will be simply called a G-orbit and an adjoint orbit in $\mathfrak{g}$, respectively.

An element X of $\mathfrak{g}$ is said to be *nilpotent* if $\mathrm{ad}(X) \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ is nilpotent as a linear operator. A G-orbit $\mathcal{O}$ in $\mathfrak{g}$ is called *nilpotent* if some of (therefore any of) elements in $\mathcal{O}$ are nilpotent. Note that the zero-orbit $\{0\} \subset \mathfrak{g}$ is also a nilpotent G-orbit in $\mathfrak{g}$. It is known that the number of nilpotent G-orbits in $\mathfrak{g}$ is finite.

One can easily check that an element $X \in \mathfrak{g}$ is nilpotent if and only if $-X \in \mathfrak{g}$ is also nilpotent. Therefore, for any nilpotent G-orbit $\mathcal{O}$ in $\mathfrak{g}$, the G-orbit $-\mathcal{O} := \{-X \mid X \in \mathcal{O}\}$ is also nilpotent. In other words, the group $\{\pm \mathrm{id}_{\mathfrak{g}}\} \subset GL_{\mathbb{R}}(\mathfrak{g})$ acts on the finite set of non-zero nilpotent G-orbits in $\mathfrak{g}$, and the quotient will be denoted by

$$\{\text{ Non-zero nilpotent } G\text{-orbits in } \mathfrak{g}\}/\{\pm \mathrm{id}_{\mathfrak{g}}\}.$$

The following is the main theorem of this talk:

Theorem 2.1. *For each semisimple linear Lie group G and its maximal compact subgroup K, there exists a bijection between the following two finite sets:*

- *The set of G-conjugacy classes of non-flat totally geodesic surfaces in G/K.*
- $\{$ *Non-zero nilpotent G-orbits in* $\mathfrak{g}\}/\{\pm \mathrm{id}_{\mathfrak{g}}\}$.

Recall that the classifications of nilpotent adjoint orbits in real semisimple Lie algebras were already known (see [1, Chapter 9] for the details). In the cases where G is connected, the classification of G-conjugacy classes of non-flat totally geodesic surfaces in G/K can be reduced to study the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action on the set of nilpotent adjoint orbits in $\mathfrak{g}$. We will study such the action in Section 3.

Even for the cases where G is not connected, the later set in Theorem 2.1 can be understand in terms of nilpotent adjoint orbits in $\mathfrak{g}$ as follows: Let us define the finite group $\mathrm{Out}_G(\mathfrak{g})$ by $\mathrm{Ad}(G)/\mathrm{Int}(\mathfrak{g})$. Note that $\mathrm{Out}_G(\mathfrak{g})$ is a subgroup of $\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ ($\mathrm{Out}_G(\mathfrak{g})$ is trivial if G is connected). Then $\mathrm{Out}_G(\mathfrak{g})$ acts on the set of non-zero nilpotent adjoint orbits in $\mathfrak{g}$ and a non-zero nilpotent G-orbit in $\mathfrak{g}$ can be written by the union of nilpotent adjoint orbits conjugate by $\mathrm{Out}_G(\mathfrak{g})$ each other. One can easily check that the actions of $\mathrm{Out}_G(\mathfrak{g})$ and $\{\pm \mathrm{id}_\mathfrak{g}\}$ on the finite set of nilpotent adjoint orbits in $\mathfrak{g}$ are commutative. Therefore, the later set in Theorem 2.1 can be written as

$$\mathrm{Out}_G(\mathfrak{g})\backslash\{\text{ non-zero nilpotent adjoint orbits in } \mathfrak{g}\,\}/\{\pm \mathrm{id}_\mathfrak{g}\}.$$

In particular, we obtain the following theorem:

Theorem 2.2 (Corollary to Theorem 2.1)**.** *Let M be a connected Riemannian symmetric space of non-compact type and denote by $\mathfrak{g}$ the Lie algebra of the isometry group* $\mathrm{Isom}(M)$ *of M. Then there exists a bijection between the following two finite sets:*

