Symmetry Breaking Operators and Branching Problems

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Branching problems of representations

1. Two examples in “nice settings”
   - A. Continuous symmetry breaking operators
   - B. Differential symmetry breaking operators
2. Finite multiplicity restrictions
3. Discretely decomposable restrictions
4. Applications

Example A. Continuous symmetry breaking operators

$G = O(n+1,1) \curvearrowright S^n$ conformally

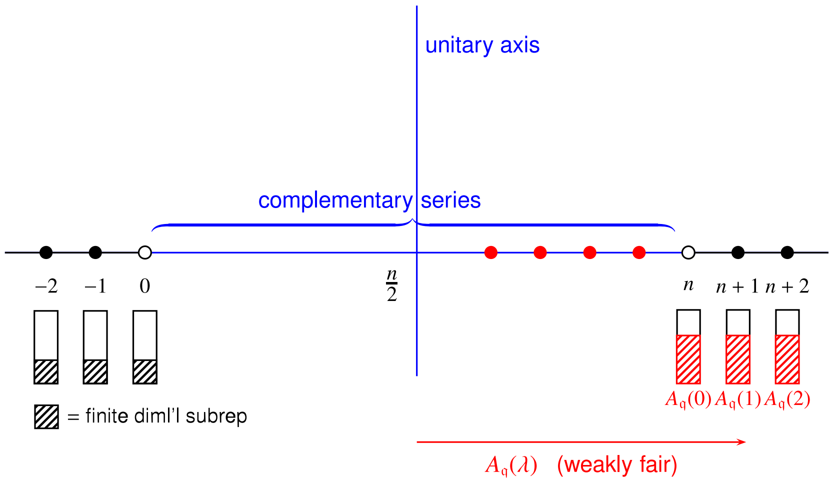
Eg. $n = 2$

$$\begin{array}{ccc} O(3,1) & \curvearrowright & S^2 \\ \shortparallel & & \wr\!\shortmid \\ SL(2,\mathbb{C}) & \curvearrowright & \mathbb{P}^1\mathbb{C} \\ & & z \mapsto \frac{az+b}{cz+d} \end{array}$$

(Möbius transform)

Parameterization for $G \curvearrowright C^\infty(G/P, \mathcal{L}_\lambda)$

$G = O(n+1,1) \curvearrowright C^\infty(G/P, \mathcal{L}_\lambda)$
spherical principal series ($\lambda \in \mathbb{C}$)



Continuous symmetry breaking operators

$$\begin{array}{ccccc} & & \mathcal{L}_\lambda & & \mathcal{L}_\nu & & \\ & & \downarrow & & \downarrow & & \\ G = O(n+1,1) & \overset{\text{conformal}}{\curvearrowright} & S^n = G/P & \hookleftarrow & G'/P' = S^{n-1} & \overset{\text{conformal}}{\curvearrowleft} & G' = O(n,1) \end{array}$$

$$\boxed{H(\lambda,\nu) := \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{L}_\lambda), C^\infty(G'/P', \mathcal{L}_\nu))}$$

$$G \circlearrowright \quad \supset \quad \circlearrowleft G'$$

Theorem A (K–Speh, to appear in Memoirs of AMS, arXiv:1310.3213)
Construction and classification of $H(\lambda,\nu)$ for all $\lambda, \nu \in \mathbb{C}$.
In particular,

$$\dim_{\mathbb{C}} H(\lambda,\nu) = \begin{cases} 1 & \text{if } (\lambda,\nu) \in \mathbb{C}^2 \setminus L_{\text{even}} \\ 2 & \text{if } (\lambda,\nu) \in L_{\text{even}} \end{cases}$$

$$L_{\text{even}} := \{(\lambda,\nu) \in \mathbb{Z}^2 : \lambda \leq \nu \leq 0,\ \lambda \equiv \nu \bmod 2\}.$$

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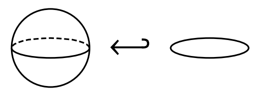

Continuous symmetry breaking operators

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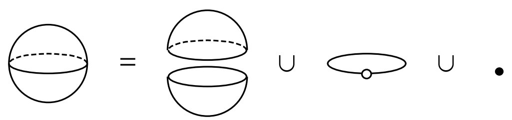
$$\boxed{H(\lambda,\nu) := \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{L}_\lambda), C^\infty(G'/P', \mathcal{L}_\nu))}$$

$\exists$3 meromorphic families of symmetry breaking operators
(i.e. continuous G'-homomorphism) corresponding to

$$\#((P \times P')\backslash(G \times G')/\Delta G') = \#(P\backslash G/P') = 3$$



Continuous symmetry breaking operators

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$$G \circlearrowright \quad \supset \quad \circlearrowleft G'$$

Problem

For which $(\lambda,\nu) \in \mathbb{C}^2$, is $H(\lambda,\nu) \neq 0$?

Continuous symmetry breaking operators

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$$\#((P \times P')\backslash(G \times G')/\Delta G') = \#(P\backslash G/P') = 3$$

open orbit	$\rightsquigarrow$	regular symmetry breaking operators
(intermediate	$\rightsquigarrow$	singular integral)
closed orbit	$\rightsquigarrow$	differential symmetry breaking operators

Continuous symmetry breaking operators

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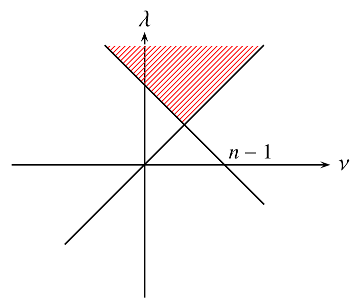
$$\boxed{H(\lambda,\nu) := \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{L}_\lambda), C^\infty(G'/P', \mathcal{L}_\nu))}$$

$\mathrm{Re}(\lambda+\nu) > n-1$ and $\mathrm{Re}(\nu-\lambda) > 0$
$\Rightarrow$ $^\exists$Regular symmetry breaking operators $\leftrightarrow$ open orbit in $P\backslash G/P'$



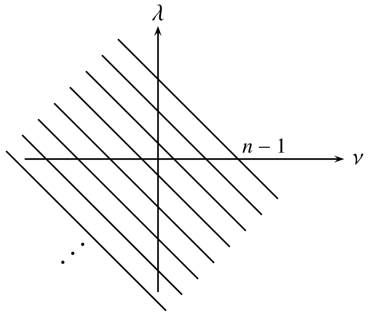
domain of convergence
$\rightsquigarrow$ meromorphic continuation
in $(\lambda,\nu) \in \mathbb{C}^2$

Continuous symmetry breaking operators

$$\begin{array}{ccccc} & \mathcal{L}_\lambda & & \mathcal{L}_\nu & \\ & \downarrow & & \downarrow & \\ G = O(n+1,1) & \overset{\text{conformal}}{\curvearrowright} \; S^n = G/P & \hookleftarrow & G'/P' = S^{n-1} \; \overset{\text{conformal}}{\curvearrowleft} & G' = O(n,1) \end{array}$$

$$\boxed{H(\lambda,\nu) := \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{L}_\lambda), C^\infty(G'/P', \mathcal{L}_\nu))}$$

$\lambda + \nu = n-1, n-3, n-5, \ldots$
$\Rightarrow$ $^\exists$Singular symmetry breaking operators $\leftrightarrow$ intermediate orbit
in $P\backslash G/P'$



Continuous symmetry breaking operators

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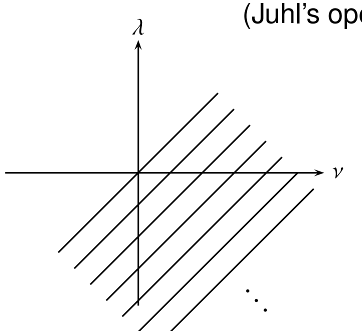
$$\boxed{H(\lambda,\nu) := \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{L}_\lambda), C^\infty(G'/P', \mathcal{L}_\nu))}$$

$\nu - \lambda = 0, 2, 4, 6, \ldots$
$\Rightarrow$ $^\exists$Differential symmetry breaking operators
(Juhl's operator in conformal geometry, 2009)
$\leftrightarrow$ closed orbit in $P\backslash G/P'$



Example A. Restriction of $A_{\mathfrak{q}}(\lambda)$-modules

$$\begin{array}{lll} G = O(n+1,1) & \curvearrowright A_{\mathfrak{q}}(i)^\infty & \supset A_{\mathfrak{q}}(i) \\ \cup & & \\ G' = O(n,1) & \curvearrowright A_{\mathfrak{q}'}(j)^\infty & \supset A_{\mathfrak{q}'}(j) \end{array}$$

$A_{\mathfrak{q}}(\lambda)^\infty \cdots$ Casselman–Wallach's smooth globalization of $A_{\mathfrak{q}}(\lambda)$

$$\boxed{\begin{array}{l} \text{Corollary (K–Speh)} \quad \text{For } i, j \in \mathbb{N} \\ \dim \mathrm{Hom}_{G'}(A_{\mathfrak{q}}(i)^\infty|_{G'}, A_{\mathfrak{q}'}(j)^\infty) \\ = \begin{cases} 1 & \text{if } i \geq j \text{ and } i \equiv j \mod 2 \\ 0 & \text{if } i < j \text{ or } i \not\equiv j \mod 2 \end{cases} \end{array}}$$

Remark. These $A_{\mathfrak{q}}(i)$ are not tempered if $n > 2$.

