

Deformations of finite groups and hypergroups

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Abstract

In this note, we introduce q -deformations of finite groups of low order, for examples, cyclic groups, symmetric groups, dihedral groups and the quaternion group in the category of hypergroups. Moreover we discuss q -deformations of certain finite hypergroups.

1. Introduction

We investigate q -deformations of finite groups and finite hypergroups in the category of hypergroup. It is known that there is no q -deformations of finite groups in the category of quantum groups. However we introduce that there are many q -deformations of finite groups in the category of hypergroups.

Hypergroups $\mathbb{Z}_q(2)$ of order two with a parameter q ($0 < q \leq 1$) are interpreted as q -deformations of the cyclic group $\mathbb{Z}_2$. This fact is our motivation that we started to investigate q -deformations of finite groups and finite hypergroups.

In section 3, we discuss q -deformations of the cyclic group $\mathbb{Z}_3$ of order three and the cyclic group $\mathbb{Z}_4$ of order four. In section 4, we discuss q -deformations of the symmetric group S_3 , the dihedral group D_4 and quaternion group Q_4 . These q -deformations are given by applying a notion of a semi-direct product hypergroup introduced by H. Heyer and S. Kawakami (see [5]).

Moreover we study q -deformations of certain finite hypergroups of low order, the orbital hypergroups $\mathcal{K}^\alpha(\mathbb{Z}_3)$ of $\mathbb{Z}_3$ and $\mathcal{K}^\alpha(\mathbb{Z}_4)$ of $\mathbb{Z}_4$, the character hypergroups $\mathcal{K}(\widehat{S}_3)$ of S_3 , $\mathcal{K}(\widehat{D}_4)$ of D_4 and $\mathcal{K}(\widehat{Q}_4)$ of Q_4 , the conjugacy class hypergroups $\mathcal{K}(S_3)$ of S_3 , $\mathcal{K}(D_4)$ of D_4 and $\mathcal{K}(Q_4)$ of Q_4 in section 5.

2. Preliminaries

For a finite set $K = \{c_0, c_1, \dots, c_n\}$, we denote by $M^b(K)$ and $M^1(K)$, the set of all complex valued measures on K and the set of all non-negative probability measures on K respectively, namely

$$M^b(K) := \left\{ \sum_{j=0}^n a_j \delta_{c_j} : a_j \in \mathbb{C} \ (j = 0, 1, 2, \dots, n) \right\},$$
$$M^1(K) := \left\{ \sum_{j=0}^n a_j \delta_{c_j} : a_j \geq 0 \ (j = 0, 1, 2, \dots, n), \sum_{j=0}^n a_j = 1 \right\}$$

where the symbol δ_c stands for the Dirac measure at $c \in K$. For $\mu = a_0 \delta_{c_0} + a_1 \delta_{c_1} + \dots + a_n \delta_{c_n} \in M^b(K)$, the *support* of μ is

$$\text{supp}(\mu) := \{c_j \in K : a_j \neq 0 \ (j = 0, 1, 2, \dots, n)\}.$$

Axiom A finite hypergroup $K = (K, M^b(K), \circ, *)$ consists of a finite set $K = \{c_0, c_1, \dots, c_n\}$ together with an associative product (called convolution) $\circ$ and an involution $*$ in $M^b(K)$ satisfying the following conditions.

- (1) The space $(M^b(K), \circ, *)$ is an associative $*$ -algebra with unit δ_{c_0} .
- (2) For $c_i, c_j \in K$, the convolution $\delta_{c_i} \circ \delta_{c_j}$ belongs to $M^1(K)$.
- (3) There exists an involutive bijection $c_i \mapsto c_i^*$ on K such that $\delta_{c_i^*} = \delta_{c_i}^*$.
Moreover $c_j = c_i^*$ if and only if $c_0 \in \text{supp}(\delta_{c_i} \circ \delta_{c_j})$ for all $c_i, c_j \in K$.

A finite hypergroup K is called *commutative* if the convolution $\circ$ on $M^b(K)$ is commutative.

