SPECTRAL ANALYSIS ON NON-RIEMANNIAN LOCALLY SYMMETRIC SPACES

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Abstract. The discrete spectrum of the Laplacian has been extensively studied on Riemannian locally symmetric spaces, but until recently little was known in the non-Riemannian setting, where the Laplacian is not an elliptic operator anymore. We will explain some recent results, joint with T. Kobayashi, in that setting. In particular, we prove that for a large class of non-Riemannian locally symmetric spaces (not necessarily of finite volume), the discrete spectrum of the Laplacian is nonempty, and contains an infinite subset which is stable under small deformations of the geometric structure.

1. Introduction

This is joint work with T. Kobayashi.

1.1. **The main objects.** Let $X = G/H$ be a semisimple symmetric space, i.e. G is a real semisimple Lie group and H an open subgroup of the set of fixed points of some involution σ of G. The space $X = G/H$ has a natural G-invariant pseudo-Riemannian structure, induced by the Killing form of the Lie algebra $\mathfrak{g}$. If Γ is a discrete subgroup of G acting properly discontinuously and freely on X, then $X_\Gamma := \Gamma\backslash X$ is a manifold (*locally symmetric space*), and there is a natural pseudo-Riemannian structure on X_Γ making the covering map $p_\Gamma : X \to X_\Gamma$ an isometry. The *Laplacian* of X_Γ is the second-order differential operator

$$\square_{X_\Gamma} = \mathrm{div}\ \mathrm{grad},$$

where the divergence is defined with respect to the pseudo-Riemannian structure. When X (or equivalently X_Γ) is non-Riemannian, $\square_{X_\Gamma}$ is *not* elliptic. We are interested in the spectral analysis of this Laplacian $\square_{X_\Gamma}$.

More generally, we consider "intrinsic" differential operators of possibly higher order on X_Γ, constructed as follows. Let $\mathbb{D}_G(X)$ be the $\mathbb{C}$-algebra of G-invariant differential operators on X. Then any operator $D \in \mathbb{D}_G(X)$ induces a differential operator D_Γ on X_Γ such that

$$D \circ p_\Gamma^* = p_\Gamma^* \circ D_\Gamma,$$

where $p_\Gamma^* : C^\infty(X_\Gamma) \to C^\infty(X)$ is the pull-back by p_Γ. As a particular case, the Laplacian $\square_X$ is G-invariant (it corresponds to the Casimir element $C_G \in U(\mathfrak{g})$ acting on X by derivation) and $(\square_X)_\Gamma = \square_{X_\Gamma}$. For any $\mathbb{C}$-algebra homomorphism $\lambda : \mathbb{D}_G(X) \to \mathbb{C}$, we denote by $L^2(X_\Gamma; \mathcal{M}_\lambda)$ the space of weak solutions $f \in L^2(X_\Gamma)$ to the system

$$D_\Gamma f = \lambda(D) f \qquad \text{for all } D \in \mathbb{D}_G(X) \tag{$\mathcal{M}_\lambda$}$$

(joint eigenfunctions for $\mathbb{D}_G(X)$). The *discrete spectrum* $\mathrm{Spec}_d(X_\Gamma)$ of X_Γ is the set of homomorphisms λ such that $L^2(X_\Gamma; \mathcal{M}_\lambda) \neq \{0\}$.

Remark 1.1. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into eigenspaces for $\mathrm{d}\sigma$. We actually consider λ as a parameter in the space $\mathfrak{j}_{\mathbb{C}}^*/W$, where $\mathfrak{j}_{\mathbb{C}}$ is a maximal semisimple abelian subspace of $\mathfrak{q}_{\mathbb{C}} = \mathfrak{q} \otimes_{\mathbb{R}} \mathbb{C}$ and W is the Weyl group of the restricted root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. Indeed, we may assume that G and H are connected, in which case there is a natural $\mathbb{C}$-algebra isomorphism

$$\Psi : \mathbb{D}_G(X) \xrightarrow{\sim} S(\mathfrak{j}_{\mathbb{C}})^W$$

(*Harish-Chandra isomorphism*, see [He]). If we identify $S(\mathfrak{j}_{\mathbb{C}})$ with the ring of polynomial functions on $\mathfrak{j}_{\mathbb{C}}^*$, then any homomorphism from $\mathbb{D}(X)$ to $\mathbb{C}$ is of the form

$$\chi_\lambda : D \longmapsto \langle \Psi(D), \lambda \rangle$$

for some $\lambda \in \mathfrak{j}_{\mathbb{C}}^*$, and $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda' \in W \cdot \lambda$. Thus we can see $\mathrm{Spec}_d(X)$ (or $\mathrm{Spec}_d(X_\Gamma)$ for any quotient X_Γ of X) as a subset of $\mathfrak{j}_{\mathbb{C}}^*/W$.

