Norm computation and analytic continuation of vector-valued holomorphic discrete series representations

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1 Introduction: Holomorphic discrete series of $SU(1,1)$

Let $D := \{w \in \mathbb{C} : |w| < 1\}$. Then $\tilde{G} := \widetilde{SU}(1,1)$ (universal covering group) acts on $\mathcal{O}(D)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(w) := (cw+d)^{-\lambda} f\left(\frac{aw+b}{cw+d}\right)$$

where $\lambda \in \mathbb{C}$. This action preserves the sesquilinear form

$$\langle f, h\rangle_\lambda := \frac{\lambda-1}{\pi}\int_D f(w)\overline{h(w)}(1-|w|^2)^{\lambda-2}dw,$$

that is, $\langle \tau_\lambda(g) f, \tau_{\bar{\lambda}}(g) h\rangle_\lambda = \langle f, h\rangle_\lambda$ holds for any $g \in \tilde{G}$. When $\operatorname{Re}\lambda > 1$, this form converges for any polynomial f, h, and therefore τ_λ defines a unitary representation of $\tilde{G}$ when $\lambda > 1$. This is called the holomorphic discrete series representation.

On the other hand, if $\operatorname{Re}\lambda \le 1$, this integral diverges for any non-zero function f. However, when $\operatorname{Re}\lambda > 1$ and $f = \sum_{m=0}^\infty a_m w^m$, we can compute as

$$\|f\|_\lambda^2 = \sum_{m=0}^\infty \frac{m!}{(\lambda)_m}|a_m|^2 \qquad \text{where} \qquad (\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1).$$

This expression is available even when $\operatorname{Re}\lambda \le 1$, and positive definite for $\lambda > 0$. Therefore τ_λ defines a unitary representation of $\tilde{G}$ when $\lambda > 0$.

This example shows that if once the norm is explicitly computed, we can treat the analytic continuation of the holomorphic discrete series representation.

2 Holomorphic discrete series of general Hermitian Lie group

Let G be a Hermitian simple Lie group, that is, its maximal compact subgroup K has a non-discrete center. We assume that G has a complexification $G^{\mathbb{C}}$. We denote their Lie algebras by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{g}^{\mathbb{C}}$, and denote the inverse of Cartan involution of $G^{\mathbb{C}}$ by $g \mapsto g^*$. Then we can take $z \in \mathfrak{z}(\mathfrak{k})$ (the center of $\mathfrak{k}$) such that the eigenvalues of $ad(z)$ are $+\sqrt{-1}$, 0, $-\sqrt{-1}$. We denote the corresponding eigenspace decomposition by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-.$$

Then there exists a domain $D \subset \mathfrak{p}^+$ such that the following diagram commutes.

$$\begin{tikzcd} G/K \arrow[r] \arrow[d, dashed, "\wr"] & G^{\mathbb{C}}/K^{\mathbb{C}}P^- \\ D \arrow[r, hook] & \mathfrak{p}^+ \arrow[u, "\exp"'] \end{tikzcd}$$

Let (τ, V) be a holomorphic representation of $K^{\mathbb{C}}$, and let χ be a suitable character of $\tilde{K}^{\mathbb{C}}$. Then we have the following isomorphism

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda})) \simeq \mathcal{O}(D, V)$$

where $\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda}))$ is the space of holomorphic sections, and $\mathcal{O}(D,V)$ is the space of V-valued holomorphic functions on D. $\tilde{G}$ acts on $\mathcal{O}(D,V)$ by the form

$$\tau_\lambda(g)f(w) = \chi(\mu(g^{-1}, w))^\lambda \tau(\mu(g^{-1}, w))^{-1} f(g^{-1}w) \qquad (g \in G,\ w \in D),$$

using some smooth map $\mu : \tilde{G} \times D \to \tilde{K}^{\mathbb{C}}$. This action preserves the sesquilinear form

$$\langle f, g\rangle_{\lambda,\tau} := \frac{c_\lambda}{\pi^n} \int_D \left(\tau(B(w)^{-1}) f(w), g(w)\right)_\tau \chi(B(w))^{\lambda - p} dw \qquad (f, g \in \mathcal{O}(D,V)),$$

where $n = \dim \mathfrak{p}^+$, $B : \mathfrak{p}^+ \supset D \to \tilde{K}^{\mathbb{C}}$ is a smooth map satisfying

$$Ad(B(w))|_{\mathfrak{p}^+} = I_{\mathfrak{p}^+} - ad([w, w^*])|_{\mathfrak{p}^+} + \frac{1}{4} ad(w) ad(w^*)|_{\mathfrak{p}^+},$$

p is an integer determined from $\mathfrak{g}$, and c_λ is a constant determined so that $\|v\|_{\lambda,\tau} = |v|_\tau$ holds for any constant function v. Then this norm converges for any nonzero polynomial if $\operatorname{Re}\lambda$ is sufficiently large.

Example 2.1. *Let* $J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$, $J' = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$, *and set*

$$G = \left\{ g \in GL(2r, \mathbb{C}) : gJ{}^t g = J, gJ' = J'\bar{g} \right\} \simeq Sp(r, \mathbb{R}),$$

Then we have

$$D = \{ w \in \mathrm{Sym}(r, \mathbb{C}) : I - ww^* \text{ is positive definite.}\},$$

and $\tilde{G}$ acts on $\mathcal{O}(D,V)$ by

$$\tau_\lambda\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \right) f(w) := \det(Cw + D)^{-\lambda} \tau\left({}^t(Cw + D)\right) f\left((Aw + B)(Cw + D)^{-1}\right).$$

This preserves

$$\langle f, g\rangle_{\lambda,\tau} := \frac{c_\lambda}{\pi^{r(r+1)/2}} \int_D \left(\tau((I - ww^*)^{-1}) f(w), g(w)\right)_\tau \det(I - ww^*)^{\lambda - (r+1)} dw.$$

We define another inner product on $\mathcal{O}(\mathfrak{p}^+, V)$.

$$\langle f, g\rangle_{F,\tau} := \frac{1}{\pi^n} \int_{\mathfrak{p}^+} (f(w), g(w))_\tau e^{-|w|^2} dw \qquad (f, g \in \mathcal{O}(\mathfrak{p}^+, V)),$$

where $|w|$ is a suitable K-invariant norm on $\mathfrak{p}^+$.