Continuous symmetry breaking operators

$$\begin{array}{ccccc} & & \mathcal{L}_\lambda & & \mathcal{L}_\nu & & \\ & & \downarrow & & \downarrow & & \\ G = O(n+1,1) & \overset{\text{conformal}}{\curvearrowright} & S^n = G/P & \hookleftarrow G'/P' = & S^{n-1} & \overset{\text{conformal}}{\curvearrowleft} & G' = O(n,1) \end{array}$$

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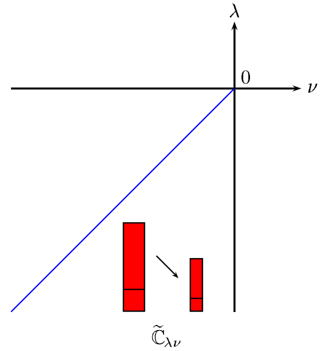

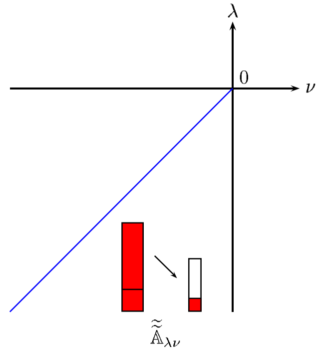

Branching problems of representations

1. Two examples in "nice settings"
   A. Continuous symmetry breaking operators
   B. Differential symmetry breaking operators
2. Finite multiplicity restrictions
3. Discretely decomposable restrictions
4. Applications

Example A. Restriction of $A_{\mathfrak{q}}(\lambda)$-modules

$$\begin{array}{lll} G = O(n+1,1) & \curvearrowright A_{\mathfrak{q}}(i)^\infty & \supset A_{\mathfrak{q}}(i) \\ \cup & & \\ G' = O(n,1) & \curvearrowright A_{\mathfrak{q}'}(j)^\infty & \supset A_{\mathfrak{q}'}(j) \end{array}$$

$A_{\mathfrak{q}}(\lambda)^\infty \cdots$ Casselman–Wallach's smooth globalization of $A_{\mathfrak{q}}(\lambda)$

$$C^\infty(G/P, \mathcal{L}_\lambda) \supset A_{\mathfrak{q}}(i)^\infty$$

$\lambda = n + i$

Example B. Differential symmetry breaking operators

$$\begin{array}{ccll} G & \supset & G' & \text{reductive groups} \\ \cup & & \cup & \\ K & \supset & K' & \text{maximal compact} \end{array}$$

$\boxed{\text{Assume } S^1 \simeq C(K) \subset K'}$

$$\begin{array}{ccc} \mathcal{V} & & \mathcal{W} \\ \downarrow & \cdots \text{holomorphic vector b'dle} \cdots & \downarrow \\ G/K & \hookleftarrow & G'/K' \\ \text{Hermitian symmetric} & \text{holomorphic} & \text{Hermitian symmetric} \end{array}$$

<u>Theorem B</u> (K–Pevzner, arXiv:1301.2111) Any continuous G'-homomorphism from $\mathcal{O}(G/K, \mathcal{V})$ to $\mathcal{O}(G'/K', \mathcal{W})$ is a differential operator.

<u>Example 1</u> $(G, G') = (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}),\ \Delta(SL(2,\mathbb{R})))$
$\cdots$ Rankin–Cohen bidifferential operator

<u>Example 2</u> $(G, G') = (SO(n,2),\ SO(n-1,2))$
$\cdots$ (a disguise of) Juhl's conformally covariant differential op (2009)

Further generalization (F-method) K–, Pevzner, Ørsted, Somberg, Souček

Example B. Differential symmetry breaking operators

G : simple linear Lie group
$\cup$
K : maximal compact subgroup

Assume G/K is a Hermitian symmetric space

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Example B. Differential symmetry breaking operators

G : simple linear Lie group
$\cup$
K : maximal compact subgroup
$\cup$
$C(K)$: center of K

Assume G/K is a Hermitian symmetric space $\Leftrightarrow C(K) \simeq S^1$

<u>Eg.</u> $G = SL(2,\mathbb{R}),\ Sp(n,\mathbb{R}),\ SU(p,q),\ SO(n,2),\ \ldots$

Finite dimensional K-module V

$\rightsquigarrow$ G-equivariant holomorphic vector b'dle $\mathcal{V} := G \times_K V$ over G/K

$\rightsquigarrow$ (highest weight) representation of G on $\mathcal{O}(G/K, \mathcal{V})$

Branching problem of representations

Setting $\quad G \supset G'$: real linear reductive groups

Branching Problem
Wish to understand how an irreducible rep π of G behaves as a rep of the subgroup G'

This setting is too wide, and branching behavior might be too wild.

Example (bad feature) Irreducible decomposition of

- the tensor product of two irred unitary reps of $SL(3,\mathbb{R})$
- restriction of irred reps of $GL(4,\mathbb{R})$ to $GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$
- restriction of irred reps of $SO(5,\mathbb{C})$ to $SO(3,2)$

may involve infinite multiplicities.

Branching problem of representations

Setting $\quad G \supset G'$: real linear reductive groups

Branching Problem
Wish to understand how an irreducible rep π of G behaves as a rep of the subgroup G'

E.g. $\quad \pi$ irreducible unitary rep of G:
branching law = irreducible decomposition of $\pi|_{G'}$.

We shall also consider non-unitary rep π

Branching problem of representations

Setting $\quad G \supset G'$: real linear reductive groups

Branching Problem
Wish to understand how an irreducible rep π of G behaves as a rep of the subgroup G'

Project
Single out 'nice framework' of branching problems in which we could go further.

Branching problem of representations

Setting $\quad G \supset G'$: real linear reductive groups

Branching Problem
Wish to understand how an irreducible rep π of G behaves as a rep of the subgroup G'

Example (tensor product)
$(G, G') = (G_1 \times G_1, \Delta G_1)$ (group case)
Tensor product $\pi' \otimes \pi''$ of two irred reps π', π'' of G_1 is the restriction

$$\pi' \otimes \pi'' = (\pi' \boxtimes \pi'')|_{\Delta G_1}$$
$$\circlearrowleft$$
$$G_1 \times G_1$$

Branching problems of representations

1. Two examples in “nice settings”
2. Finite multiplicity restrictions
   - admissible representations (review)
   - general problems
   - real spherical homogeneous spaces
   - (PP) and (BB) — criteria for finite-multiplicities
3. Discretely decomposable restrictions
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‘Nice behaviors’ of branching laws

Try to reveal underlying representation-theoretic / geometric properties

	Example A (K–Speh)	Example B (K–Pevzner)
	$\Uparrow$	$\Uparrow$
Rep Theory (branching)	Bounded multiplicities	Discretely decomposable restriction + finite multiplicities
	$\Uparrow$	$\Uparrow$
geometry	(real) spherical varieties	singularity spectrum / projection of nilpotent orbits

Harish-Chandra’s admissibility theorem (review)

$$\underset{\text{real reductive}}{G} \supset \underset{\text{max compact}}{K}$$

$\boxed{\begin{array}{l}\underline{\text{Fact}}\ \text{(Harish-Chandra's admissibility theorem)}\\ \quad \text{For any irreducible unitary rep } \pi \text{ of } G,\\ \quad \dim \mathrm{Hom}_K(\tau, \pi|_K) < \infty \quad \text{for } {}^\forall\tau \in \widehat{K}\end{array}}$

$\rightsquigarrow$

Nice property of branching for $G \downarrow K$

$\rightsquigarrow$

$\boxed{\begin{array}{l}\text{Analytic sutdy of unitary reps } \pi\\ \qquad \Downarrow\\ \text{Algebraic study of } (\mathfrak{g}, K)\text{-modules } \pi_K\end{array}}$

Intertwining operators for restriction

$$\begin{array}{ccccc} G & \supset & G' & & \text{reductive groups}\\ \text{irred} \circlearrowleft & & \circlearrowright \text{irred} & & \\ \pi & \begin{array}{c}\longrightarrow\\ \hookleftarrow\end{array} & \tau & & \end{array}$$

$\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K)$	$\mathrm{Hom}_{G'}(\tau, \pi\vert_{G'})$	$\mathrm{Hom}_{G'}(\pi\vert_{G'}, \tau)$
algebraic hom	continuous hom	continuous hom

- Nice framework for branching 1
  $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ (finite-multiplicity restriction)
- Nice framework for branching 2
  $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K) \neq 0$ (discretely decomposable restriction)

Branching problems of representations

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Admissible smooth representations (review)

Consider non-unitary reps π, too.