1.2. **A program.** In [KK1, KK2] we initiated the following program:

A) Construct joint L^2-eigenfunctions on X_Γ corresponding to $\mathrm{Spec}_d(X_\Gamma)$;

B) Understand the behavior of $\mathrm{Spec}_d(X_\Gamma)$ under small deformations of Γ inside G.

By a small deformation we mean a homomorphism close enough to the natural inclusion in the compact-open topology on $\mathrm{Hom}(\Gamma, G)$.

Problems A and B have been studied extensively in the following two cases.

• Assume that H is compact. Then X is Riemannian and the Laplacian $\square_X$ is elliptic. If X_Γ is compact, then the discrete spectrum of $\square_{X_\Gamma}$ is infinite. If furthermore Γ is irreducible, then Weil's local rigidity theorem [We1] states that nontrivial deformations exist only when X is the hyperbolic plane $\mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$, in which case compact quotients X_Γ have an interesting deformation space modulo conjugation, namely their Teichmüller space. Viewed as a "function" on the Teichmüller space, the discrete spectrum varies analytically [BC] and nonconstantly. On the other hand, for noncompact X_Γ the discrete spectrum $\mathrm{Spec}_d(X_\Gamma)$ may be considerably different depending on whether Γ is arithmetic or not (see Selberg [S], Phillips–Sarnak [PS1, PS2], Wolpert [Wp], etc.).

• Assume that Γ is trivial. Then the group G naturally acts on $L^2(X_\Gamma) = L^2(X)$, and so representation-theoretic methods may be used. Spectral analysis on the semisimple symmetric space X with respect to $\mathbb{D}(X)$ is essentially equivalent to finding a Plancherel-type theorem for the irreducible decomposition of the regular representation of G on $L^2(X)$: see van den Ban–Schlichtkrull [BS], Delorme [D], Oshima [Os1], Oshima–Sekiguchi [OS], as a far-reaching generalization of Harish-Chandra's earlier work [Ha] on the regular representation $L^2(G)$ for group manifolds. Flensted-Jensen [F] and Matsuki–Oshima [MO] showed that $\mathrm{Spec}_d(X) \neq \emptyset$ if and only if the rank condition

$$\mathrm{rank}\, G/H = \mathrm{rank}\, K/K \cap H \tag{1.1}$$

is satisfied (where K is a maximal compact subgroup of G defined by a Cartan involution of G commuting with σ), in which case they gave an explicit description of $\mathrm{Spec}_d(X)$. The rest of the spectrum (tempered representations for X, see [Ber]) is constructed from the discrete spectrum of smaller symmetric spaces by induction.

On the other hand, Problems A and B have not been much studied when H is noncompact, Γ is nontrivial, and Γ acts properly discontinuously on $X = G/H$, except in the group manifold case when X_Γ identifies with ${}^\backprime\Gamma\backslash{}^\backprime G$ for some semisimple Lie group ${}^\backprime G$ and some discrete subgroup ${}^\backprime\Gamma$.

1.3. **Difficulties in the general case.** The fact that H is noncompact and Γ nontrivial implies new difficulties from several perspectives:

1. Analysis: the Laplacian on X_Γ is not an elliptic operator anymore;
2. Geometry: an arbitrary discrete subgroup Γ of G does not necessarily act properly discontinuously on X;
3. Representation theory: a discrete subgroup Γ of G acting properly discontinuously on X always has infinite covolume in G; moreover, G does not act on $L^2(X_\Gamma)$ and $L^2(X_\Gamma) \neq L^2(\Gamma\backslash G)^H$ since H is noncompact.

In particular, point (1) makes Problem A of Section 1.2 nontrivial: we do not know a priori whether or not $\mathrm{Spec}_d(X_\Gamma) \neq \emptyset$, even for compact X_Γ.

Point (2) creates some underlying difficulty to Problem B: we need to consider locally symmetric spaces X_Γ for which the proper discontinuity of the action of Γ on X is preserved under small deformations of Γ in G. This is nontrivial. The question was first studied by Kobayashi [Ko4] in the late 1990s; we refer to [Cn] for an account of some recent developments. An interesting aspect of the case of noncompact H is that there are more examples where nontrivial deformations of compact quotients exist than for compact H (see Section 2.3).