- *The set of* $\mathrm{Isom}(M)$*-conjugacy classes of non-flat totally geodesic surfaces in M.*
- $\mathrm{Out}(\mathfrak{g})\backslash\{$ *non-zero nilpotent adjoint orbits in* $\mathfrak{g}\,\}/\{\pm \mathrm{id}_\mathfrak{g}\}$*, where* $\mathrm{Out}(\mathfrak{g})$ *denotes* $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$*.*

3 The $\{\pm \mathrm{id}_\mathfrak{g}\}$-action on the set of nilpotent adjoint orbits in $\mathfrak{g}$

Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$. In this section, we recall Djokovic's classification of nilpotent adjoint orbits in $\mathfrak{g}$ and study the $\{\pm \mathrm{id}_\mathfrak{g}\}$-action on the set of such orbits.

Let us denote by $\mathfrak{g}_\mathbb{C}$, $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{p}_\mathbb{C}$ the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$, respectively. We fix a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}$, and denote by $\mathfrak{t}_\mathbb{C}$ the complexification of $\mathfrak{t}$. Since $\mathfrak{k}_\mathbb{C}$ is a complex reductive Lie algebra, the root system $\Delta(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C}) \subset \sqrt{-1}\mathfrak{t}^*$ and its Weyl group $W(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})$ are defined, where $\sqrt{-1}\mathfrak{t}^*$ is the dual space of the real vector space $\sqrt{-1}\mathfrak{t}$. Note that in the cases where $\mathfrak{k}$ has a non-trivial center, then the root system $\Delta(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})$ does not span the vector space $\sqrt{-1}\mathfrak{t}^*$. Let us fix a positive system $\Delta^+(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})$ of $\Delta(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})$. Then the closed Weyl chamber

$$\sqrt{-1}\mathfrak{t}_+ := \{X \in \sqrt{-1}\mathfrak{t} \mid \alpha(X) \geq 0 \quad \text{for any } \alpha \in \Delta^+(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})\}$$

is a fundamental domain of $\sqrt{-1}\mathfrak{t}$ for the $W(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})$-action.

By the results in Kostant–Rallis [7] and the Kostant–Sekiguchi correspondence [12], the following two fact holds:

Fact 3.1 (cf. [1, Chapter 9.4 and 9.5]). *For each nilpotent adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$, there uniquely exists an element $H_{\mathcal{O}} \in \sqrt{-1}\mathfrak{t}_+$ such that there exists $X', Y' \in \mathfrak{p}_{\mathbb{C}}$ satisfying that*

$$[H_{\mathcal{O}}, X'] = 2X', \ [H_{\mathcal{O}}, Y'] = -2Y', \ [X', Y'] = H_{\mathcal{O}}$$
$$\textit{and} \quad -\sqrt{-1}H_{\mathcal{O}} + X' + Y' \in \mathcal{O}.$$

Fact 3.2 (cf. [1, Chapter 9.4 and 9.5]). *Two nilpotent adjoint orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathfrak{g}$ are the same if and only if $H_{\mathcal{O}_1} = H_{\mathcal{O}_2}$.*

Then the set of nilpotent adjoint orbits in $\mathfrak{g}$ has a bijection to the set

$$\{H_{\mathcal{O}} \in \sqrt{-1}\mathfrak{t} \mid \mathcal{O} \text{ is a nilpotent adjoint orbit in } \mathfrak{g}\}.$$

In this talk, we say that the element $H_{\mathcal{O}} \in \sqrt{-1}\mathfrak{t}_+$ is *the defining vector* of a nilpotent adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$.