G real reductive linear Lie group
K maximal compact subgroup

π: continuous representation of finite length on a Banach space $\mathcal{H}$

Def π is admissible $\iff \dim \mathrm{Hom}_K(\tau, \pi|_K) < \infty$ for any $\tau \in \widehat{K}$

π^∞: representation on the space $\mathcal{H}^\infty$ of smooth vectors.

Def We say π^∞ is an admissible smooth representation.

Finite multiplicity restriction (plan)

$$\begin{array}{ccc} G & \supset & G' \\ \text{irred} \circlearrowleft & & \circlearrowright \text{irred} \\ \pi & \dashrightarrow & \tau \end{array}$$

General Problem (first project) Find a criterion for
1) $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ to be finite-dimensional, ${}^\forall \pi \in \widehat{G}_{\mathrm{adm}}$ and ${}^\forall \tau \in \widehat{G'}_{\mathrm{adm}}$.
2) $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ to be uniformly bounded w.r.t. π and τ.

$\Downarrow$ $\leftarrow$ criterion (explain later)

Classification (second project)
Find all the pairs (G, G') satisfying 1) (or more strongly, 2)).

Casselman–Wallach globalization theory (review)

$$\begin{array}{ccccc} \pi_K & & \pi^\infty & & \pi \\ \mathcal{H}_K & \underset{\text{dense}}{\subset} & \mathcal{H}^\infty & \underset{\text{dense}}{\subset} & \mathcal{H} \end{array}$$

Fact (Casselman–Wallach) Canonical equivalence of categories
$\{(\mathfrak{g}, K)$-modules π_K of finite length$\}$
$\longleftrightarrow$ {smooth admissible representations π^∞}

$\widehat{G}_{\mathrm{adm}}$ = {irreducible smooth admissible representations}/ $\sim$

Basic question in non-commutative harmonic analysis

G : real reductive linear group $\curvearrowright$ X

Non-commutative harmonic analysis
$\cdots$ wish to understand "functions on X"
by means of irreducible reps of G.

Basic Question

For which G-spaces X, does the following condition hold?
1) $\mathrm{Hom}_G(\pi, C^\infty(X))$ is finite-dimensional for all $\pi \in \widehat{G}_{\mathrm{adm}}$.
2) $^\exists C > 0$ such that $\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \le C$
for all $\pi \in \widehat{G}_{\mathrm{adm}}$.

X : reductive symmetric space $\Rightarrow$ 2) $\overset{\text{obvious}}{\Rightarrow}$ 1)

Finite multiplicity restriction (plan)

$$\begin{array}{ccc} G & \supset & G' \\ \text{irred} \circlearrowleft & & \circlearrowright \text{irred} \\ \pi & \dashrightarrow & \tau \end{array}$$

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2) $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ to be uniformly bounded w.r.t. π and τ.

⇝ 'good framework'

Further Problem
Analyze not only abstract representations but
explicitly symmetry breaking operators
(analytic refinement of branching laws)

e.g. Theorem A (K–Speh), Theorem B (K–Pevzner)

Spherical vs real spherical

$G_{\mathbb{C}}$ complex reductive $\supset H_{\mathbb{C}}$ subgroup

Definition $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if Borel subgroup B
has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$.

$\Longleftrightarrow \#(B\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$ (Brion, Vinberg) (~1986)

G real reductive $\supset H$ subgroup

Definition (K–) G/H is real spherical if
minimal parabolic P has an open orbit in G/H.

$\Longleftrightarrow \#(P\backslash G/H) < \infty$ (Matsuki et al.) (~1990s)

G/H symmetric space $\underset{\not\Leftarrow}{\Rightarrow}$ $G_{\mathbb{C}}/H_{\mathbb{C}}$ spherical $\underset{\not\Leftarrow}{\Rightarrow}$ G/H real spherical

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Answer to basic questions on $C^\infty(G/H)$

$G \supset H$ real reductive linear groups

Theorem C (K–T. Oshima, Adv. Math. 2013) (i) and (ii) are equivalent.
(i) (Finite multiplicities)
$$\dim \mathrm{Hom}_G(\pi, C^\infty(G/H)) < \infty$$
for any admissible smooth rep π of G.
(ii) G/H is real spherical.

Corollary (1) and (2) are equivalent
(1) (Bounded multiplicities) ${}^\exists C > 0$ s.t.
$$\dim \mathrm{Hom}_G(\pi, C^\infty(G/H)) \le C,$$
for any admissible smooth rep π of G.
(2) $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.

Point The property of "bounded multiplicities" in $C^\infty(G/H)$ depends only on the complexification of G/H.

Answer to basic questions on $C^\infty(G/H)$

$G \supset H$ real reductive linear groups

Theorem C (K–T. Oshima, Adv. Math. 2013) (i) and (ii) are equivalent.
(i) (Finite multiplicities)
$$\dim \mathrm{Hom}_G(\pi, C^\infty(G/H)) < \infty$$
for any admissible smooth rep π of G.
(ii) G/H is real spherical.

Methods of proof

(ii) $\Rightarrow$ (i)
Equivariant compactification + micro-local analysis
("hyperfunction-valued boundary maps")

(i) $\Rightarrow$ (ii)
Construction of integral intertwining operators
(Cf. Knapp–Stein, Poisson–Fourier, Jacquet integral, ...)

Multiplicities in $C^\infty(G/H)$ and branching $G \downarrow G'$

In summary

(Geometry)

G/H symmetric $\underset{\not\Leftarrow}{\Rightarrow}$ (2) $G_{\mathbb{C}}/H_{\mathbb{C}}$ spherical $\underset{\not\Leftarrow}{\Rightarrow}$ (ii) G/H real spherical

$\Updownarrow$ Theorem C $\qquad$ $\Updownarrow$ Corollary

(1) bounded multiplicities $\Rightarrow$ (i) finite multiplicities

(Regular representation on $C^\infty(G/H)$)

Answer to basic questions on $C^\infty(G/H)$

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(i) (Finite multiplicities)
$$\dim \mathrm{Hom}_G(\pi, C^\infty(G/H)) < \infty$$
for any admissible smooth rep π of G.
(ii) G/H is real spherical.

Remark.
1) Theorem holds in a more general setting:
- non-reductive H,
- sections for any G-equivariant vector bundle.

2) Upper/lower estimates of the multiplicities are obtained.

Eg. Kostant–Lynch theory for Whittaker functions ($H = N$)

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Geometric property (PP)

Let $G \supset G'$ be real reductive linear groups.

$$\begin{array}{ccc} G & \supset & G' \\ \cup & & \cup \\ P & & P' \end{array}$$

P P' minimal parabolic subgroups

Definition–Lemma We say (G, G') satisfies (PP) if one of the following equivalent conditions are satisfied:

(PP1) $(G \times G')/\Delta G'$ is real spherical as a $(G \times G')$-space.
(PP2) G/P' is real spherical as a G-space.
(PP3) G/P is real spherical as a G'-space.
(PP4) G has an open orbit in $G/P \times G/P'$ via the diagonal action.
(PP5) $\#(\Delta G \backslash (G \times G)/(P \times P')) = \#(P \backslash G/P') < \infty$.