1.4. **The main result.** The goal of this talk is to explain the following result, proved in [KK2].

Theorem 1.2. *Let $X = G/H$ be a semisimple symmetric space satisfying the rank condition* (1.1), *and let Γ be a discrete subgroup of G acting "strongly properly discontinuously" on X. Then*

(1) $\mathrm{Spec}_d(X_\Gamma)$ *is infinite; it contains an explicit infinite subset I depending only on quantitative estimates for proper discontinuity;*
(2) if X_Γ is compact and "standard", then there is a neighborhood $\mathcal{U} \subset$ $\mathrm{Hom}(\Gamma, G)$ *of the natural inclusion such that for any $\varphi \in \mathcal{U}$, the action of $\varphi(\Gamma)$ on X is properly discontinuous and $I \subset$* $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$.

We will give definitions of "strongly properly discontinuously" and "standard" in Section 2. After this, we will be able to state a complementary result to Theorem 1.2, about the existence of a "universal spectrum" for standard quotients (Theorem 3.1).

Note that in Theorem 1.2.(1) we do not assume the quotient X_Γ to be compact, nor of finite volume. Theorem 1.2.(2) describes a new rigidity phenomenon, which has no counterpart in the Riemannian world. The fact that the action on X remains properly discontinuous after any small deformation was proved in [Ka3].

2. Properly discontinuous actions

Recall that a discrete group Γ is said to act *properly discontinuously* on X if any compact subset of X meets only finitely many of its Γ-translates.

From now on we shall always assume H to be noncompact. In this case, not all discrete subgroups Γ of G act properly discontinuously on $X = G/H$: for instance, infinite discrete subgroups of H do not, since they fix a point in X. In fact, determining which groups act properly discontinuously on X is a very delicate question, which was first considered in full generality by Kobayashi in the early 1990s; we refer to [KY] for an excellent survey.

2.1. Strong proper discontinuity.

In order to quantify proper discontinuity, we use a properness criterion due to Benoist [Ben] and Kobayashi [Ko3]. Fix a Cartan involution θ of the semisimple group G, defining a maximal compact subgroup $K = G^\theta$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into eigenspaces of $\mathrm{d}\theta$. The Cartan decomposition $G = K(\exp \overline{\mathfrak{a}_+})K$ holds, where $\overline{\mathfrak{a}_+}$ is a closed Weyl chamber in a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Any $g \in G$ belongs to Ke^aK for a *unique* $a \in \overline{\mathfrak{a}_+}$: we set $\mu(g) := a \in \overline{\mathfrak{a}_+}$. This defines a map $\mu : G \to \overline{\mathfrak{a}_+}$ (*Cartan projection*); it is continuous, proper, and surjective. Fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}$, invariant under the Weyl group of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$, and let d be the corresponding metric on $\mathfrak{a}$.

Properness criterion (Benoist, Kobayashi)**.** *A discrete subgroup Γ of G acts properly discontinuously on $X = G/H$ if and only if for any $R > 0$, we have $d(\mu(\gamma), \mu(H)) \geq R$ for almost all $\gamma \in \Gamma$.*

By "almost all" we mean "all but finitely many". In other words, Γ acts properly discontinuously on $X = G/H$ if and only if the set $\mu(\Gamma)$ "goes away from $\mu(H)$ at infinity".

The following refinement was introduced in [KK2] (see Figure 1).

Definition 2.1. A discrete subgroup Γ of G acts *strongly properly discontinuously* (or *sharply*) on $X = G/H$ if there exist $C, C' > 0$ such that for all $\gamma \in \Gamma$,

$$d(\mu(\gamma), \mu(H)) \geq C \, \|\mu(\gamma)\| - C'.$$

The positive numbers C, C' are called *sharpness constants* for Γ.

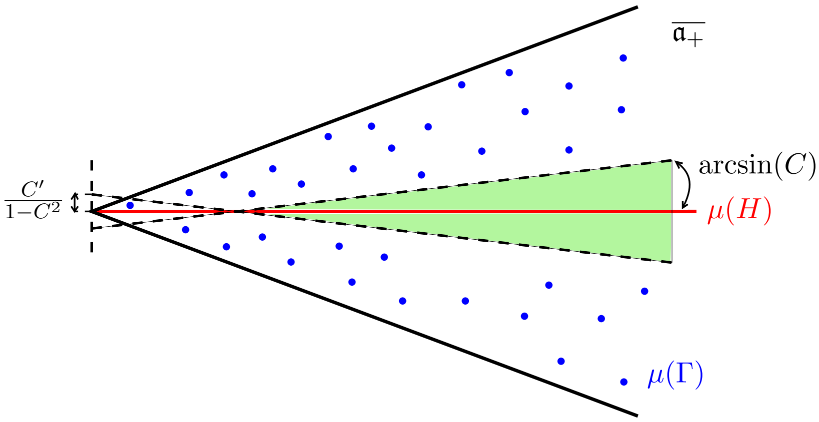

FIGURE 1. The Cartan projection of a discrete group Γ acting strongly properly discontinuously on X, with sharpness constants C, C'