Remark 3.3. *The Cayley transform of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ and the Kostant–Sekiguchi correspondence have ambiguities from outer-automorphisms of $\mathfrak{sl}(2,\mathbb{R})$. In Fact 3.1, we fix a Kostant–Sekiguchi correspondence. If we take another one, then we obtain another fact similar to Fact 3.1, and the definition of the defining vector of nilpotent adjoint orbit will be changed. However, to study the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action on the set of nilpotent adjoint orbits in $\mathfrak{g}$, such the ambiguity is not effective. We omit the details here.*

For exceptional simple Lie algebras $\mathfrak{g}$, such the sets were classified by Djokovic in [9, 10]. One can find the classification tables in [1, Chapter 9.6].

Remark 3.4. *For classical simple Lie algebras $\mathfrak{g}$, one can find classifications of nilpotent adjoint orbits in $\mathfrak{g}$ in terms of singed Young diagrams in [1, Chapter 9.3]. To obtain the information of the defining vector of each nilpotent adjoint orbit, we need some computations.*

To study the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action on the set of nilpotent adjoint orbits $\mathcal{O}$ in $\mathfrak{g}$, we only need to study $H_{-\mathcal{O}}$ for each $\mathcal{O}$. The theorem below plays an important role to do it:

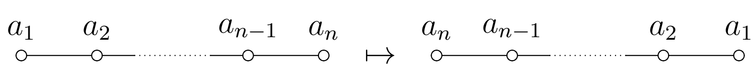
Theorem 3.5. *Let us denote w_0 by the longest element of the Weyl group $W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ acting on $\sqrt{-1}\mathfrak{t}$ with respect to the positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then for each nilpotent adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$ with the defining vector $H_{\mathcal{O}}$, the defining vector $H_{-\mathcal{O}}$ of the nilpotent adjoint orbit $-\mathcal{O}$ coincides with the vector $-w_0(H_{\mathcal{O}}) \in \sqrt{-1}\mathfrak{t}_+$.*

In general, for an inner-product space V, a reduced irreducible root system Δ realized as a spaning subset of the dual space V^* and its symple system Π, the map $-w_0 : V \to V,\ H \mapsto -w_0(H)$, where w_0 is the longest element of the Weyl group of Δ with respect to Π, can be computed as follows: For each $H \in V$, the map $\Phi_H : \Pi \to \mathbb{R},\ \alpha \mapsto \alpha(H)$ is called *the weighted Dynkin diagram* for (Δ, Π) of $H \in V$. The correspondence $H \mapsto \Phi_H$ gives a bijection between V and the weighted Dynkin diagrams for (Δ, Π). Note that the closed Weyl chamber of V with respect to Π correspondes to weighted Dynkin diagrams without negative weights.

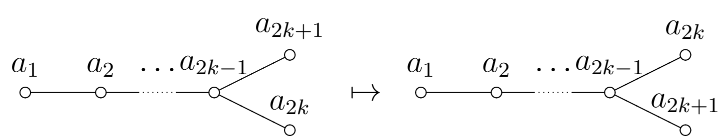
We denote by ι the linear transformation on the set of weighted Dynkin diagrams for (Δ, Π) corresponding to the linear transformaion $-w_0$ on V. Then the fact below holds:

Fact 3.6 (cf. [11, Theorem 6.3]). *The endomorphism ι is non-trivial if and only if the root system Δ is of type A_n, D_{2k+1} or E_6 $(n \geq 2,\ k \geq 2)$. In such cases, the forms of ι are the following:*

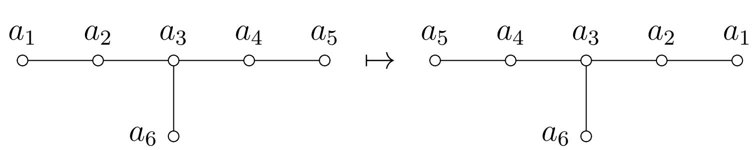
For type A_n $(n \geq 2)$



For type D_{2k+1} $(k \geq 2)$



For type E_6



In the cases where $\mathfrak{g}$ is non-Hermitian simple Lie algebra, then $\mathfrak{k}_{\mathbb{C}}$ is semisimple. Thus, by combining with Theorem 3.5 with Fact 3.6, we can compute the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action on the set of nilpotent adjoint orbits in $\mathfrak{g}$. In particular, we have the following:

Corollary 3.7 (Corollary to Theorem 3.5)**.** *Let $\mathfrak{g}$ be a non-Hermitian simple Lie algebra $\mathfrak{g}$, and assume that any simple factor of the complex semisimple Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is not of type A_n, D_{2k+1} or E_6 $(n \geq 2,\ k \geq 2)$. Then the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action on the set of nilpotent adjoint orbits is trivial.*

For Hermitian simple Lie algebras $\mathfrak{g}$, the computations of the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-actions are more complecated since $\mathfrak{k}_{\mathbb{C}}$ have non-trivial center. We omit such the cases in this note.

4 For some non-Hermitian cases

In this section, we study the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-actions on the set of nilpotent adjoint orbits in some non-Hermitian simple Lie algebras $\mathfrak{g}$.

By Corollary 3.7, we have the following:

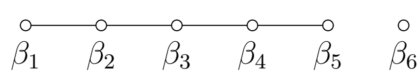
Theorem 4.1. *If $\mathfrak{g}$ is isomorphic to one of the following non-Hermitian simple Lie algebras, then the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action on the set of nilpotent adjoint orbits in $\mathfrak{g}$ is trivial.*

$$\mathfrak{sl}(n,\mathbb{R}) \ (\textit{with } n \equiv 0,1,3 \bmod 4),\ \mathfrak{su}^*(2n),\ \mathfrak{so}(p,q) \ (\textit{with } p,q \equiv 0,1,3 \bmod 4),$$
$$\mathfrak{sp}(p,q),\ \mathfrak{e}_{6(6)},\ \mathfrak{e}_{6(-26)},\ \mathfrak{e}_{7(-5)},\ \mathfrak{e}_{8(8)},\ \mathfrak{e}_{8(-24)},\ \mathfrak{f}_{4(4)},\ \mathfrak{f}_{4(-20)},\ \mathfrak{g}_{2(2)}.$$

In particular, by Theorem 2.1, if G is a non-Hermitian connected simple Lie group such that $\mathrm{Lie}(G)$ is isomorphic to one of the algebras listed in Theorem 4.1, then there exists a bijection between the set of totally geodesic surfaces in the Riemannian symmetric space G/K and the set of non-zero nilpotent adjoint orbits in $\mathfrak{g}$, where K is a maximal compact subroup of G.

As an exmaple that the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action is non-trivial, we study the case $\mathfrak{g} = \mathfrak{e}_6(2)$ as follows.

Example 4.2. *Let $\mathfrak{g} = \mathfrak{e}_{6(2)}$. Then $\mathfrak{k}_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. We denote by $\Pi(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ the set of simple roots in $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. We draw the Dynkin diagram of $\Pi(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ as*



Then we have a bijection between the set of weighted Dynkin diagrams without negative weights on the diagram above and $\sqrt{-1}\mathfrak{t}_+$.

The classification of nilpotent adjoint orbits $\mathcal{O}$ in $\mathfrak{g} = \mathfrak{e}_{6(2)}$ is described as the classification of weighted Dynkin diagrams $(\beta_1(H_{\mathcal{O}}), \dots, \beta_5(H_{\mathcal{O}}), \beta_6(H_{\mathcal{O}}))$ of the defining vectors $H_{\mathcal{O}} \in \sqrt{-1}\mathfrak{t}_+$ of nilpotent adjoint orbits $\mathcal{O}$ in $\mathfrak{g}$ as in Table 1 below (cf. [1, Chapter 9.6]):