Geometric property (BB)

Let $G \supset G'$ be real reductive linear groups.

$$\begin{array}{ccc} G & \supset & G' \\ \cap & & \cap \ \text{complexification} \\ G_{\mathbb{C}} & \supset & G'_{\mathbb{C}} \\ \cup & & \cup \\ B & & B' \end{array}$$

B B' Borel subgroups

Definition–Lemma We say (G, G') satisfies (BB) if one of the following equivalent conditions are satisfied:

(BB1) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\Delta G'_{\mathbb{C}}$ is spherical as a $(G_{\mathbb{C}} \times G'_{\mathbb{C}})$-space.
(BB2) $G_{\mathbb{C}}/B'$ is spherical as a $G_{\mathbb{C}}$-space.
(BB3) $G_{\mathbb{C}}/B$ is spherical as a $G'_{\mathbb{C}}$-space.
(BB4) $G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/B \times G_{\mathbb{C}}/B'$ via the diagonal action.
(BB5) $\#(\Delta G_{\mathbb{C}} \backslash (G_{\mathbb{C}} \times G_{\mathbb{C}})/(B \times B')) = \#(B \backslash G_{\mathbb{C}}/B') < \infty$.

Geometric properties (PP) and (BB)

Easy to see

$$\text{(BB)} \quad \Longrightarrow \quad \text{(PP)}$$

Multiplicities of the restriction of reps $G \downarrow G'$

In summary

$$\begin{array}{ccc}
\text{(complex geometry)} & & \text{(real geometry)} \\
\text{(BB)} & \Longrightarrow & \text{(PP)} \\
\Updownarrow \text{Theorem D} & & \Updownarrow \text{Corollary} \\
\text{Bounded multiplicities} & \Longrightarrow & \text{Finite multiplicities}
\end{array}$$

(the restriction of irred reps of G to G')

Finiteness Theorem in branching problem

Theorem D ([Adv.Math.2003] + Progr.Math. (to appear)) The following three conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:

- (i) (Symmetry breaking) $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ for any admissible smooth reps π of G and any τ of G'.
- (ii) (Invariant bilinear forms) $\dim \mathrm{Hom}_{G'}(\pi|_{G'} \widehat{\otimes} \tau, \mathbb{C}) < \infty$ for any admissible smooth π of G and any τ of G'.
- (iii) (Geometric property (PP)) $^{\exists}$minimal parabolic subgroups $P \subset G$ and $P' \subset G'$ such that PP' is open in G.

Best possible constant C

$$C := \sup_{\pi \in \widehat{G}_{\mathrm{adm}}} \sup_{\tau \in \widehat{G'}_{\mathrm{adm}}} \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$$

Example $(G, G') = (GL(n+1, \mathbb{F}),\ GL(n, \mathbb{F}) \times GL(1, \mathbb{F}))$

1) (Aizenbud–Gourevitch et al., Sun–Zhu)
$C = 1$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

2) If $\mathbb{F} = \mathbb{H}$, then (BB) fails but (PP) holds.
$\Rightarrow$ Each multiplicity (the dimension of symmetry breaking operators) is finite, but is not uniformly bounded (i.e. "$C = \infty$").

Uniformly bounded theorem

Corollary The following three conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:

- (i) (Symmetry breaking) $^{\exists}C > 0$ such that $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \le C$ ($^{\forall}\pi \in \widehat{G}_{\mathrm{adm}}$, $^{\forall}\tau \in \widehat{G'}_{\mathrm{adm}}$).
- (ii) (Invariant bilinear forms) $^{\exists}C > 0$ such that $\dim \mathrm{Hom}_{G'}(\pi|_{G'} \widehat{\otimes} \tau, \mathbb{C}) \le C$ ($^{\forall}\pi \in \widehat{G}_{\mathrm{adm}}$, $^{\forall}\tau \in \widehat{G'}_{\mathrm{adm}}$).
- (iii) (Geometric property (BB)) $^{\exists}$Borel subgroups $B \subset G_{\mathbb{C}}$ and $B' \subset G'_{\mathbb{C}}$ such that BB' is open in $G_{\mathbb{C}}$.

(iii) $\Rightarrow$ (i), (ii): follows also from the classification of the pairs (G, G') satisfying (BB) and the multiplicity-one theorem by Aizenbud–Gourevitch et al., Sun–Zhu (Ann Math 2012)

Examples of finite/bounded multiplicities

Example (cf. Gross–Prasad conjecture for (1) and (2))

1) $(O(p+1,q),\ O(p,q)\times O(1))$ (BB) holds.
2) $(U(p+1,q),\ U(p,q)\times U(1))$ (BB) holds.
3) $(Sp(p+1,q),\ Sp(p,q)\times Sp(1))$ (BB) fails, (PP) holds.

Example $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8,1))$ (PP) holds.
$(\mathfrak{e}_{6(-26)}, \mathfrak{o}(9,1)+\mathbb{R})$ (PP) holds.

Theorem E (K–Matsuki, Trans Groups (2014)—Dynkin's 90th birthday)
Gave a complete classification of the symmetric pairs (G, G') satisfying (PP) (and also that of (BB)).

Tri-linear invariant forms

G : a simple Lie group

Corollary 2 Equivalent on $\mathfrak{g}$:

(1) For any triple of irred reps π_1, π_2, and $\pi_3 \in \widehat{G}_{\mathrm{adm}}$
$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) < \infty$

(1)′ For any triple of irred reps π_1, π_2, and $\pi_3 \in \widehat{G}_{\mathrm{adm}}$
$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2, \pi_3) < \infty$

(2) $\mathfrak{g} \simeq \mathfrak{o}(n,1)$ $(n \geq 2)$ or G is compact

Cf. For $\mathfrak{g} = \mathfrak{su}(n,1)$ $(n \geq 2)$, $\mathfrak{sp}(n,1)$, $\mathfrak{f}_{4(-20)}$
$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) = \infty$
for some π_1, π_2 and π_3.

Classification of (G, G') with (PP)

Theorem E Let (G, G') be a reductive symmetric pair. Then, (iv) ⇔ (v).

(iv) $(G \times G')/\Delta G'$ is real spherical (i.e. (G, G') satisfies (PP)).

(v) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

(A) (Easy case)

(A–1) $\mathfrak{g} = \mathfrak{g}'$
(A–2) $\mathfrak{g} = \mathbb{R}$
(A–3) G is compact
(A–4) $G' = K$

(B) (Strong Gelfand pairs and real forms)

(B–1) $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$ $(n \geq 2)$.
(B–2) $(\mathfrak{o}(n+1,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))$ $(n \geq 2)$.
(B–3) $(\mathfrak{sl}(n+1,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$ $(n \geq 1)$.
(B–4) $(\mathfrak{su}(p+1,q), \mathfrak{u}(p,q))$ $(p+q \geq 1)$.
(B–5) $(\mathfrak{o}(p+1,q), \mathfrak{o}(p,q))$ $(p+q \geq 2)$.

(C) (Split rank one case)

(C–1) $(\mathfrak{o}(p+q,1), \mathfrak{o}(p)+\mathfrak{o}(q,1))$ $(p+q \geq 2)$.
(C–2) $(\mathfrak{su}(p+q,1), \mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(q,1)))$ $(p+q \geq 1)$.
(C–3) $(\mathfrak{sp}(p+q,1), \mathfrak{sp}(p)+\mathfrak{sp}(q,1))$ $(p+q \geq 1)$.
(C–4) $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8,1))$.

(D) (Other cases)

(D–1) $(\mathfrak{o}(n,1)+\mathfrak{o}(n,1), \mathrm{diag}\,\mathfrak{o}(n,1))$ $(n \geq 2)$.
(D–2) $(\mathfrak{su}^*(2n+2), \mathfrak{su}(2)+\mathfrak{su}^*(2n)+\mathbb{R})$.
(D–3) $(\mathfrak{o}^*(2n+2), \mathfrak{o}(2)+\mathfrak{o}^*(2n))$ $(n \geq 1)$.
(D–4) $(\mathfrak{sp}(p+1,q), \mathfrak{sp}(p,q)+\mathfrak{sp}(1))$.
(D–5) $(\mathfrak{o}(2n,2), \mathfrak{u}(n,1))$.
(D–6) $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1)+\mathbb{R})$.