In other words, Γ acts strongly properly discontinuously on $X = G/H$ if the set $\mu(\Gamma)$ goes away from $\mu(H)$ *at least linearly*. This gives a way to quantify proper discontinuity: the larger C is, the "most properly discontinuous" the action is; the situation becomes delicate when C comes close to 0.

Not all properly discontinuous actions on X are strongly properly discontinuous. However, we shall now see that many are.

2.2. **Standard quotients.** A large and important class of examples is constructed as follows (see [Ko1]).

Definition 2.2. A quotient $X_\Gamma = \Gamma\backslash X$ of X by a discrete subgroup of G is called *standard* if Γ is contained in some reductive subgroup L of G acting properly on X.

It is easy to see that if X_Γ is standard, then the action Γ on X is strongly properly discontinuous in the sense of Definition 2.1 (see [KK2, Ex. 4.10]), and so we may apply Theorem 1.2 as soon as the rank condition (1.1) is satisfied.

Here are a few examples, taken from [KY, Cor. 3.3.7], of triples (G, H, L) where G is a simple Lie group, $X = G/H$ a semisimple symmetric space satisfying the rank condition (1.1), and L a reductive subgroup of G acting properly and cocompactly on X. We denote by m and n any integers ≥ 1 with m even. See also [Ok] for many noncocompact examples.

	G	H	L
(i)	$\mathrm{SO}(2,2n)$	$\mathrm{SO}(1,2n)$	$\mathrm{U}(1,n)$
(ii)	$\mathrm{SO}(2,2m)$	$\mathrm{U}(1,m)$	$\mathrm{SO}(1,2m)$
(iii)	$\mathrm{SO}(4,4n)$	$\mathrm{SO}(3,4n)$	$\mathrm{Sp}(1,n)$
(iv)	$\mathrm{SU}(2,2n)$	$\mathrm{U}(1,2n)$	$\mathrm{Sp}(1,n)$
(v)	$\mathrm{SO}(8,8)$	$\mathrm{SO}(7,8)$	$\mathrm{Spin}(1,8)$

Note that the standard quotient X_Γ is compact if and only if Γ is a uniform lattice in L and L acts cocompactly on X. Almost all known examples of compact quotients of reductive homogeneous spaces are standard, and conjecturally [KY, Conj. 3.3.10] any reductive homogeneous space admitting compact quotients admits standard ones. See [KK2, § 4] for more details.

2.3. **Deformation of compact quotients.** In our setting where H is noncompact, there are more examples with nontrivial deformations of compact quotients than for compact H. Here are two examples.

In (i) above, where $X = \mathrm{SO}(2,2n)/\mathrm{SO}(1,2n)$, the group $L = \mathrm{U}(1,n)$ has a nontrivial center $Z(L)$, isomorphic to $\mathrm{U}(1)$. For certain uniform lattices Γ of L, small nontrivial deformations of Γ inside $G = \mathrm{SO}(2,2n)$ can be obtained by considering homomorphisms of the form $\gamma \mapsto \gamma\psi(\gamma)$ with $\psi \in \mathrm{Hom}(\Gamma, Z(L))$ (see [Ko4]). By [R, We2], any small deformation of Γ inside G is actually of this form, up to conjugation. The locally symmetric spaces corresponding to these nontrivial deformations remain standard, but the existence of a stable discrete spectrum given by Theorem 1.2.(2) is not obvious even in this case.

In Example (ii) above, where $X = \mathrm{SO}(2,2m)/\mathrm{U}(1,m)$, it is actually possible to deform certain standard compact quotients of X into *nonstandard* ones. Indeed, using a geometric construction (*bending*) due to Johnson–Millson [JM], one can obtain small *Zariski-dense* deformations inside $G = \mathrm{SO}(2,2m)$ of certain arithmetic uniform lattices Γ of $L = \mathrm{SO}(1,2m)$: by [Ka3], this yields a continuous family of compact quotient manifolds X_Γ with Γ Zariski-dense in G (i.e. not contained in any proper algebraic subgroup of G).