Let K and L be finite hypergroups. A mapping $\varphi : K \rightarrow L$ is called a (*hypergroup*) *homomorphism* of K into L if there exists a $*$ -homomorphism $\tilde{\varphi}$ of $M^b(K)$ into $M^b(L)$ as $*$ -algebras such that $\delta_{\varphi(c)} = \tilde{\varphi}(\delta_c)$. If $\tilde{\varphi}$ is bijective, φ is called an *isomorphism* of K onto L . In the case that $L = K$, an isomorphism $\varphi : K \rightarrow K$ is called an *automorphism* of K . The set of all automorphisms of K becomes a group and it is denoted by $\text{Aut}(K)$. Let G be a finite group. A homomorphism $\alpha : G \rightarrow \text{Aut}(K)$ is called an action of G on K .

For a commutative hypergroup K , a complex-valued function χ on K is called a *character* if χ is linearly extendable on $M^b(K)$ to be $\tilde{\chi}(\delta_{c_i}) = \chi(c_i)$ and satisfying that $\tilde{\chi}(\delta_{c_0}) = 1$, $\tilde{\chi}(\delta_{c_i} \circ \delta_{c_j}) = \tilde{\chi}(\delta_{c_i})\tilde{\chi}(\delta_{c_j})$ and $\tilde{\chi}(\delta_{c_i}^*) = \overline{\tilde{\chi}(\delta_{c_i})}$ for all $c_i, c_j \in K$. We denote the trivial character by χ_0 . Let $\hat{K}$ be the set of all characters of K . A convolution on $\hat{K}$ is defined by multiplication of functions on K . Then $\hat{K}$ becomes a signed hypergroup and the duality $\hat{\hat{K}} \cong K$ holds.

(1) **Conjugacy class hypergroup** Let G be a finite group. For $g \in G$, put $\alpha_g(k) = Ad_g(k) = gkg^{-1}$ ($k \in G$). Then α is an action of G on G . Hence we obtain the orbital hypergroup $\mathcal{K}^\alpha(G)$ which we denote by $\mathcal{K}(G)$ which is called a conjugacy class hypergroup of G .

(2) **Character hypergroup** For a finite group G , $\hat{G} = \{\pi_0, \pi_1, \dots, \pi_m\}$ is the set of the all equivalence classes of irreducible representations of G . For $\pi_j \in \hat{G}$, a character χ_j associated with π_j is defined by

$$\chi_j(g) = \frac{1}{\dim \pi_j} \text{tr}(\pi_j(g)).$$

Then $\mathcal{K}(\hat{G}) = \{\chi_0, \chi_1, \dots, \chi_m\}$ becomes a commutative hypergroup with unit χ_0 by the multiplication of functions on G .

(3) **Hypergroup join** For two finite hypergroups $H = \{h_0, h_1, \dots, h_m\}$ and $L = \{\ell_0, \ell_1, \dots, \ell_k\}$, a hypergroup join

$$H \vee L = \{h_0, h_1, \dots, h_m, \ell_1, \dots, \ell_k\}$$

is defined by the convolution $\diamond$ whose structure equations are

$$\begin{aligned}\delta_{h_i} \diamond \delta_{h_j} &= \delta_{h_i} \circ \delta_{h_j}, & \delta_{h_i} \diamond \delta_{\ell_j} &= \delta_{\ell_j}, \\ \delta_{\ell_i} \diamond \delta_{\ell_j} &= \delta_{\ell_i} \circ \delta_{\ell_j} \text{ when } \ell_j \neq \ell_i^*, \\ \delta_{\ell_i} \diamond \delta_{\ell_i}^* &= n_i^0 \omega(H) + \sum_{j=1}^k n_i^j \delta_{\ell_j}\end{aligned}$$

where $\delta_{\ell_i} \circ \delta_{\ell_i}^* = n_i^0 \delta_{\ell_0} + \sum_{j=1}^k n_i^j \delta_{\ell_j}$ and $\omega(H)$ is the normalized Haar measure of H .