Tri-linear invariant forms

G : a simple Lie group

Corollary 2 Equivalent on $\mathfrak{g}$:

(1) For any triple of irred reps π_1, π_2, and $\pi_3 \in \widehat{G}_{\mathrm{adm}}$
$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) < \infty$

(1)′ For any triple of irred reps π_1, π_2, and $\pi_3 \in \widehat{G}_{\mathrm{adm}}$
$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2, \pi_3) < \infty$

(2) $\mathfrak{g} \simeq \mathfrak{o}(n,1)$ $(n \geq 2)$ or G is compact

Corollary 3 $\sup_{\pi_1,\pi_2,\pi_3 \in \widehat{G}_{\mathrm{adm}}} \dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) < \infty$
$\iff \mathfrak{g} \simeq \mathfrak{o}(2,1)$, $\mathfrak{o}(3,1)$, or $\mathfrak{o}(3)$

- Pukánszky, Williams, Repka (Decomposition of $\pi_1 \otimes \pi_2$ for $SL(2,\mathbb{R})$) Note: $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{o}(2,1)$
- Bernstein–Reznikov integral (Clerc–K–Ørsted–Pevzner 2011)
$\mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C})$ for $G = O(n,1)$

Idea of proof for Theorem E

Extend the idea for the group case

Proposition (K– '95) G simple

$(G \times G \times G)/\Delta G$ real spherical

$\overset{\text{linearization}}{\Longleftrightarrow}$ MA has an open orbit in $\mathfrak{n}$

$\overset{\text{root system}}{\Longleftrightarrow}$ G is compact or $\mathfrak{g} \simeq \mathfrak{so}(n,1)$

For general case $(G \times G')/\Delta G'$, we use

linearization + root system + α (prehomogeneous spaces)

- invariant theory for quivers
- replace (PP) by (QP)
- ...

Classification of (G, G') with (BB)

Theorem E Let (G, G') be a reductive symmetric pair. Then, (iv) ⇔ (v).

(iv) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\Delta G'_{\mathbb{C}}$ is spherical (i.e. (G, G') satisfies (BB)).

(v) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

(A) (Easy case)

(A–1) $\mathfrak{g} = \mathfrak{g}'$

(A–2) $\mathfrak{g} = \mathbb{R}$

(A–3) G is compact

(A–4) $G' = K$

(B) (Strong Gelfand pairs and real forms)

(B–1) $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$ $(n \geq 2)$.

(B–2) $(\mathfrak{o}(n+1,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))$ $(n \geq 2)$.

(B–3) $(\mathfrak{sl}(n+1,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$ $(n \geq 1)$.

(B–4) $(\mathfrak{su}(p+1,q), \mathfrak{u}(p,q))$ $(p+q \geq 1)$.

(B–5) $(\mathfrak{o}(p+1,q), \mathfrak{o}(p,q))$ $(p+q \geq 2)$.

(C) (Split rank one case)

(C–1) $(\mathfrak{o}(p+q,1), \mathfrak{o}(p)+\mathfrak{o}(q,1))$ $(p+q \geq 2)$.

(C–2) $(\mathfrak{su}(p+q,1), \mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(q,1)))$ $(p+q \geq 1)$.

(C–3) $(\mathfrak{sp}(p+q,1), \mathfrak{sp}(p)+\mathfrak{sp}(q,1))$ $(p+q \geq 1)$.

(C–4) $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8,1))$.

(D) (Other cases)

(D–1) $(\mathfrak{o}(n,1)+\mathfrak{o}(n,1), \operatorname{diag}\mathfrak{o}(n,1))$ $(n \geq 2)$.

(D–2) $(\mathfrak{su}^*(2n+2), \mathfrak{su}(2)+\mathfrak{su}^*(2n)+\mathbb{R})$.

(D–3) $(\mathfrak{o}^*(2n+2), \mathfrak{o}(2)+\mathfrak{o}^*(2n))$ $(n \geq 1)$.

(D–4) $(\mathfrak{sp}(p+1,q), \mathfrak{sp}(p,q)+\mathfrak{sp}(1))$.

(D–5) $(\mathfrak{o}(2n,2), \mathfrak{u}(n,1))$.

(D–6) $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1)+\mathbb{R})$.

Finite multiplicity restriction

$$\begin{array}{ccccl} G & \supset & G' & & \text{reductive groups} \\ \text{irred} \circlearrowleft & & \circlearrowright \text{irred} & & \\ \pi & \rightleftarrows & \tau & & \end{array}$$

$\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K)$	$\mathrm{Hom}_{G'}(\tau, \pi\vert_{G'})$	$\mathrm{Hom}_{G'}(\pi\vert_{G'}, \tau)$
algebraic hom	continuous hom	continuous hom

- Nice framework for branching 1
  $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ (finite-multiplicity restriction)
- Nice framework for branching 2
  $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K) \neq 0$ (discretely decomposable restriction)

Classification in special cases

Some cases in Theorem E were classified previously.

1) Complex symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$, $\mathfrak{g}_{\mathbb{C}}$: simple

(PP)

⇕ obvious

(BB)

⇕ Kostant, Krämer (1976)

$(\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}) = (\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$ or $(\mathfrak{o}(n+1,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))$

2) Group case $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$, $\mathfrak{g}$: simple

(PP)

⇕

$(G \times G \times G)/\Delta G$ is real spherical

⇕ K– (1995)

G is compact or $\mathfrak{g} \simeq \mathfrak{so}(n,1)$

‘Nice framework’ 2 (discrete decomposability)

$G \supset G'$ real reductive, $\pi \in \widehat{G}$ irred. unitary

Definition ('93) We say π is G'-admissible if
$\pi|_{G'} \simeq \sum^{\oplus}_{\tau \in \widehat{G'}} m_\pi(\tau)\tau$ (discrete Hilbert sum)
with multiplicity $m_\pi(\tau) < \infty$ for all $\tau \in \widehat{G'}$

Nice framework of branching for $G \downarrow G'$

$\wr$

Analytic study of branching problems
of unitary reps for $G \downarrow G'$

$\Downarrow \; \Uparrow$

Algebraic study of branching
for $(\mathfrak{g}, K)$-modules to $(\mathfrak{g}', K')$-modules

Branching problems of representations

1. Two examples in “nice settings”
2. Finite multiplicity restrictions
3. Discretely decomposable restrictions
   - criterion (review)
   - classification ($A_{\mathfrak{q}}(\lambda)$-module case)
4. Applications

‘Nice framework’ 2 (discrete decomposability)

$G \supset G'$ real reductive, $\pi \in \widehat{G}$ irred. unitary

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with multiplicity $m_\pi(\tau) < \infty$ for all $\tau \in \widehat{G'}$

as a ‘good framework’ for branching problem

Criterion $\cdots$ K– ’90s
Classification $\cdots$ K–Y. Oshima (2012, 2014)
Explicit branching laws in this framework
$\cdots$ K– ’88, ’93, Gross–Wallach ’94, J.-S. Li, J.-S. Huang–P. Pandžić–G. Savin, H.-Y. Loke, W. Bertram–J. Hilgert, K–Ørsted, B. Speh, M. Duflo–J. Vargas, G. Zhang, H. Sekiguchi, Yoshiki Oshima, J. Möllers, K–F. Kassel, . . .

Discretely decomposable restriction $G \downarrow G'$

$G \supset G'$ (reductive groups) $\quad \pi$: irreducible unitary rep of G

Def 1 Restriction $\pi|_{G'}$ is discretely decomposable if

$$\pi|_{G'} \simeq \sum^{\oplus}_{\tau \in \widehat{G'}} n_\pi(\tau)\tau \quad \text{(discrete Hilbert sum).}$$

$\wr$ algebraic analog including non-unitary case

Def 2 ([Invent 97]) Restriction $X|_{\mathfrak{g}'}$ of an irreducible $\mathfrak{g}$-module X is discretely decomposable if $^{\exists}$increasing filtration $\{X_j\}$ of $\mathfrak{g}'$-modules of finite length s.t. $X = \bigcup_j X_j$

To be precise, if the underlying $(\mathfrak{g}, K)$-module π_K of a unitary rep π of G satisfies Def 2, then

$$\pi_K|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\tau \in \widehat{G'}} n_\pi(\tau)\tau_{K'} \quad \text{(algebraic direct sum)}$$

$$\pi|_{G'} \simeq \sum^{\oplus} n_\pi(\tau)\tau \quad \text{(Hilbert direct sum)}$$

Criterion for discretely decomposable restrictions

$G \supset G'$ real reductive groups
$K \supset K'$ maximal compact subgroups

Fact (K– Invent 97, Ann Math 98) Let X be an irreducible $(\mathfrak{g}, K)$-module.
1) $X|_{\mathfrak{g}'}$ is discretely decomposable as a $\mathfrak{g}'$-module $\Rightarrow$ ①
2) ② $\Rightarrow X|_{\mathfrak{g}'}$ is discretely decomposable & finite multiplicity.