3. General strategy for the proof of Theorem 1.2

In order to prove Theorem 1.2, we construct explicit L^2-eigenfunctions on X_Γ using Flensted-Jensen's discrete series for X under the rank condition (1.1), as follows. For any $\lambda \in \mathfrak{j}_{\mathbb{C}}^*/W$ (see Remark 1.1) and any $f \in L^2(X;\mathcal{M}_\lambda)_K$ (K-finite element in $L^2(X;\mathcal{M}_\lambda)$), we consider the "generalized Poincaré series"

$$f^\Gamma := \left(X_\Gamma \ni \Gamma x \longmapsto \sum_{\gamma\in\Gamma} f(\gamma\cdot x) \right). \tag{3.1}$$

We prove that f^Γ is well defined on X_Γ (i.e. the series converges) and belongs to $L^2(X_\Gamma;\mathcal{M}_\lambda)$ as soon as the spectral parameter $\lambda \in \mathfrak{j}_{\mathbb{C}}^*/W$ is regular enough, in the sense that the distance from λ to the walls of $\mathfrak{j}_{\mathbb{C}}^*$ is large enough. Moreover, in the case that f is some explicit function constructed by Flensted-Jensen, we prove that f^Γ is nonzero for λ regular enough.

Addressing these convergence problems requires understanding how joint eigenfunctions $f \in L^2(X;\mathcal{M}_\lambda)_K$ decay at infinity, in relation with how Γ-orbits are distributed in X: thus there two complementary aspects, analytic and geometric. How regular the spectral parameter λ must be turns out to depend only on the geometry of X and on the sharpness constants of Γ (Definition 2.1); since these vary semicontinuously starting from a cocompact standard situation [Ka3], we obtain the deformation results of Theorem 1.2.

We actually prove the existence of a *universal spectrum* for standard quotients:

Theorem 3.1. *Let $X = G/H$ be a semisimple symmetric space satisfying the rank condition* (1.1)*, and let L be a reductive subgroup of G acting properly on X. Then there is an explicit infinite set I such that $I \subset \mathrm{Spec}_d(X_\Gamma)$ for* all *discrete subgroups Γ of L.*

4. The case of $X = \mathrm{AdS}^3$

We now give a more precise idea of the proof in the special case where X is the 3-dimensional *anti-de Sitter space* $\mathrm{AdS}^3 = \mathrm{SO}(2,2)/\mathrm{SO}(1,2) \simeq (\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R}))/\operatorname{Diag}(\mathrm{SL}_2(\mathbb{R}))$. Note that $X = \mathrm{AdS}^3$ is a model space for 3-dimensional Lorentzian manifolds of constant negative curvature (Lorentzian analogues of real hyperbolic 3-manifolds). Here X has rank 1, hence the discrete spectrum of X (or of its quotients X_Γ) identifies with the discrete spectrum of its Laplacian. The discrete spectrum of X is well known, and properly discontinuous actions on X are well understood since recently:

Fact 4.1. $\mathrm{Spec}_d(\Box_{\mathrm{AdS}^3}) = \{k(k-2) \mid k \in \mathbb{N}\}$.

Fact 4.2 ([Ka1, GK])**.** *Any finitely generated group Γ acting properly discontinuously on $X = \mathrm{AdS}^3$ acts strongly properly discontinuously.*

The sharpness constants C, C' (Definition 2.1) have a geometric interpretation in terms of equivariant Lipschitz maps from the hyperbolic plane $\mathbb{H}^2$ to itself (see [Ka1, GK]).

Using Facts 4.1 and 4.2, Kobayashi and I prove the following in [KK2].

Theorem 4.3. *For any finitely generated discrete subgroup Γ of $G = \mathrm{SO}(2,2)$ acting properly discontinuously on $X = \mathrm{AdS}^3$,*

$$\mathrm{Spec}_d(\Box_X) \supset \left\{ k(k-2) \;\middle|\; k \in \mathbb{N},\ k > \frac{10}{C^3} \right\},$$

where C, C' are sharpness constants for Γ.

Note that the first sharpness constant satisfies $C \leq 1/\sqrt{2}$, and equality may hold if and only if X_Γ is standard (Definition 2.2).

4.1. **Geometric aspects.** We wish to quantify the distribution of Γ-orbits in X. However, since X is not Riemannian, we do not have a metric to measure distances in X. The solution is to consider a "pseudo-distance to the origin" $\|\nu\|$ in X, as follows.