Let $\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+, V) = \bigoplus_i W_i$ be an irreducible decomposition under K such that each subspace is orthogonal to other subspaces with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$. Then since $\|\cdot\|_{\lambda,\tau}$ and $\|\cdot\|_{F,\tau}$ are both K-invariant, the ratio of two norms are constant on W_i. We denote this ratio by $R_i(\lambda)$. Moreover, if $W_i \perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$ implies $W_i \perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{\lambda,\tau}$ (for example, if $\mathcal{P}(\mathfrak{p}^+, V)$ is K-multiplicity free), then we have

$$\|f\|_{\lambda,\tau}^2 = \sum_i R_i(\lambda) \|f_i\|_{F,\tau}^2 \qquad (f \in \mathcal{O}(\mathfrak{p}^+, V))$$

where f_i is the orthogonal projection of f onto W_i. Although $R_i(\lambda)$ are initially defined when $\operatorname{Re}\lambda$ is sufficiently large, these are meromorphically continued for whole $\mathbb{C}$, and $(\tau_\lambda, \mathcal{O}(\mathfrak{p}^+, V))$ becomes unitarizable if all $R_i(\lambda)$ are positive or infinity. The underlying $(\mathfrak{g}, \tilde{K})$ module of the corresponding unitary subrepresentation is the sum of W_i's such that the corresponding $R_i(\lambda)$ is finite. The goal of this talk is to calculate this ratio $R_i(\lambda)$ for the cases in the table.

G	K	V
$Sp(r,\mathbb{R})$	$U(r)$	$\bigwedge^k(\mathbb{C}^r)^\vee \quad (0 \le k \le r-1)$
$SO^*(2s)$	$U(s)$	$S^k(\mathbb{C}^s)^\vee$ $S^k(\mathbb{C}^s)\otimes \det^{-k/2} \quad (k\in\mathbb{N})$
$SU(q,r)$	$S(U(q)\times U(r))$	$\mathbb{C}\boxtimes V' \quad (V'$: any irrep of $U(r))$
$Spin_0(2,n)$	$(Spin(2)\times Spin(n))/\mathbb{Z}_2$	$\mathbb{C}_k\boxtimes V_{(k,\ldots,k,\pm k)} \quad (k\in\frac{1}{2}\mathbb{N}) \quad (n:\text{even})$ $\mathbb{C}_k\boxtimes V_{(k,\ldots,k)} \quad (k\in\{0,\frac{1}{2}\}) \quad (n:\text{odd})$
$(E_{6(-14)})$	$(SO(2)\times Spin(10))$	$(\mathcal{H}^k(\mathbb{C}^{16}) \quad (k\in\mathbb{N}))$
$(E_{7(-25)})$	$(SO(2)\times E_6)$	$(\mathbb{C})$

Remark 2.2. *The question of when the analytic continuation of the holomorphic discrete series representation is unitarizable is studied by e.g. Berezin [1], Vergne-Rossi [10], and Wallach [11], and completely classified by Enright-Howe-Wallach [2] and Jakobsen [6] by different methods.*

Remark 2.3. *The results on norm computation are already proved for several settings.*

- *B. Ørsted (1980, [7]) for $G = SU(r,r)$, scalar type.*
- *J. Faraut and A. Korányi (1990, [3]) for G any Hermitian Lie group, scalar type.*
- *B. Ørsted and G. Zhang (1994, 1995, [8][9]) for $G = Sp(r,\mathbb{R})$, $V = (\mathbb{C}^r)^\vee$, $G = SU(r,r)$, $V = \mathbb{C}\boxtimes\mathbb{C}^r$, $G = SO^*(4r)$, $V = (\mathbb{C}^{2r})^\vee$.*
- *S. Hwang, Y. Liu and G. Zhang (2004, [5]) for $G = SU(n,1)$, $V = \bigwedge^p(\mathbb{C}^n)^\vee\boxtimes\mathbb{C}$, $\bigwedge^q\mathbb{C}^n\boxtimes\mathbb{C}$.*

3 Main results

Theorem 3.1. *When $(G,K,V) = (Sp(r,\mathbb{R}), U(r), \bigwedge^k(\mathbb{C}^r)^\vee)$ $(0\le k\le r-1)$, we have*

$$\mathcal{P}(\mathfrak{p}^+, V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}} \bigoplus_{\substack{\mathbf{k}\in\{0,1\}^r,\ |\mathbf{k}|=k\\ m_j+k_j\le m_{j-1}}} V^\vee_{(2m_1+k_1,2m_2+k_2,\ldots,2m_r+k_r)},$$

and for $f\in V^\vee_{(2m_1+k_1,\ldots,2m_r+k_r)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{\prod_{j=1}^k\left(\lambda-\frac{1}{2}(j-1)\right)}{\prod_{j=1}^r\left(\lambda-\frac{1}{2}(j-1)\right)_{m_j+k_j}}$$
$$= \frac{1}{\prod_{j=1}^k\left(\lambda-\frac{1}{2}(j-1)+1\right)_{m_j+k_j-1}\prod_{j=k+1}^r\left(\lambda-\frac{1}{2}(j-1)\right)_{m_j+k_j}}.$$

Theorem 3.2. *(1) When $(G,K,V) = (SO^*(4r), U(2r), S^k(\mathbb{C}^{2r})^\vee)$ $(k\in\mathbb{N})$, we have*

$$\mathcal{P}(\mathfrak{p}^+, V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}} \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^r,\ |\mathbf{k}|=k\\ m_j+k_j\le m_{j-1}}} V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\ldots,m_r+k_r,m_r)},$$

and for $f\in V^\vee_{(m_1+k_1,m_1,\ldots,m_r+k_r,m_r)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{(\lambda)_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}}$$
$$= \frac{1}{(\lambda+k)_{m_1+k_1-k}\prod_{j=2}^r(\lambda-2(j-1))_{m_j+k_j}}.$$

(2) When $(G,K,V) = (SO^*(4r), U(2r), S^k(\mathbb{C}^{2r})\otimes\det^{-k/2})$ $(k\in\mathbb{N})$, we have

$$\mathcal{P}(\mathfrak{p}^+, V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}} \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^r,\ |\mathbf{k}|=k\\ m_j-k_j\ge m_{j+1}}} V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r)+(k/2,\ldots,k/2)},$$

and for $f \in V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r)+(k/2,\dots,k/2)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{\prod_{j=1}^{r-1}(\lambda-2(j-1))_k}{\prod_{j=1}^{r}(\lambda-2(j-1))_{m_j-k_j+k}} = \frac{1}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2(r-1))_{m_r-k_r+k}}.$$

Theorem 3.3. *(1) When $(G,K,V)=(SO^*(4r+2),U(2r+1),S^k(\mathbb{C}^{2r+1})^\vee)$ $(k\in\mathbb{N})$, we have*

$$\mathcal{P}(\mathfrak{p}^+,V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}} \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^{r+1},\ |\mathbf{k}|=k\\ m_j+k_j\le m_{j-1}}} V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\dots,m_r+k_r,m_r,k_{r+1})},$$

and for $f\in V^\vee_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r,k_{r+1})}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{(\lambda)_k}{\prod_{j=1}^{r}(\lambda-2(j-1))_{m_j+k_j}(\lambda-2r)_{k_{r+1}}} = \frac{1}{(\lambda+k)_{m_1+k_1-k}\prod_{j=2}^{r}(\lambda-2(j-1))_{m_j+k_j}(\lambda-2r)_{k_{r+1}}}.$$