① $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}(X)) \subset \mathcal{N}_{\mathfrak{g}'_{\mathbb{C}}}$

② $\mathrm{AS}_K(X) \cap \mu(T^*(K/K')) = \{0\}$

① $\mathcal{V}(X)$ (associated variety) $\subset \mathcal{N}_{\mathfrak{g}_{\mathbb{C}}}$ (nilpotent cone) $\cap (\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$ (Joseph, Vogan)
$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}_{\mathbb{C}}^* \to (\mathfrak{g}'_{\mathbb{C}})^*$ projection

Algebraic proof : $\mathcal{V}(X)$ is an "approximation" of X.

Criterion for discretely decomposable restrictions

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2) ② $\Rightarrow X|_{\mathfrak{g}'}$ is discretely decomposable & finite multiplicity.

Finite multiplicity
$\underset{\text{def}}{\iff} \dim \mathrm{Hom}_{\mathfrak{g}',K'}(Y, X|_{\mathfrak{g}'}) < \infty$ for any irreducible $(\mathfrak{g}', K')$-module Y.

Criterion for discretely decomposable restrictions

$G \supset G'$ real reductive groups
$K \supset K'$ maximal compact subgroups

Fact (K– Invent 97, Ann Math 98) Let X be an irreducible $(\mathfrak{g}, K)$-module.
1) $X|_{\mathfrak{g}'}$ is discretely decomposable as a $\mathfrak{g}'$-module $\Rightarrow$ ①
2) ② $\Rightarrow X|_{\mathfrak{g}'}$ is discretely decomposable & finite multiplicity.

① $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}(X)) \subset \mathcal{N}_{\mathfrak{g}'_{\mathbb{C}}}$

② $\mathrm{AS}_K(X) \cap \mu(T^*(K/K')) = \{0\}$

② $\mathrm{AS}_K(X)$: asymptotic K-support (Kashiwara–Vergne)
$\mu : T^*(K/K') \to \sqrt{-1}\mathfrak{t}^*$ moment map

Idea : use singularity spectrum of hyperfunction characters

Criterion for discretely decomposable restrictions

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$K \supset K'$ maximal compact subgroups

Fact (K– Invent 97, Ann Math 98) Let X be an irreducible $(\mathfrak{g}, K)$-module.
1) $X|_{\mathfrak{g}'}$ is discretely decomposable as a $\mathfrak{g}'$-module $\Rightarrow$ ①
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② $\mathrm{AS}_K(X) \cap \mu(T^*(K/K')) = \{0\}$

Classification (extremal case) for $G \downarrow G^\sigma$

$\mathfrak{g}$: non-compact simple Lie algebra $/\mathbb{R}$
$\sigma \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\sigma^2 = \mathrm{id}$ and $\sigma(K) = K$.

Theorem F (K–Yoshiki Oshima, Crelles J. (online first))
(i) and (ii) are equivalent:
(i) $^\exists$infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X s.t. X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.
(ii) $\sigma\beta \neq -\beta$ where β is the highest non-compact root of $\mathfrak{g}$.

Corollary Gave a complete classification of all the symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ satisfying (i).

Restriction for symmetric pairs $G \downarrow G^\sigma$

$\mathfrak{g}$: reductive Lie algebra, $K \subset G$
$\sigma \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\sigma^2 = \mathrm{id}$, $\sigma(K) = K$

Consider the restriction of an irreducible $(\mathfrak{g}, K)$-module X to the subalgebra $\mathfrak{g}^\sigma := \{u \in \mathfrak{g} : \sigma u = u\}$.

Classification (extremal case) for $G \downarrow G^\sigma$

$\mathfrak{g}$: non-compact simple Lie algebra $/\mathbb{R}$
$\sigma \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\sigma^2 = \mathrm{id}$ and $\sigma(K) = K$.

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(i) and (ii) are equivalent:
(i) $^\exists$infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X s.t. X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.
(ii) $\sigma\beta \neq -\beta$ where β is the highest non-compact root of $\mathfrak{g}$.

Special case $G_{\mathbb{C}} \downarrow G_{\mathbb{R}}$
i.e. $\mathfrak{g}$ is complex Lie alg, and σ is complex conjugation.

(ii) $\iff$ $\mathfrak{g}_{\mathbb{R}}$ meets the minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$
$\iff$ $\mathfrak{g}_{\mathbb{R}}$ is compact, $\mathfrak{su}^*(2n)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(m,n)$, $\mathfrak{f}_{4(-20)}$, or $\mathfrak{e}_{6(-26)}$

(The last equivalence $\cdots$ Brylinski–Kostant, Okuda)

Restriction for symmetric pairs $G \downarrow G^\sigma$

$\mathfrak{g}$: reductive Lie algebra, $K \subset G$
$\sigma \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\sigma^2 = \mathrm{id}$, $\sigma(K) = K$

Consider the restriction of an irreducible $(\mathfrak{g}, K)$-module X to the subalgebra $\mathfrak{g}^\sigma := \{u \in \mathfrak{g} : \sigma u = u\}$.

Example 1 σ = Cartan involution
$\Longrightarrow X|_{\mathfrak{g}^\sigma}$ is always discretely decomposable.

Example 2 $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{R}))$ σ = complex conjugation
$\Longrightarrow X|_{\mathfrak{g}^\sigma}$ is never discretely decomposable
if $\dim X = \infty$.

'Nice framework' 2 (discrete decomposability)

$G \supset G'$ real reductive, $\pi \in \widehat{G}$ irred. unitary

Definition ('93) We say π is G'-admissible if

$$\pi|_{G'} \simeq \sum_{\tau \in \widehat{G'}}^{\oplus} m_\pi(\tau)\tau \text{ (discrete Hilbert sum)}$$

with multiplicity $m_\pi(\tau) < \infty$ for all $\tau \in \widehat{G'}$

as a 'good framework' for branching problem

Criterion ··· K– '90s

Classification ··· K–Y. Oshima (2012, 2014)

Explicit branching laws in this framework

··· K– '88, '93, Gross–Wallach '94, J.-S. Li, J.-S. Huang–P. Pandžić–G. Savin, H.-Y. Loke, W. Bertram–J. Hilgert, K–Ørsted, B. Speh, M. Duflo–J. Vargas, G. Zhang, H. Sekiguchi, Yoshiki Oshima, J. Möllers, K–F. Kassel, ...

Criterion for discretely decomposable restrictions

$G \supset G'$ real reductive groups

$K \supset K'$ maximal compact subgroups

Fact (K– '97, '98) Let X be an irreducible $(\mathfrak{g}, K)$-module.

1) $X|_{\mathfrak{g}'}$ is discretely decomposable as a $\mathfrak{g}'$-module $\Rightarrow$ ①

2) ② $\Rightarrow X|_{\mathfrak{g}'}$ is discretely decomposable & finite multiplicity.

① $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}(X)) \subset \mathcal{N}_{\mathfrak{g}'_{\mathbb{C}}}$

② $\mathrm{AS}_K(X) \cap \mu(T^*(K/K')) = \{0\}$

Geometry of Zuckerman's $A_{\mathfrak{q}}(\lambda)$-modules (review)

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \subset \mathfrak{g}^*}$ elliptic orbit through $\lambda \in \mathfrak{g}^*$

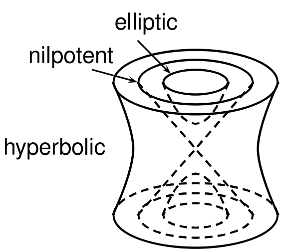
$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \simeq \mathfrak{g}$

reductive Lie gp

$\wr$

$\boxed{\pi_\lambda}$ unitary rep of G

(Vogan, Wallach '84)



Geometric aspect of $A_{\mathfrak{q}}(\lambda)$

Discrete decomposability of the restriction of $A_{\mathfrak{q}}(X)$

The criteria ①, ② are computable (and equivalent), for instance, if

$X = A_{\mathfrak{q}}(\lambda)$ Zuckerman derived functor module

$(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{g}, \mathfrak{g}^\sigma)$ symmetric pair

Corollary (K– Invent 97) Let $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ be a θ-stable parabolic subalg of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$, and $\sigma \in \mathrm{Aut}(\mathfrak{g})$ with $\sigma^2 = \mathrm{id}$.
Then (i), (i)′, and (ii) are equivalent:

(i) The restriction $X|_{\mathfrak{g}^\sigma}$ is discretely decomposable.