Up to conjugation, we may assume that the Cartan involution θ defining K in Section 2.1 commutes with the involution σ fixing H. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into eigenspaces of $\mathrm{d}\sigma$ as in Remark 1.1, let $\mathfrak{b} \subset \mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, and let $\overline{\mathfrak{b}_+}$ be the closed positive Weyl chamber associated with a system $\Sigma^+(\mathfrak{k}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{q},\mathfrak{b})$ of positive restricted roots. Then the *polar decomposition* (or *generalized Cartan decomposition*) $G = K(\exp\overline{\mathfrak{b}_+})H$ holds (see [F]). Any $g \in G$ belongs to Ke^bH for a unique $b \in \overline{\mathfrak{b}_+}$: we set $\nu(g) := b \in \overline{\mathfrak{b}_+}$. This defines a map $\nu : G \to \overline{\mathfrak{b}_+} \subset \mathfrak{a}$ (*polar projection*); it is continuous, surjective, and right-H-invariant, hence induces a map $\nu : X = G/H \to \overline{\mathfrak{b}_+} \subset \mathfrak{a}$. Geometrically, $\|\nu(x)\|$ can be interpreted as a "pseudo-distance" from the origin $x_0 = eH \in X$ to $x \in X$: in order to go from x_0 to x in X, one can first travel along the flat sector $(\exp\overline{\mathfrak{b}_+})\cdot x_0$, then along some (compact) K-orbit; $\|\nu(x)\|$ measures how far one must go in $(\exp\overline{\mathfrak{b}_+})\cdot x_0$.

Here is a concrete formula in the case of AdS^3. We can realize $X = \mathrm{AdS}^3$ as the quadric $\{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}$. Then $\|\nu\| : X \to \mathbb{R}^+$ is the function

$$x \longmapsto \operatorname{arcosh}\big(x_1^2 + x_2^2 + x_3^2 + x_4^2\big),$$

measuring the pseudo-distance to the origin $x_0 := (1,0,0,0)$. We thus see that the "pseudo-balls centered at the origin"

$$B_R := \{x \in X \mid \|\nu(x)\| \leq R\},$$

for $R > 0$, are compact and exhaust X. We prove the following.

Lemma 4.4. *Let Γ be a torsion-free discrete subgroup of $G = \mathrm{SO}(2,2)$ acting strongly properly discontinuously on $X = \mathrm{AdS}^3$, with sharpness constants C, C'. For any $x \in X$ there is a constant $c_x > 0$ such that for any $R > 0$,*

$$\#\big\{\gamma \in \Gamma \mid \|\nu(\gamma\cdot x)\| \leq R\big\} \leq c_x\, e^{R/C}.$$

4.2. **Analytic aspects.** We consider explicit eigenfunctions $f \in L^2(X, \mathcal{M}_\lambda)_K$ that were constructed by Flensted-Jensen [F] under the rank condition (1.1), using some duality principle and the Poisson transform. Applying deep results of microlocal analysis and hyperfunction theory [KKM+], Oshima and Matsuki [MO, Os2] gave a detailed analysis of the asymptotic behavior at infinity of these eigenfunctions. In [KK2] we use their estimates and express them in terms of

- the "weighted distance" $d(\lambda)$ of the spectral parameter λ to the walls of $\mathfrak{j}^*$ (which measures the regularity of λ),
- the "pseudo-distance from the origin" $\|\nu(x)\|$ of $x \in X$ (which measures how x goes to infinity).

Concretely, for $X = \mathrm{AdS}^3$, Flensted-Jensen's functions associated with the eigenvalue $k(k-2)$ (see Fact 4.1) are given by

$$\psi_k^{\pm}(x) := \big(x_1 \pm \sqrt{-1}\,x_2\big)^{-k} \in L^2(X)_K.$$

In this case it is elementary to check the following.

Lemma 4.5. *The functions $\psi_k^{\pm} : X \to \mathbb{C}$ satisfy $\psi_k^{\pm}(x_0) = 1$ and, for any $x \in X$,*

$$|\psi_k^{\pm}(x)| \leq \cosh\Big(\frac{\|\nu(x)\|}{2}\Big)^{-k/2} \leq\, 2^k\, e^{-k\|\nu(x)\|/2}.$$

4.3. **Conclusion.** It is elementary to see from Lemmas 4.4 and 4.5 that if Γ has sharpness constants C, C', then the generalized Poincaré series $(\psi_k^{\pm})^{\Gamma}$ (see (3.1)) is well defined for $k > 2/C$.