(2) When $(G,K,V)=(SO^*(4r+2),U(2r+1),S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2})$ $(k\in\mathbb{N})$, we have

$$\mathcal{P}(\mathfrak{p}^+,V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}} \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^{r+1},\ |\mathbf{k}|=k\\ m_j-k_j\ge m_{j+1}}} V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r,-k_{r+1})+(k/2,\dots,k/2)},$$

and for $f\in V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r,-k_{r+1})+(k/2,\dots,k/2)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{\prod_{j=1}^{r}(\lambda-2(j-1))_k}{\prod_{j=1}^{r}(\lambda-2(j-1))_{m_j-k_j+k}(\lambda-2r+1)_{k-k_{r+1}}} = \frac{1}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_{k-k_{r+1}}}.$$

Theorem 3.4. *When $(G,K,V)=(U(q,r),U(q)\times U(r),\mathbb{C}\boxtimes V^{(r)}_{\mathbf{k}})$ $(\mathbf{k}\in\mathbb{N}^r,\ k_1\ge\cdots\ge k_r\ge 0)$, we have*

$$\mathcal{P}(\mathfrak{p}^+,V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^{\min\{q,r\}}\\ m_1\ge\cdots\ge m_{\min\{q,r\}}\ge 0}} V^{(q)\vee}_{\mathbf{m}}\boxtimes(V^{(r)}_{\mathbf{m}}\otimes V^{(r)}_{\mathbf{k}}) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^{\min\{q,r\}}\\ m_1\ge\cdots\ge m_{\min\{q,r\}}\ge 0}} \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}^r\\ n_1\ge\cdots\ge n_r\ge 0}} c^{\mathbf{n}}_{\mathbf{k},\mathbf{m}} V^{(q)\vee}_{\mathbf{m}}\boxtimes V^{(r)}_{\mathbf{n}},$$

and for $f\in V^\vee_{\mathbf{m}}\boxtimes V_{\mathbf{n}}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{\prod_{j=1}^{r}(\lambda-(j-1))_{k_j}}{\prod_{j=1}^{r}(\lambda-(j-1))_{n_j}} = \frac{1}{\prod_{j=1}^{r}(\lambda-(j-1)+k_j)_{n_j-k_j}}.$$

Theorem 3.5. *When $(G,K,V)=(Spin_0(2,n),(Spin(2)\times Spin(n))/\mathbb{Z}_2,\mathbb{C}_k\boxtimes V_{(k,\dots,k,\pm k)})$ ($k\in\frac{1}{2}\mathbb{N}$ and both signs are allowed if n is even, $k\in\{0,\frac{1}{2}\}$ and only plus sign is allowed if n is odd), we have*

$$\mathcal{P}(\mathfrak{p}^+,V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^2\\ m_1\ge m_2\ge 0}} \bigoplus_{l=\max\{-k,k-m_1+m_2\}}^{k} \mathbb{C}_{m_1+m_2+k}\boxtimes V_{(m_1-m_2+l,k,\dots,k,\pm l)},$$

and for $\mathbb{C}_{m_1+m_2+k}\boxtimes V_{(m_1-m_2+l,k,\dots,k,\pm l)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{(\lambda)_{2k}}{(\lambda)_{m_1+k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}} = \frac{1}{(\lambda+2k)_{m_1-k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}.$$

4 Proof of main results

4.1 Preliminaries

Let G be a Hermitian simple Lie group, with its complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-$ as before. We take a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$. Then this automatically becomes a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We denote its root system by $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) = \Delta_{\mathfrak{p}^+} \cup \Delta_{\mathfrak{k}^{\mathbb{C}}} \cup \Delta_{\mathfrak{p}^-}$, and we take a maximal set of mutually strongly orthogonal roots $\{\gamma_1, \ldots, \gamma_r\} \subset \Delta_{\mathfrak{p}^+}$ suitably. We fix $e_j \in \mathfrak{p}^+_{\gamma_j}$ such that $[[e_j, e_j^*], e_j] = 2e_j$ for each j, and set

$$h_j := [e_j, e_j^*] \in \mathfrak{h}^{\mathbb{C}}, \quad \mathfrak{a}_{\mathfrak{l}} := \bigoplus_{j=1}^r \mathbb{R}h_j \subset \mathfrak{h}^{\mathbb{C}}, \quad e := \sum_{j=1}^r e_j \in \mathfrak{p}^+, \quad h := \sum_{j=1}^r h_j \in \mathfrak{a}_{\mathfrak{l}}.$$

Then $ad(h)|_{\mathfrak{p}^+}$ has eigenvalues 2 and 1. We set

$$\mathfrak{p}^+_t := \{x \in \mathfrak{p}^+ : [h, x] = 2x\}, \qquad \mathfrak{p}^-_t := (\mathfrak{p}^+_t)^*, \qquad \mathfrak{k}^{\mathbb{C}}_t := [\mathfrak{p}^+_t, \mathfrak{p}^-_t],$$
$$\mathfrak{g}^{\mathbb{C}}_t := \mathfrak{p}^+_t \oplus \mathfrak{k}^{\mathbb{C}}_t \oplus \mathfrak{p}^-_t, \qquad \mathfrak{g}_t := \mathfrak{g}^{\mathbb{C}}_t \cap \mathfrak{g}, \qquad \mathfrak{k}_t := \mathfrak{k}^{\mathbb{C}}_t \cap \mathfrak{k}.$$

Then $\mathfrak{p}^+_t$ becomes a complex Jordan algebra with the product

$$x \cdot y := \frac{1}{2}[[x, e^*], y] \qquad (x, y \in \mathfrak{p}^+_t).$$

Let $K^{\mathbb{C}}_t$, G_t, K_t be the connected subgroups of $G^{\mathbb{C}}$ corresponding to $\mathfrak{k}^{\mathbb{C}}_t$, $\mathfrak{g}_t$, $\mathfrak{k}_t$ respectively. Also, we set

$${}^cG_t := \mathrm{Int}(e^{\frac{\pi i}{4}(e+e^*)})G_t, \qquad L := {}^cG_t \cap K^{\mathbb{C}}_t, \qquad K_L := L \cap K_t.$$

We define the symmetric cone $\Omega \subset \mathfrak{p}^+_t \cap {}^c\mathfrak{g}_t$ by

$$\Omega := Ad(L)e \subset \mathfrak{p}^+_t \cap {}^c\mathfrak{g}_t.$$

Then there exists a split solvable subgroup $A_L N_L^- \subset L$ which acts simply transitively on Ω. For each $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{C}^r$ let $\Delta_{\mathbf{m}}$ be the relative invariant function on $\mathfrak{p}^+_t$:

$$\Delta_{\mathbf{m}}(Ad(an)x) = e^{2(a_1m_1+\cdots+a_rm_r)}\Delta_{\mathbf{m}}(x) \qquad (x \in \mathfrak{p}^+_t,\ a = e^{a_1h_1+\cdots+a_rh_r} \in A_L,\ n \in N_L^-),$$
$$\Delta_{\mathbf{m}}(e) = 1.$$