(i)′ The restriction $X|_{\mathfrak{g}^\sigma}$ is discretely decomposable with finite multiplicities.

(ii) $\mathbb{R}_+$-span $\Delta(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) \cap \{\alpha : \sigma\alpha = -\alpha\} = \{0\}$.

$\Longrightarrow$ complete classification of the triple

$$(\mathfrak{g}, \mathfrak{q}, \sigma)$$

satisfying one of (all of) the conditions (i), (i)′, and (ii) (K–Y.O 2012)

$K_{\mathbb{C}}$-orbits on the flag variety $G_{\mathbb{C}}/B$

$$\begin{array}{ccccc} G & \subset & G_{\mathbb{C}} & \supset & B \ \text{Borel} \\ \cup & & \cup & & \\ K & \subset & K_{\mathbb{C}} & & \end{array}$$

$$\begin{array}{ccc} \{\theta\text{-stable parabolic subalgebras in } \mathfrak{g}_{\mathbb{C}}\} & \longrightarrow & K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B \\ \Downarrow \!\!\!\!\! \in & & \Downarrow \!\!\!\!\! \in \\ \mathfrak{q} & \longmapsto & \mathcal{O} \end{array}$$

such that

$$\overline{\bigcup_{\mathfrak{b}\in\mathcal{O}} \mathfrak{b}} = \mathrm{Ad}(K_{\mathbb{C}})\cdot\mathfrak{q}$$

Geometry of Zuckerman's $A_{\mathfrak{q}}(\lambda)$-modules (review)

$G_{\mathbb{C}}/Q$

$\Uparrow$ open — ($^{\exists}Q \subset G_{\mathbb{C}}$ parabolic)

$\boxed{O_\lambda = \mathrm{Ad}^*(G)\lambda \subset \mathfrak{g}^*}$ — elliptic orbit through $\lambda \in \mathfrak{g}^*$ — $G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \simeq \mathfrak{g}$, G reductive Lie gp

$\Downarrow$ complex structure

$\boxed{\mathcal{L}_{\lambda+\rho} \to O_\lambda}$ — G-equiv. holo. line b'dle

$\Downarrow$

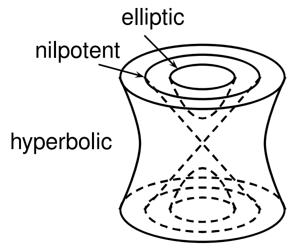
$\boxed{H^*_{\bar\partial}(O_\lambda, \mathcal{L}_{\lambda+\rho})}$ — G-module(Fréchet space) maximal globalization (Schmid, Wong '91)

$\cup$ dense

$\boxed{^{\exists 1}\pi_\lambda}$ — unitary rep of G (Vogan, Wallach '84)

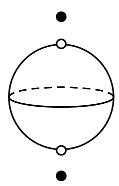
$\cup$ dense

$A_{\mathfrak{q}}(\lambda-\rho)$



Geometric aspect of $A_{\mathfrak{q}}(\lambda)$

$GL(1,\mathbb{C}) \times GL(1,\mathbb{C}) \curvearrowright GL(2,\mathbb{C})/B$



$= X$

$= U$

$= Y$

$\rightsquigarrow$

$$\begin{array}{ccc|c} X & & Y & \text{dimension } 0 \\ & U & & 1 \end{array}$$

$\mathbb{P}^1\mathbb{C} \simeq GL(2,\mathbb{C})/B$

Basic properties of Zuckerman's $A_{\mathfrak{q}}(\lambda)$-modules (review)

$\boxed{O_\lambda = \mathrm{Ad}^*(G)\cdot\lambda \text{ elliptic orbit } \rightsquigarrow \pi_\lambda := \overline{A_{\mathfrak{q}}(\lambda)} \text{ unitary rep of } G}$

Geometry

$\mathfrak{g}^* \ni \lambda \rightsquigarrow Q \underset{\text{parabolic}}{\subset} G_{\mathbb{C}}$

$G_{\mathbb{C}}/Q \underset{\text{closed}}{\supset} K_{\mathbb{C}}\cdot o \rightsquigarrow \mathcal{D}\text{-module } i_+(O_{K_{\mathbb{C}}\cdot o}(\lambda))$

$\updownarrow$ Hecht–Milicic–Schmid–Wolf duality

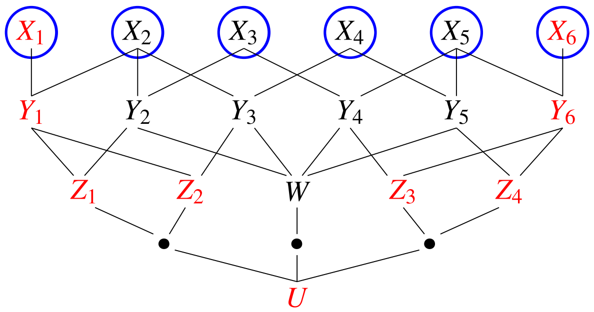
$G_{\mathbb{C}}/Q \underset{\text{open}}{\supset} \boxed{G\cdot o \simeq O_\lambda} \underset{\text{geometric quantization}}{\rightsquigarrow} H^S_{\bar\partial}(O_\lambda, \mathcal{L}_{\lambda+\rho})$

$\updownarrow$

$A_{\mathfrak{q}}(\lambda)$ algebraic construction

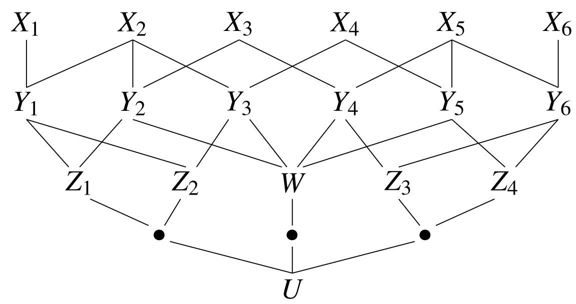
- Enright–Varadarajan (Ann Math '75), Parthasarathy (Comp Math '78)
- Zuckerman (lectures at IAS), Vogan's Green Book ('81)

$K_{\mathbb{C}} \curvearrowright G_{\mathbb{C}}/B \Rightarrow$ **reps of** $G = U(2,2)$



○ discrete series representations
• unitary highest weight modules
⊙ holomorphic (anti-holomorphic) discrete series

$GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \curvearrowright GL(4,\mathbb{C})/B$



dimension
2
3
4
5
6

Discrete decomposability of the restriction of $A_{\mathfrak{q}}(X)$

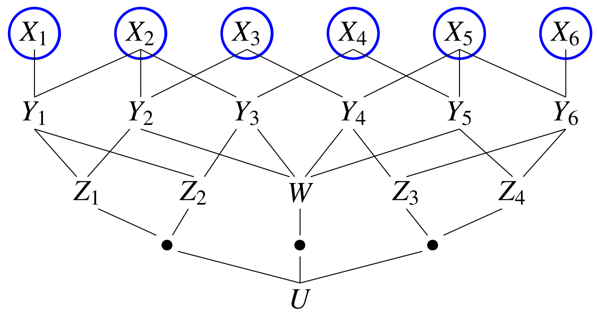
The criteria ①, ② are computable (and equivalent), for instance, if

$X = A_{\mathfrak{q}}(\lambda)$ Zuckerman derived functor module
$(\mathfrak{g}, \mathfrak{g}')$ symmetric pair

Corollary (K– '97) Let $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ be a θ-stable parabolic subalg of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$, and $\sigma \in \mathrm{Aut}(\mathfrak{g})$ with $\sigma^2 = \mathrm{id}$.
Then (i), (i)′, and (ii) are equivalent:

(i) The restriction $X|_{\mathfrak{g}^\sigma}$ is discretely decomposable.
(i)′ The restriction $X|_{\mathfrak{g}^\sigma}$ is discretely decomposable with finite multiplicities.
(ii) $\mathbb{R}_+$-span $\Delta(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) \cap \{\sigma\alpha = -\alpha\} = \{0\}$.