To see that $(\psi_k^{\pm})^{\Gamma}$ is L^2 on X_{Γ}, we prove that it is both L^1 and L^{∞}. The L^1 part is an easy application of the Fubini lemma. For the L^{∞} part, we establish a refinement of Lemma 4.4 with a *uniform* constant c_x for all $x \in X$ in some fundamental domain for the action of Γ.

To see that $(\psi_k^{\pm})^{\Gamma}$ is nonzero on X_{Γ}, we prove that

$$\sum_{\gamma \in \Gamma \smallsetminus \{1\}} |\psi_k^{\pm}(\gamma \cdot x_0)| < 1 = \psi_k^{\pm}(x_0).$$

This requires even finer estimates, not only at infinity, but also *near the origin* x_0. We refer to [KK2, § 7–8] for details.

For more about $X = \mathrm{AdS}^3$, see [KK2, § 9]; for some other detailed examples, see [KK2, § 10] and [Ko5].

5. Perspectives

The approach described so far is not the only possible way to try to construct discrete spectrum. In [KK3] we describe another approach, and propose to divide the discrete spectrum into type **I** and type **II**.

5.1. **Discrete spectrum of type I and type II.** The discrete spectrum constructed in Theorem 1.2 is of type **I**, in the following sense. Let $\mathcal{D}'(X)$ be the space of distributions on X and $p_{\Gamma}^* : L^2(X_{\Gamma}) \to \mathcal{D}'(X)$ the map induced by the projection $p_{\Gamma} : X \to X_{\Gamma}$. For any $\lambda \in \mathfrak{j}_{\mathbb{C}}^*/W$ (see Remark 1.1), we define $L^2(X_{\Gamma}; \mathcal{M}_{\lambda})_{\mathbf{I}}$ to be the preimage, under p_{Γ}^*, of the closure of $L^2(X; \mathcal{M}_{\lambda})$ in $\mathcal{D}'(X)$. Let $L^2(X_{\Gamma}; \mathcal{M}_{\lambda})_{\mathbf{II}}$ be the orthogonal complement of $L^2(X_{\Gamma}; \mathcal{M}_{\lambda})_{\mathbf{I}}$ in $L^2(X_{\Gamma}; \mathcal{M}_{\lambda})$, so that

$$L^2(X_{\Gamma}; \mathcal{M}_{\lambda}) = L^2(X_{\Gamma}; \mathcal{M}_{\lambda})_{\mathbf{I}} \oplus L^2(X_{\Gamma}; \mathcal{M}_{\lambda})_{\mathbf{II}}$$

(Hilbert space decomposition). For $i \in \{\mathbf{I}, \mathbf{II}\}$, we set

$$\mathrm{Spec}_d(X_{\Gamma})_i := \{\lambda \in \mathrm{Spec}_d(X_{\Gamma}) \mid L^2(X_{\Gamma}; \mathcal{M}_{\lambda})_i \neq \{0\}\}.$$

Then

$$\mathrm{Spec}_d(X_{\Gamma}) = \mathrm{Spec}_d(X_{\Gamma})_{\mathbf{I}} \cup \mathrm{Spec}_d(X_{\Gamma})_{\mathbf{II}}.$$

By construction, $\mathrm{Spec}_d(X_{\Gamma})_{\mathbf{I}}$ is contained in $\mathrm{Spec}_d(X) = \mathrm{Spec}_d(X)_{\mathbf{I}}$, hence is nonempty only if the rank condition (1.1) holds [F, MO]. By Theorem 1.2, the set $\mathrm{Spec}_d(X_{\Gamma})_{\mathbf{I}}$ is actually nonempty (and even infinite) as soon as (1.1) holds and the action of Γ on X is strongly properly discontinuous.

5.2. **Constructing discrete spectrum of type II for standard quotients.** In [KK3] we construct discrete spectrum of type **II** (i.e. *not* coming from discrete series for X). We work in a situation that is both more general than Section 1.4, in the sense that we do not assume the rank condition (1.1) to be satisfied, and more specific: we take Γ to be standard (Definition 2.2), contained in a reductive subgroup L of G acting properly *and* *transitively* on X.