Then $\Delta := \Delta_{(1,\ldots,1)}$ is also relatively invariant under $K^{\mathbb{C}}_t = L^{\mathbb{C}}$:

$$\Delta(Ad(l)x) = \chi(l)^2\Delta(x) \qquad (l \in K^{\mathbb{C}}_t,\ x \in \mathfrak{p}^+_t).$$

Using these, we define

$$\Gamma_\Omega(\mathbf{m}) := \int_\Omega \Delta_{\mathbf{m}}(x)\Delta(x)^{-\frac{n_t}{r}} e^{-\operatorname{tr}(x)} dx$$

where $n_t = \dim_{\mathbb{C}} \mathfrak{p}^+_t = \dim_{\mathbb{R}} \Omega$, $r = \mathrm{rank}_{\mathbb{R}} G$, and $\operatorname{tr}(x) := r\frac{(x,e^*)_{\mathfrak{g}}}{(e,e^*)_{\mathfrak{g}}}$. Then it is known that

$$\Gamma_\Omega(\mathbf{m}) = (2\pi)^{\frac{n_t-r}{2}} \prod_{j=1}^r \Gamma\left(m_j - \frac{d}{2}(j-1)\right),$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$ if $\Omega = \mathrm{Herm}_+(r, \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and $d = n-2$ if Ω is the light cone in $\mathbb{R}^{1,n-1}$.

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{p}^+$	$\mathfrak{g}_t$	$\mathfrak{l}$	$\mathfrak{k}_{\mathfrak{l}}$	$\mathfrak{p}^+_t \cap {}^c\mathfrak{g}_t$
$\mathfrak{sp}(r,\mathbb{R})$	$\mathfrak{u}(r)$	$\mathrm{Sym}(r,\mathbb{C})$	$\mathfrak{sp}(r,\mathbb{R})$	$\mathfrak{gl}(r,\mathbb{R})$	$\mathfrak{o}(r)$	$\mathrm{Sym}(r,\mathbb{R})$
$\mathfrak{so}^*(2s)$	$\mathfrak{u}(s)$	$\mathrm{Skew}(s,\mathbb{C})$	$\mathfrak{so}^*(4\lfloor\frac{s}{2}\rfloor)$	$\mathfrak{gl}(\lfloor\frac{s}{2}\rfloor,\mathbb{H})$	$\mathfrak{sp}(\lfloor\frac{s}{2}\rfloor)$	$\mathrm{Herm}(\lfloor\frac{s}{2}\rfloor,\mathbb{H})$
$\mathfrak{su}(q,r)$ $(q \geq r)$	$\mathfrak{s}(\mathfrak{u}(q)\oplus\mathfrak{u}(r))$	$M(q,r;\mathbb{C})$	$\mathfrak{su}(r,r)$	$\mathfrak{gl}(r,\mathbb{C})/\mathfrak{u}(1)$	$\mathfrak{su}(r)$	$\mathrm{Herm}(r,\mathbb{C})$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2)\oplus\mathfrak{so}(n)$	$\mathbb{C}^n$	$\mathfrak{so}(2,n)$	$\mathbb{R}\oplus\mathfrak{so}(1,n-1)$	$\mathfrak{so}(n-1)$	$\mathbb{R}^{1,n-1}$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(2)\oplus\mathfrak{so}(10)$	$M(2,1;\mathbb{O})_{\mathbb{C}}$	$\mathfrak{so}(2,8)$	$\mathbb{R}\oplus\mathfrak{so}(1,7)$	$\mathfrak{so}(7)$	$\mathbb{R}^{1,7}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(2)\oplus\mathfrak{e}_6$	$\mathrm{Herm}(3,\mathbb{O})_{\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$	$\mathbb{R}\oplus\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$	$\mathrm{Herm}(3,\mathbb{O})$

4.2 Key theorem

Let $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ be a K-irreducible subspace, and for $f \in W$ we set $R_W(\lambda) := \|f\|^2_{\lambda,\tau}/\|f\|^2_{F,\tau}$. Our aim is to calculate this ratio $R_W(\lambda)$. In this subsection we state the key theorem.

Let $V|_{K_t^{\mathbb{C}}} = \bigoplus_i V_i$ be the decomposition of the $K^{\mathbb{C}}$-module V into $K_t^{\mathbb{C}}$-irreducible submodules, and for each i we denote by $\overline{V_i}$ the complex conjugate representation of V_i with respect to the real form $L \subset K_t^{\mathbb{C}}$. Let rest : $\mathcal{P}(\mathfrak{p}^+, V) \to \mathcal{P}(\mathfrak{p}_t^+, V) = \bigoplus_i \mathcal{P}(\mathfrak{p}_t^+, V_i)$ be the restriction map, and for each i we take $K_t^{\mathbb{C}}$-submodules $W_{ij} \subset \mathcal{P}(\mathfrak{p}_t^+, V_i)$ such that $\mathrm{rest}(W) \subset \bigoplus_i \bigoplus_j W_{ij}$. Then the key theorem claims that

Theorem 4.1. *Let $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ be a K-irreducible subspace, with $\mathrm{rest}(W) \subset \bigoplus_i \bigoplus_j W_{ij} \subset \bigoplus_i \mathcal{P}(\mathfrak{p}_t^+, V_i)$ as above. We denote by $-\frac{k_{1,i}}{2}\gamma_1 - \cdots - \frac{k_{r,i}}{2}\gamma_r\Big|_{\mathfrak{a}_{\mathfrak{l}}}$ the restricted lowest weight of V_i. We assume*

(A1) $V_i|_{K_L}$ still remains irreducible for each i.

(A2) All the K_L-spherical irreducible subspaces in $W_{ij} \otimes \overline{V_i}$ have the same lowest weight $-n_{ij,1}\gamma_1 - \cdots - n_{ij,r}\gamma_r$.