$K_{\mathbb{C}} \curvearrowright G_{\mathbb{C}}/B \Leftarrow$ **reps of** $G = U(2,2)$

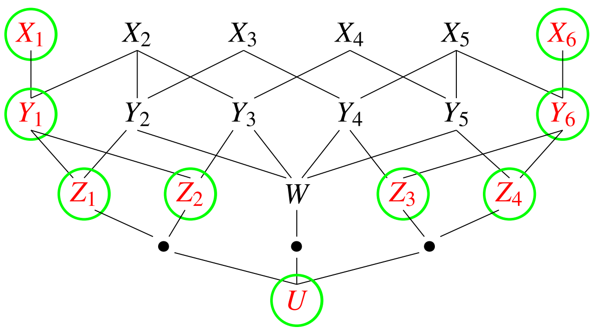


discrete ser.

trivial rep

○ discrete series for $L^2(G)$

$U(2,2) \downarrow U(2,1) \times U(1)$



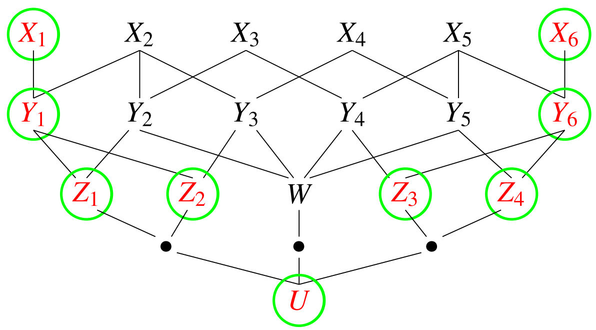
discrete ser.

trivial rep

○ restriction of $\overline{A_{\mathfrak{q}}(\lambda)}$ to $U(2,1) \times U(1)$ is discretely decomposable

- unitary highest weight modules

$U(2,2) \downarrow U(2,1) \times U(1)$



discrete ser.

trivial rep

○ restriction of $\overline{A_{\mathfrak{q}}(\lambda)}$ to $U(2,1) \times U(1)$ is discretely decomposable

- unitary highest weight modules

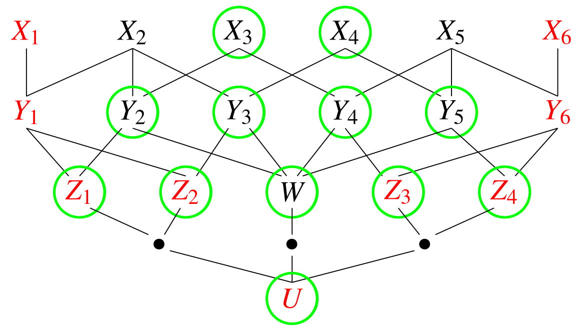
Classification of $(A_{\mathfrak{q}}(\lambda), G, G')$

With Yoshiki Oshima (Adv. Math 2012) we have given a complete classification of the triples

$$\mathfrak{q} \underset{\theta\text{-stable parabolic}}{\subset} \mathfrak{g}_{\mathbb{C}} \underset{\text{symmetric pairs}}{\supset} \mathfrak{g}'_{\mathbb{C}}$$

such that $A_{\mathfrak{q}}(\lambda)|_{\mathfrak{g}'}$ is discretely decomposable.

$U(2,2) \downarrow Sp(1,1)$ $(\approx SO(4,2) \downarrow SO(4,1))$



discrete ser.

trivial rep

○ restriction of $A_{\mathfrak{q}}(\lambda)$ to $Sp(1,1)$ is discretely decomposable

Note • unitary highest weight modules

Basic properties of Zuckerman's $A_{\mathfrak{q}}(\lambda)$-modules (review)

$\boxed{O_\lambda = \mathrm{Ad}^*(G)\cdot\lambda \text{ elliptic orbit } \rightsquigarrow \pi_\lambda := \overline{A_{\mathfrak{q}}(\lambda)} \text{ unitary rep of } G}$

- λ: sufficiently 'positive', 'integral'
  $\Longrightarrow \overline{A_{\mathfrak{q}}(\lambda)} \neq 0$, irreducible unitary (Zuckerman, Vogan, Wallach)
- π irred unitary rep with integral, regular inf character
  $\Longrightarrow \pi \simeq {}^{\exists}\overline{A_{\mathfrak{q}}(\lambda)}$ (Vogan–Salamanca-Riba)
- non-zero $(\mathfrak{g}, K)$-cohomologies (Vogan–Zuckerman)

'Nice framework' 2 (discrete decomposability)

$G \supset G'$ real reductive, $\pi \in \widehat{G}$ irred. unitary

Definition ('93) We say π is G'-admissible if
$\pi|_{G'} \simeq \sum^{\oplus}_{\tau\in\widehat{G'}} m_\pi(\tau)\tau$ (discrete Hilbert sum)
with multiplicity $m_\pi(\tau) < \infty$ for all $\tau \in \widehat{G'}$

as a 'good framework' for branching problem

Criterion ··· K– '90s
Classification ··· K–Y. Oshima (2012, 2014)
Explicit branching laws in this framework
··· K– '88, '93, Gross–Wallach '94, J.-S. Li,
J.-S. Huang–P. Pandžić–G. Savin, H.-Y. Loke,
W. Bertram–J. Hilgert, K–Ørsted, B. Speh,
M. Duflo–J. Vargas, G. Zhang, H. Sekiguchi,
Yoshiki Oshima, J. Möllers, K–F. Kassel, . . .

Cf. Theorem A (K–Speh) discussed $\mathrm{Hom}_{G'}(A_{\mathfrak{q}}(i)^\infty|_{G'}, A_{\mathfrak{q}'}(j)^\infty)$
when the restriction is not discretely decomposable.

Basic properties of Zuckerman's $A_{\mathfrak{q}}(\lambda)$-modules (review)

$\boxed{O_\lambda = \mathrm{Ad}^*(G)\cdot\lambda \text{ elliptic orbit } \rightsquigarrow \pi_\lambda := \overline{A_{\mathfrak{q}}(\lambda)} \text{ unitary rep of } G}$

$G_\lambda = \{g : \mathrm{Ad}^*(g)\lambda = \lambda\}$

G compact	··· Borel–Weil–Bott construction
G_λ compact torus	··· π_λ = Harish-Chandra's discrete series

Basic properties of Zuckerman's $A_{\mathfrak{q}}(\lambda)$-modules (review)

$\boxed{O_\lambda = \mathrm{Ad}^*(G)\cdot\lambda \text{ elliptic orbit } \rightsquigarrow \pi_\lambda := \overline{A_{\mathfrak{q}}(\lambda)} \text{ unitary rep of } G}$

- λ: sufficiently 'positive', 'integral'
  $\Longrightarrow \overline{A_{\mathfrak{q}}(\lambda)} \neq 0$, irreducible unitary (Zuckerman, Vogan, Wallach)

Applications of discretely decomposable restrictions

Previous/recent applications include:

- modular varieties — vanishing thm of cohomologies
- construction of (new) discrete series for non-symmetric spaces
- spectral analysis on non-Riemannian locally symmetric spaces $\Gamma\backslash G/H$
- parabolic geometry — construction of covariant differential operators

Analytic approach to branching problems
discrete decomposability $\wr\wr$
Algebraic approach to branching problems

Non-Riemannian locally symmetric space

M : compact anti-de Sitter mfd of dimension $2n+1$.

Question How about spectral analysis of Δ_M on $L^2(M)$.
hyperbolic operator

Example Take a torsion-free cocompact lattice Γ of $U(n,1)$
$M := \Gamma\backslash X = \Gamma\backslash SO(2n,2)/SO(2n,1)$

Construction of non-zero eigenfunction of Δ (F. Kassel–K, arXiv:1209.4075)

$$A_{\mathfrak{q}}(\lambda) \hookrightarrow \mathrm{Ker}(\Delta_X - \lambda) \subset L^1(X) \cap C^\infty(X)$$
$$\downarrow \text{ generalized Poincaré series}$$
$$\mathrm{Ker}(\Delta_M - \lambda) \subset L^1(M)$$

Non-Riemannian locally symmetric space

M : compact anti-de Sitter mfd of dimension $2n+1$.

Question How about spectral analysis of Δ_M on $L^2(M)$.
hyperbolic operator

Example Take a torsion-free cocompact lattice Γ of $U(n,1)$
$M := \Gamma\backslash X = \Gamma\backslash SO(2n,2)/SO(2n,1)$

Branching for $G \downarrow \overline{\Gamma} \quad = \quad SO(2n,2) \downarrow U(n,1)$

Further spectral analysis $\cdots$ use that the restriction of $A_{\mathfrak{q}}(\lambda)$ to $\overline{\Gamma}$ (Zariski closure) is discretely decomposable and multiplicity-free.

Thank you very much!

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