Theorem 5.1. *Let $X = G/H$ be a reductive homogeneous space and L a reductive subgroup of G acting properly and transitively on X. Assume G is simple and the complexification $X_{\mathbb{C}}$ is $L_{\mathbb{C}}$-spherical. Then for any torsion-free discrete subgroup Γ of L,*

1. *the Laplacian $\square_{X_\Gamma}$ extends to a self-adjoint operator on $L^2(X_\Gamma)$,*
2. *for any $\lambda \in \mathrm{Spec}_d(X_\Gamma)$, the space $L^2(X_\Gamma; \mathcal{M}_\lambda)$ contains real analytic functions as a dense subset,*
3. *if $\mathrm{Spec}_d(Y_\Gamma)$ is infinite (e.g. if Y_Γ is compact), then $\mathrm{Spec}_d(X_\Gamma)_{\mathbf{II}}$ is infinite.*

By "$L_{\mathbb{C}}$-spherical" we mean that a Borel subgroup of $L_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. Theorem 5.1 applies to the examples in the following table. Note that in (ii) for n odd, and in (vii), the rank condition (1.1) is not satisfied. We allow X_Γ to have infinite volume.

	G	H	L
(i)	$\mathrm{SO}(2n, 2)$	$\mathrm{SO}(2n, 1)$	$\mathrm{U}(n, 1)$
(ii)	$\mathrm{SO}(4n, 2)$	$\mathrm{U}(2n, 1)$	$\mathrm{SO}(4n, 1)$
(iii)	$\mathrm{SU}(2n, 2)$	$\mathrm{U}(2n, 1)$	$\mathrm{Sp}(n, 1)$
(iv)	$\mathrm{SU}(2n, 2)$	$\mathrm{Sp}(n, 1)$	$\mathrm{U}(2n, 1)$
(v)	$\mathrm{SO}(4n, 4)$	$\mathrm{SO}(4n, 3)$	$\mathrm{Sp}(1) \times \mathrm{Sp}(n, 1)$
(vi)	$\mathrm{SO}(8, 8)$	$\mathrm{SO}(8, 7)$	$\mathrm{Spin}(8, 1)$
(vii)	$\mathrm{SO}(8, \mathbb{C})$	$\mathrm{SO}(7, \mathbb{C})$	$\mathrm{Spin}(7, 1)$
(viii)	$\mathrm{SO}(4, 4)$	$\mathrm{Spin}(4, 3)$	$\mathrm{SO}(4, 1) \times \mathrm{SO}(3)$

Remark 5.2. We prove that Conditions (1), (2), (3) of Theorem 5.1 also hold in the following different setting:

- $X = ({}'G \times {}'G)/\mathrm{Diag}({}'G)$ where ${}'G$ is a noncompact semisimple Lie group,
- $L = {}'G \times {}'K$ where ${}'K$ a maximal compact subgroup of ${}'G$,
- Γ is any torsion-free discrete subgroup of L.

In the classical case where $\Gamma = {}'\Gamma \times \{1\}$, we have $L^2(X_\Gamma) = L^2({}'\Gamma \backslash {}'G)$. The novelty here is that Γ does not need to be contained in one factor of ${}'G \times {}'G$. Examples are obtained by taking the graph, in $L = {}'G \times {}'K$, of a homomorphism $\rho : {}'\Gamma \to {}'K$, and in fact all examples are of this form [Ko2, Ka2]. Such homomorphisms ρ exist in many situations, for instance when ${}'\Gamma$ is a free group, or a uniform lattice in $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ [Ko2].

To prove Theorem 5.1, we set $L_H := L \cap H$ and let L_K be a maximal compact subgroup of L containing L_H. The transitivity of the action of L on X means that X fibers over the Riemannian symmetric space $Y = L/L_K$ of L, with compact fiber $F := L_K/L_H$. Consider the bundle

$$X = G/H \simeq L/L_H \xrightarrow{F} L/L_K = Y.$$

There are three natural subalgebras of $\mathbb{D}_L(X)$, namely $\mathbb{D}_G(X)$, the image of the center $Z(\mathfrak{l}_{\mathbb{C}})$ of the enveloping algebra $U(\mathfrak{l}_{\mathbb{C}})$ under the action by left differentiation, and the image of $\mathbb{D}_{L_K}(F)$ under its natural embedding into $\mathbb{D}_G(X)$. In [KK4], we establish certain linear relations between generators of these three subalgebras. In [KK3], we then prove that these relations imply the spectral properties of Theorem 5.1. This reduces the spectral analysis of the pseudo-Riemannian locally symmetric space X_Γ, for $\Gamma \subset L$, to that of the Riemannian locally symmetric space $Y_\Gamma = \Gamma\backslash L/L_K$.

We are also able to construct *continuous* spectrum on X_Γ by transferring continuous spectrum from Y_Γ. Details will soon be available in [KK3].

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