Then we have

$$R_W(\lambda) = \frac{c_\lambda}{\sum_{ij} a_{ij}} \sum_{ij} a_{ij} \frac{\Gamma_\Omega\left(\lambda + \mathbf{k}_i - \frac{n}{r}\right)}{\Gamma_\Omega(\lambda + \mathbf{n}_{ij})}$$

where $n = \dim \mathfrak{p}^+$, $r = \mathrm{rank}_{\mathbb{R}} G$, and a_{ij} are some non-negative numbers, with

$$c_\lambda^{-1} = \frac{1}{\dim V} \sum_i (\dim V_i) \frac{\Gamma_\Omega\left(\lambda + \mathbf{k}_i - \frac{n}{r}\right)}{\Gamma_\Omega(\lambda + \mathbf{k}_i)}.$$

In the rest of subsection we prove the above theorem. Let $K_W(z, w) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V))$ be the reproducing kernel of W with respect to $\|\cdot\|_{F,\tau}$. Then easily we have

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\int_D \mathrm{Tr}_V\left(\tau(B(w)^{-1})K_W(w,w)\right)\chi(B(w))^{\lambda-p}dw}{\displaystyle\int_{\mathfrak{p}^+} \mathrm{Tr}_V(K_W(w,w))e^{-|w|^2}dw}.$$

By using Weyl's integration formula twice, we can show

$$\begin{aligned} R_W(\lambda) &= \frac{c_\lambda \displaystyle\int_K \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_V\left(\tau(B(kx^{1/2})^{-1})K_W(kx^{1/2}, kx^{1/2})\right)\chi(B(kx^{1/2}))^{\lambda-p}\Delta(x)^b dxdk}{\displaystyle\int_K \int_\Omega \mathrm{Tr}_V\left(K_W(kx^{1/2}, kx^{1/2})\right)e^{-|kx^{1/2}|^2}\Delta(x)^b dxdk} \\ &= \frac{c_\lambda \displaystyle\int_K \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_V\left(\tau(kP(e-x)^{-1}k^{-1})K_W(kx^{1/2}, kx^{1/2})\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dxdk}{\displaystyle\int_K \int_\Omega \mathrm{Tr}_V\left(K_W(kx^{1/2}, kx^{1/2})\right)e^{-\mathrm{tr}(x)}\Delta(x)^b dxdk} \end{aligned}$$

where $x^{1/2}$ is the square root with respect to the Jordan algebra structure on $\mathfrak{p}_t^+$, $kx^{1/2}$ is the abbreviation of $Ad(k)x^{1/2}$, $b := \frac{1}{r}(\dim \mathfrak{p}^+ - \dim \mathfrak{p}_t^+)$, and $P : \Omega \to L$ is the map satisfying

$$P(x)y = 2(xy)x - (xx)y \qquad (x, y \in \Omega).$$

Then it follows from the invariance of $K_W(z, w)$ that

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_V\left(\tau(P(e-x)^{-1})K_W(x,e)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx}{\displaystyle\int_\Omega \mathrm{Tr}_V\left(K_W(x,e)\right)e^{-\mathrm{tr}(x)}\Delta(x)^b dx}.$$

By the assumption, we can write

$$K_W(z,w) = \sum_{ij} \tilde{a}_{ij} K_{W_{ij}}(z,w) \in \mathcal{P}(\mathfrak{p}_t^+ \times \overline{\mathfrak{p}_t^+}, \mathrm{End}(V)) \qquad (z,w \in \mathfrak{p}_t^+)$$

where $K_{W_{ij}}(z,w)$ is the reproducing kernel of W_{ij}, and $\tilde{a}_{ij}$ are non-negative numbers. Therefore we have

$$R_W(\lambda) = \frac{c_\lambda \sum_{ij} \tilde{a}_{ij} \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(e-x)^{-1})K_{W_{ij}}(x,e)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx}{\sum_{ij} \tilde{a}_{ij} \int_\Omega \mathrm{Tr}_{V_i}\left(K_{W_{ij}}(x,e)\right) e^{-\mathrm{tr}(x)}\Delta(x)^b dx}.$$

The theorem follows from the following lemma.

Lemma 4.2.

$$\begin{aligned}&\int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(e-x)^{-1})K_{W_{ij}}(x,e)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx\\ =&\frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{m}_{ij})}\int_\Omega \mathrm{Tr}_{V_i}\left(K_{W_{ij}}(x,e)\right)e^{-\mathrm{tr}(x)}\Delta(x)^b dx.\end{aligned}$$

Proof. Since $K_{W_{ij}}(\cdot,e) \in (W_{ij}\otimes\overline{V_i})^{K_L} \subset \mathcal{P}(\mathfrak{p}_t^+, V_i\otimes\overline{V_i})^{K_L}$, by assumption (A2) there exists $F(x)\in\mathcal{P}(\mathfrak{p}_t^+, V_i\otimes\overline{V_i})$ such that

$$\begin{aligned}\tau(l^{-1})F(lx)\tau({}^tl^{-1}) &= \Delta_{\mathbf{m}_{ij}}(le)F(x) && (x\in\mathfrak{p}_t^+,\ l\in A_LN_L^-),\\ \int_{K_L}\tau(k^{-1})F(kx)\tau({}^tk^{-1})dk &= K_{W_{ij}}(x,e) && (x\in\mathfrak{p}_t^+).\end{aligned}$$

We calculate

$$J := \int_{y\in\Omega}\int_{z\in\Omega\cap(y-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(y-z)^{-1})F(z)\right)\Delta(y-z)^{\lambda-p}\Delta(z)^b dz\, e^{-\mathrm{tr}(y)}dy$$

in two ways. We take $l\in A_LN_L^-$ such that $y=le$, and put $z=lx$. Then we can show

$$\begin{aligned}J &= \int_\Omega \Delta_{\mathbf{m}_{ij}}(y)\Delta(y)^{\lambda-\frac{n_t}{r}}e^{-\mathrm{tr}(y)}dy \int_{\Omega\cap(e-\Omega)}\mathrm{Tr}_{V_i}\left(\tau_i(P(e-x)^{-1})F(x)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx\\ &= \Gamma_\Omega(\lambda+\mathbf{m}_{ij})\int_{\Omega\cap(e-\Omega)}\mathrm{Tr}_{V_i}\left(\tau_i(P(e-x)^{-1})K_{W_{ij}}(x,e)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx.\end{aligned}$$

On the other hand, we put $y-z=:x$. Then we have

$$J = \mathrm{Tr}_{V_i}\left(\int_\Omega \tau_i(P(x)^{-1})\Delta(x)^{\lambda-p}e^{-\mathrm{tr}(x)}dx \int_\Omega F(z)\Delta(z)^b e^{-\mathrm{tr}(z)}dz\right).$$

Since $\int_\Omega \tau_i(P(x)^{-1})\Delta(x)^{\lambda-p}e^{-\mathrm{tr}(x)}dx$ commutes with K_L-action, by assumption (A1) this is proportional to the identity map. By testing with the lowest weight vector, we can show

$$\int_\Omega \tau_i(P(x)^{-1})\Delta(x)^{\lambda-p}e^{-\mathrm{tr}(x)}dx = \Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)I_{V_i},$$

and therefore we have

$$J = \Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)\int_\Omega \mathrm{Tr}_{V_i}(K_{W_{ij}}(z))\Delta(z)^b e^{-\mathrm{tr}(z)}dz.$$

Thus the lemma follows by comparing two expressions of J. $\square$

From this lemma the theorem follows by putting $a_{ij} := \tilde{a}_{ij}\int_\Omega \mathrm{Tr}_{V_i}(K_{W_{ij}}(z))\Delta(z)^b e^{-\mathrm{tr}(z)}dz$. In order to determine the normalizing constant c_λ, we consider the constant function. Then since $K_V(z,w)=I_V$, $K_{V_i}(z,w)=I_{V_i}$, we can show $a_{ij} = (\dim V_i)\Gamma_\Omega\left(\frac{n}{r}\right)$, and the claim on c_λ follows.