By combining Theorem 3.5 with Fact 3.6, the $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action, i.e. the correspondence $\mathcal{O}$ to $-\mathcal{O}$, can be computed by

$$\begin{aligned}(\beta_1(H_{-\mathcal{O}}),&\beta_2(H_{-\mathcal{O}}), \beta_3(H_{-\mathcal{O}}), \beta_4(H_{-\mathcal{O}}), \beta_5(H_{-\mathcal{O}}), \beta_6(H_{-\mathcal{O}})) \\ &= (\beta_5(H_{\mathcal{O}}), \beta_4(H_{\mathcal{O}}), \beta_3(H_{\mathcal{O}}), \beta_2(H_{\mathcal{O}}), \beta_1(H_{\mathcal{O}}), \beta_6(H_{\mathcal{O}})).\end{aligned}$$

In particular, for the nilpotent adjoint orbits $\mathcal{O}$ in $\mathfrak{g}$ labeled 9, 12, 27 *and* 29 *in Table 1, the labels of the nilpotent adjoint orbits $-\mathcal{O}$ are* 10, 13, 28 *and* 30*, respectively. For the other all nilpotent adjoint orbits $\mathcal{O}$ in $\mathfrak{g}$, the equality $\mathcal{O} = -\mathcal{O}$ holds since $(\beta_1(H_{\mathcal{O}}), \dots, \beta_5(H_{\mathcal{O}}))$ are symmetric.*

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Table 1: List of nilpotent adjoint orbits in $\mathfrak{e}_6(2)$

label of $\mathcal{O}$	$(\beta_1(H_{\mathcal{O}}), \dots, \beta_5(H_{\mathcal{O}}),$	$\beta_6(H_{\mathcal{O}}))$	non-trivial $\{\pm \mathrm{id}_{\mathfrak{g}}\}$-action
0	$(0,0,0,0,0,$	$0)$	
1	$(0,0,1,0,0,$	$1)$	
2	$(1,0,0,0,1,$	$2)$	
3	$(0,1,0,1,0,$	$0)$	
4	$(0,0,1,0,0,$	$3)$	
5	$(1,0,1,0,1,$	$1)$	
6	$(0,0,0,0,0,$	$4)$	
7	$(2,0,0,0,2,$	$0)$	
8	$(0,0,2,0,0,$	$2)$	
9	$(2,1,0,0,1,$	$1)$	$\star$
10	$(1,0,0,1,2,$	$1)$	$\star$
11	$(0,2,0,2,0,$	$0)$	
12	$(3,0,1,0,0,$	$0)$	$\star$
13	$(0,0,1,0,3,$	$0)$	$\star$
14	$(1,1,0,1,1,$	$2)$	
15	$(1,0,2,0,1,$	$4)$	
16	$(0,1,2,1,0,$	$2)$	
17	$(1,1,1,1,1,$	$1)$	
18	$(1,0,3,0,1,$	$1)$	
19	$(1,1,1,1,1,$	$3)$	
20	$(0,0,4,0,0,$	$0)$	
21	$(0,2,0,2,0,$	$4)$	
22	$(2,0,2,0,2,$	$2)$	
23	$(0,0,4,0,0,$	$8)$	
24	$(2,0,4,0,2,$	$4)$	
25	$(4,0,0,0,4,$	$4)$	
26	$(2,2,0,2,2,$	$0)$	
27	$(1,2,1,1,3,$	$1)$	$\star$
28	$(3,1,1,2,1,$	$1)$	$\star$
29	$(3,1,3,1,0,$	$4)$	$\star$
30	$(0,1,3,1,3,$	$4)$	$\star$
31	$(1,3,1,3,1,$	$3)$	
32	$(2,2,2,2,2,$	$2)$	
33	$(0,4,0,4,0,$	$4)$	
34	$(2,2,4,2,2,$	$4)$	
35	$(4,0,4,0,4,$	$8)$	
36	$(4,4,0,4,4,$	$4)$	
37	$(4,4,4,4,4,$	$8)$	