4.3 Tube type case

For a $K^{\mathbb{C}}$-module V, if $V|_{K_t^{\mathbb{C}}}$ still remains irreducible, the previous theorem becomes easier.

Corollary 4.3. *Let V be a $K^{\mathbb{C}}$-module such that $V|_{K_t^{\mathbb{C}}}$ still remains irreducible. We denote by $-\frac{k_1}{2}\gamma_1 - \cdots - \frac{k_r}{2}\gamma_r\big|_{\mathfrak{a}_{\mathfrak{l}}}$ the restricted lowest weight of V. Let $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ be a K-irreducible subspace. We assume*

(A1) $V|_{K_L}$ still remains irreducible.

(A2) All the K_L-spherical irreducible subspaces in $W \otimes \overline{V}$ have the same lowest weight $-n_1\gamma_1 - \cdots - n_r\gamma_r$.

Then we have

$$c_\lambda = \frac{\Gamma_\Omega(\lambda + \mathbf{k})}{\Gamma_\Omega\left(\lambda + \mathbf{k} - \frac{n}{r}\right)}, \qquad R_W(\lambda) = \frac{\Gamma_\Omega(\lambda + \mathbf{k})}{\Gamma_\Omega(\lambda + \mathbf{n})} = \frac{\prod_{j=1}^r \left(\lambda - \frac{d}{2}(j-1)\right)_{k_j}}{\prod_{j=1}^r \left(\lambda - \frac{d}{2}(j-1)\right)_{n_j}}.$$

This is applicable if

- G : any, $V = \mathbb{C}$ 1-dimensional,
- G satisfies $G = G_t$ i.e. G is of tube type, or
- $G = SU(q, r)$ $(q \geq r)$, $V = \mathbb{C} \boxtimes V'$ with V' : any,

and this proves Theorems 3.1 ($Sp(r, \mathbb{R})$), 3.2 ($SO^*(4r)$), 3.4 ($SU(q, r),\ q \geq r$), and 3.5 ($Spin_0(2, n)$) (and also for $G = E_{7(-25)}$ case).

4.4 Non-tube type: Easy case

If G is of non-tube type i.e. $G \neq G_t$, determining the ratio of norms $R_W(\lambda)$ is not easy because determining numbers a_{ij} is not easy. However, in some cases we can compute $R_W(\lambda)$ easily by embedding $\mathcal{O}(D, V)$ into another representation of a larger group.

Let $G = SO^*(4r+2)$, $V = S^k(\mathbb{C}^{2r+1})^\vee = \mathcal{P}_k(\mathbb{C}^{2r+1})$, and let $G' = SO^*(4r+4)$, $V' = \mathbb{C}$. Then the universal covering group $\tilde{G}$ acts on $\mathcal{O}(D, \mathcal{P}_k(\mathbb{C}^{2r+1}))$, $\tilde{G}'$ acts on $\mathcal{O}(D')$, and we can show that the standard embedding

$$\iota : \mathcal{O}(D, \mathcal{P}_k(\mathbb{C}^{2r+1})) \to \mathcal{O}(D')$$

which corresponds to the decomposition

$$\mathfrak{p}'^+ = \mathrm{Skew}(2r+2, \mathbb{C}) = \mathrm{Skew}(2r+1, \mathbb{C}) \oplus \mathbb{C}^{2r+1} = \mathfrak{p}^+ \oplus \mathbb{C}^{2r+1}$$

intertwines the $\tilde{G}$-action, and is an isometry with respect to $\|\cdot\|_{F,\tau;G}$ and $\|\cdot\|_{F;G'}$. Now we have

$$\begin{aligned}
\mathcal{P}(\mathfrak{p}'^+)\big|_K &= \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}^{r+1} \\ n_1 \geq \cdots \geq n_{r+1} \geq 0}} V'^{\vee}_{(n_1, n_1, n_2, n_2, \ldots, n_{r+1}, n_{r+1})}\Big|_K \\
&= \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}^{r+1} \\ n_1 \geq \cdots \geq n_{r+1} \geq 0}} \bigoplus_{\substack{\mathbf{m} \in \mathbb{N}^r \\ n_j \geq m_j \geq n_{j+1}}} V^\vee_{(n_1, m_1, n_2, m_2, \ldots, m_r, n_{r+1})}, \\
\mathcal{P}(\mathfrak{p}^+, \mathcal{P}_k(\mathbb{C}^{2r+1})) &= \bigoplus_{\substack{\mathbf{m} \in \mathbb{N}^r \\ m_1 \geq \cdots \geq m_r \geq 0}} \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}^{r+1} \\ m_{j-1} \geq n_j \geq m_j \\ |\mathbf{n} - (\mathbf{m}, 0)| = k}} V^\vee_{(n_1, m_1, n_2, m_2, \ldots, m_r, n_{r+1})}.
\end{aligned}$$

Therefore we have

$$\iota(V^\vee_{(n_1, m_1, n_2, m_2, \ldots, m_r, n_{r+1})}) \subset V'^{\vee}_{(n_1, n_1, n_2, n_2, \ldots, n_{r+1}, n_{r+1})},$$

and thus, for $f \in V^\vee_{(n_1,m_1,n_2,m_2,\dots,m_r,n_{r+1})}$,

$$\frac{\|\iota(f)\|^2_{\lambda;G'}}{\|f\|^2_{F,\tau;G}} = \frac{\|\iota(f)\|^2_{\lambda;G'}}{\|\iota(f)\|^2_{F;G'}} = \frac{1}{\prod_{j=1}^{r+1}(\lambda-2(j-1))_{n_j}}.$$

Since ι intertwines the $\tilde{G}$-action, this is an isometry up to scalar with respect to $\|\cdot\|_{\lambda,\tau;G}$ and $\|\cdot\|_{\lambda;G'}$. By the assumption that two norms $\|\cdot\|_{\lambda,\tau;G}$ and $\|\cdot\|_{F,\tau;G}$ coincide for constant functions, we get, for $f \in V^\vee_{(n_1,m_1,n_2,m_2,\dots,m_r,n_{r+1})}$,

$$\frac{\|f\|^2_{\lambda,\tau;G}}{\|f\|^2_{F,\tau;G}} = \frac{(\lambda)_k}{\prod_{j=1}^{r+1}(\lambda-2(j-1))_{n_j}},$$

and this proves Theorem 3.3(1). Similarly, by embedding the representation $V_{\mathbf{k}}^{(p)\vee}\boxtimes\mathcal{O}(D,\mathbb{C}\boxtimes V_{\mathbf{k}}^{(r)})$ of $U(p)\times U(q,r)$ into the scalar type representation of $U(p+q,r)$, we can show Theorem 3.4.

4.5 Non-tube type: Difficult case

When $(G,V)=(SO^*(4r+2),S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2})$ (or $G=E_{6(-14)}$), computation of ratio of norms is more difficult than the previous subsection. In this case, we combine the key theorem and another information.

Let $(G,V)=(SO^*(4r+2),S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2})$. Then we have $G_t=SO^*(4r)$. First, since $V|_{K_t^{\mathbb{C}}}=\bigoplus_{l=0}^k S^l(\mathbb{C}^{2r})\otimes\det^{-k/2}$, the normalizing constant c_λ is given by

$$\begin{aligned}
c_\lambda^{-1} &= \frac{1}{\binom{2r+k}{k}}\sum_{l=0}^k\binom{2r+l-1}{l}\frac{\Gamma_\Omega\left(\lambda+k-(2r+1)-(0,\dots,0,l)\right)}{\Gamma_\Omega(\lambda+k-(0,\dots,0,l))} \\
&= \frac{1}{\prod_{j=1}^{r-1}(\lambda+k-(2r+1)-2(j-1))_{2r+1}(\lambda-4r+1)_{2r}(\lambda+k-2r+1)} \\
&= \frac{\Gamma_\Omega(\lambda+(k,\dots,k,0)-(2r+1))(\lambda-2r+1)_k}{\Gamma_\Omega(\lambda+(k,\dots,k,k))}.
\end{aligned}$$

Now we compute the ratio of norms $R_{\mathbf{mk}}(\lambda)$ on the $K^{\mathbb{C}}$-irreducible subspace

$$V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r,-k_{r+1})+\frac{k}{2}} \subset V^\vee_{(m_1,m_1,\dots,m_r,m_r,0)}\otimes V^\vee_{(0,\dots,0,-k)+\frac{k}{2}} \subset \mathcal{P}(\mathfrak{p}^+,V).$$

using the key theorem. Then since we can show

$$\begin{aligned}
\operatorname{rest}(V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r,-k_{r+1})+\frac{k}{2}}) &\subset \bigoplus_{l=k-k_{r+1}}^k \bigoplus_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l \\ k_j\le l_j\le m_{j+1}-m_j}} V^{t\vee}_{(m_1,m_1-l_1,\dots,m_r,m_r-l_r)+\frac{k}{2}} \\
&\subset \bigoplus_{l=k-k_{r+1}}^k V^{t\vee}_{(m_1,m_1,\dots,m_r,m_r)}\otimes V^{t\vee}_{(0,\dots,0,-l)+\frac{k}{2}} \subset \bigoplus_{l=k-k_{r+1}}^k \mathcal{P}(\mathfrak{p}_t^+,S^l(\mathbb{C}^{2r})\otimes\det{}^{-k/2}),
\end{aligned}$$

by the key theorem, the ratio of norms is written as

$$\begin{aligned}
R_{\mathbf{mk}}(\lambda) &= \frac{c_\lambda}{\sum_{\mathbf{l}} a_{\mathbf{mkl}}}\sum_{l=k-k_{r+1}}^k \sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l \\ k_j\le l_j\le m_{j+1}-m_j}} a_{\mathbf{mkl}}\frac{\Gamma_\Omega\left(\lambda+(k,\dots,k,k-l)-(2r+1)\right)}{\Gamma_\Omega(\lambda+k+\mathbf{m}-\mathbf{l})} \\
&= \frac{1}{\sum_{\mathbf{l}} a_{\mathbf{mkl}}}\sum_{l=k-k_{r+1}}^k \sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l \\ k_j\le l_j\le m_{j+1}-m_j}} \frac{a_{\mathbf{mkl}}(\lambda-4r+1)_{k-l}}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-l_j}(\lambda-2r+1)_k}
\end{aligned}$$

with some non-negative numbers $a_{\mathbf{mkl}}$. It is difficult to know the exact values of $a_{\mathbf{mkl}}$, but at least this proves

Lemma 4.4.
$$R_{\mathbf{m}\mathbf{k}}(\lambda)=\frac{(\textit{monic poly. of degree } k_{r+1})}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_k}.$$

Next, let $G':=SU(1,2r)$, and consider the (non-irreducible) representation
$$\mathcal{O}(D',\mathbb{C}\boxtimes(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\otimes S^k(\mathbb{C}^{2r})\otimes\det{}^{-k})).$$
Then the embedding
$$\iota:\mathcal{O}(D',\mathbb{C}\boxtimes(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\otimes S^k(\mathbb{C}^{2r})\otimes\det{}^{-k}))\to\mathcal{O}(D,S^k(\mathbb{C}^{2r})\otimes\det{}^{-k/2})$$
which corresponds to the decomposition of the base space and the inclusion of fibers
$$\mathfrak{p}^+=\mathrm{Skew}(2r+1,\mathbb{C})=\mathbb{C}^{2r}\oplus\mathrm{Skew}(2r,\mathbb{C})=\mathfrak{p}'^+\oplus\mathrm{Skew}(2r,\mathbb{C}),$$
$$\mathbb{C}\boxtimes(S^k(\mathbb{C}^{2r})\otimes\det{}^{-k})\simeq\mathbb{C}_{-k/2}\boxtimes(S^k(\mathbb{C}^{2r})\otimes\det{}^{-k/2})\subset S^k(\mathbb{C}^{2r+1})\otimes\det{}^{-k/2}=V$$
intertwines the G'-action. We regard $\mathcal{O}(D',\mathbb{C}\boxtimes(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\otimes S^k(\mathbb{C}^{2r})\otimes\det{}^{-k}))$ as the subspace of $\mathcal{O}(D,S^k(\mathbb{C}^{2r})\otimes\det{}^{-k/2})$ through ι. Using this embedding and the previous lemma, we want to show that on $V^\vee_{(m_1,m_1-k_1,\ldots,m_r,m_r-k_r,-k_{r+1})+\frac{k}{2}}$ the ratio is given by
$$R_{\mathbf{m}\mathbf{k}}(\lambda)=\frac{1}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_{k-k_{r+1}}}$$
by induction on $\min\{j:m_j=0\}$.

First, when $\mathbf{m}=\mathbf{0}$ i.e. on $V^\vee_{(0,\ldots,0,-k)+\frac{k}{2}}$ this is clear. Next, we assume this holds when $m_j=0$, and prove on $V^\vee_{(m_1,m_1-k_1,\ldots,m_r,m_r-k_r,-k_{r+1})+\frac{k}{2}}$ when $m_{j+1}=0$. To do this, we consider
$$\begin{aligned}W'&:=V'^\vee_{(m_1;m_1-k_1,m_2,\ldots,m_r,m_r-k_r,-k_{r+1})+(0;k)}\\&\subset V'^\vee_{(m_1;m_1,m_2,m_2,m_3,\ldots,m_r,m_r,0)}\otimes V'^\vee_{(0;k,\ldots,k,0)}\\&\subset\mathcal{P}(\mathfrak{p}'^+\oplus\mathrm{Skew}(2r,\mathbb{C}))\otimes(\mathbb{C}\boxtimes(S^k(\mathbb{C}^{2r})\otimes\det{}^{-k})).\end{aligned}$$
Then we have
$$\iota(W')\subset V^\vee_{(m_1,m_1-k_1,\ldots,m_r,m_r-k_r,-k_{r+1})+\frac{k}{2}},$$
so it suffices to compute the ratio on W'. Next we determine which $\tilde{G}'$-submodules W' lies on. Since we have
$$\begin{aligned}W'&\subset V'^\vee_{(m_1;m_1,0,\ldots,0)}\otimes V'^\vee_{(0;m_2,m_2,m_3,m_3,\ldots,m_r,m_r,0,0)}\otimes V'^\vee_{(0;k,\ldots,k,0)}\\&\subset\mathcal{P}(\mathfrak{p}'^+)\otimes\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\otimes(\mathbb{C}\boxtimes(S^k(\mathbb{C}^{2r})\otimes\det{}^{-k})),\end{aligned}$$
we can show
$$\begin{aligned}W'&\subset\bigoplus_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=k\\ l_j\le k_{j+1},\ l_r\ge k_{r+1}}}V'^\vee_{(m_1;m_1,0,\ldots,0)}\otimes V'^\vee_{(0;m_2,m_2-l_1,\ldots,m_r,m_r-l_{r-1},0,-l_r)+(0;k)}\\&\subset\bigoplus_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=k\\ l_j\le k_{j+1},\ l_r\ge k_{r+1}}}\mathcal{P}(\mathfrak{p}'^+,V'^\vee_{(0;m_2,m_2-l_1,\ldots,m_r,m_r-l_{r-1},0,-l_r)+(0;k)}).\end{aligned}$$
Next, we check how $\|\cdot\|_{\lambda,\tau}$ is normalized on each summand. For each summand, the space of 'constant functions' satisfies
$$\iota(V'^\vee_{(0;m_2,m_2-l_1,\ldots,m_r,m_r-l_{r-1},0,-l_r)+(0;k)})\subset\bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^r,\ |\mathbf{n}|=k\\ n_j\le l_j,\ n_r\ge l_r}}V^\vee_{(m_2,m_2-n_1,\ldots,m_r,m_r-n_{r-1},0,0,-n_r)+\frac{k}{2}}.$$

Let $\mathrm{proj}_{\mathbf{mn}} : \mathcal{P}(\mathfrak{p}^+, S^k(\mathbb{C}^{2r}) \otimes \det^{-k/2}) \to V^\vee_{(m_2,m_2-n_1,\dots,m_r,m_r-n_{r-1},0,0,-n_r)+\frac{k}{2}}$ be the projection. Then there exist constants $b_{\mathbf{lmn}}$ such that, for $f \in V'^\vee_{(0;m_2,m_2-l_1,\dots,m_r,m_r-l_{r-1},0,-l_r)+(0;k)}$, $\|\mathrm{proj}_{\mathbf{mn}}(f)\|^2_{F,\tau} = b_{\mathbf{lmn}}\|f\|^2_{F,\tau}$ holds. Also, by the induction hypothesis we have

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r,\ |\mathbf{n}|=k \\ n_j\le l_j,\ n_r\ge l_r}} \frac{\|\mathrm{proj}_{\mathbf{mn}}(f)\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r,\ |\mathbf{n}|=k \\ n_j\le l_j,\ n_r\ge l_r}} b_{\mathbf{lmn}} R_{(m_2,\dots,m_r,0),\mathbf{n}}(\lambda) \\ &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r,\ |\mathbf{n}|=k \\ n_j\le l_j,\ n_r\ge l_r}} \frac{b_{\mathbf{lmn}}}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}-n_j}(\lambda-2r+1)_{k-n_r}} \\ &= \frac{(\text{monic poly. of degree } k-l_r)}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}}(\lambda-2r+1)_{k-l_r}}.\end{aligned}$$

Therefore, on each $\iota(\mathcal{O}(D', V'^\vee_{(0;m_2,m_2-l_1,\dots,m_r,m_r-l_{r-1},0,-l_r)+(0;k)})) \subset \mathcal{O}(D, S^k(\mathbb{C}^{2r}) \otimes \det^{-k/2})$, the norm $\|\cdot\|_{\lambda,\tau}$ is normalized so that the above formula holds for 'constant functions'. Finally, let $\mathrm{proj}'_{\mathbf{lm}} : \mathcal{P}(\mathfrak{p}'^+, \mathbb{C}\boxtimes(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\otimes S^k(\mathbb{C}^{2r})\otimes\det^{-k})) \to \mathcal{P}(\mathfrak{p}'^+, V'^\vee_{(0;m_2,m_2-l_1,\dots,m_r,m_r-l_{r-1},0,-l_r)+(0;k)})$ be the projection. Then there exist numbers $c_{\mathbf{klm}}$ such that, for $f \in W'$, $\|\mathrm{proj}'_{\mathbf{lm}}(f)\|^2_{F,\tau} = c_{\mathbf{klm}}\|f\|_{F,\tau}$ holds. Moreover, by the above results on normalization and the result on $SU(1,2r)$ (Theorem 3.4), we have, for $f \in W'$,

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} &= \sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=k \\ l_j\le k_{j+1},\ l_r\ge k_{r+1}}} c_{\mathbf{klm}} \frac{\|\mathrm{proj}'_{\mathbf{lm}}(f)\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} \\ &= \sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=k \\ l_j\le k_{j+1},\ l_r\ge k_{r+1}}} \Biggl(c_{\mathbf{klm}} \frac{(\text{monic poly. of degree } k-l_r)}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}}(\lambda-2r+1)_{k-l_r}} \\ &\quad\times \frac{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}}\prod_{j=2}^{r}(\lambda+k-2(j-1)+1)_{m_j-l_{j-1}}(\lambda-2r+1)_{k-l_r}}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}\prod_{j=2}^{r}(\lambda+k-2(j-1)+1)_{m_j}(\lambda-2r+1)_{k-k_{r+1}}} \Biggr) \\ &= \frac{(\text{monic poly. of degree } k_2+\cdots+k_r)}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}\prod_{j=2}^{r}(\lambda+k+m_j-k_j-2(j-1)+1)_{k_j}(\lambda-2r+1)_{k-k_{r+1}}}.\end{aligned}$$

Combining this with the Lemma

$$R_{\mathbf{mk}}(\lambda) = \frac{(\text{monic poly. of degree } k_{r+1})}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_k},$$

we get the desired formula.

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