3. Deformations of finite abelian groups

Let $K = \{c_0, c_1\}$ be a hypergroup of order two. Then the structure of K is determined by

$$\delta_{c_1} \circ \delta_{c_1} = q\delta_{c_0} + (1-q)\delta_{c_1}$$

where $0 < q \leq 1$. We denote it by $\mathbb{Z}_q(2)$ which is interpreted as a q -deformation of $\mathbb{Z}_2$. Stimulating by this fact, we have started to study q -deformations of finite groups.

3.1 Deformation $\mathbb{Z}_q(3)$ of $\mathbb{Z}_3$

First of all we discuss a q -deformation of $\mathbb{Z}_3$. It is easy to check the following proposition directly and this fact is also described in the paper (Wildberger [21]).

Proposition 3.1 Let $K = \{c_0, c_1, c_2\}$ be a hypergroup of order three. For each q ($0 < q \leq 1$) there exists a unique hypergroup of order three such that $\delta_{c_1}^* = \delta_{c_2}$ and $\delta_{c_1} \circ \delta_{c_2} = q\delta_{c_0} + \dots$.

We denote the above K by $\mathbb{Z}_q(3)$, which is interpreted as a q -deformation of $\mathbb{Z}_3$. The structure equations of $\mathbb{Z}_q(3) = \{c_0, c_1, c_2\}$ ($0 < q \leq 1$) are determined by

$$\begin{aligned}\delta_{c_1} \circ \delta_{c_2} &= q\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2}, \\ \delta_{c_1} \circ \delta_{c_1} &= \frac{1-q}{2}\delta_{c_1} + \frac{1+q}{2}\delta_{c_2}, \\ \delta_{c_2} \circ \delta_{c_2} &= \frac{1+q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2}.\end{aligned}$$

Put $\widehat{\mathbb{Z}_q(3)} = \{\chi_0, \chi_1, \chi_2\}$. Then the character table of $\mathbb{Z}_q(3)$ is

	c_0	c_1	c_2
χ_0	1	1	1
χ_1	1	ω_q	$\overline{\omega_q}$
χ_2	1	$\overline{\omega_q}$	ω_q

where $\omega_q = \frac{-q+i\sqrt{q^2+2q}}{2}$.

By the symmetry of the character table we see that $\widehat{\mathbb{Z}_q(3)} \cong \mathbb{Z}_q(3)$.

3.2 Deformation $\mathbb{Z}_{(p,q)}(4)$ of $\mathbb{Z}_4$

We investigated several kinds of extension problem in the category of commutative hypergroups, refer to [6], [8], [10], [11], [12], [13], [14], [15], [16], [17], [18]. The cyclic group $\mathbb{Z}_4$ of order four is a non-splitting extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. Then one can consider a non-splitting extension $\mathbb{Z}_{(p,q)}(4)$ ($0 < p \leq 1$, $0 < q \leq 1$) of $\mathbb{Z}_q(2)$ by $\mathbb{Z}_p(2)$ as follows.

Proposition 3.2 (Example 4.2 in [14]) For (p, q) ($0 < p \leq 1$, $0 < q \leq 1$) there exists a unique hypergroup $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ of order four, which is an extension hypergroup of $\mathbb{Z}_q(2)$ by $\mathbb{Z}_p(2) = \{c_0, c_2\}$ such that $c_1^* = c_3$.

The structure of $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ ($0 < p \leq 1, 0 < q \leq 1$) is given by

$$\begin{aligned}\delta_{c_1} \circ \delta_{c_1} &= \delta_{c_3} \circ \delta_{c_3} = \frac{1-q}{2}\delta_{c_1} + q\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\ \delta_{c_2} \circ \delta_{c_2} &= p\delta_{c_0} + (1-p)\delta_{c_2}, \quad \delta_{c_1} \circ \delta_{c_2} = \frac{1-p}{2}\delta_{c_1} + \frac{1+p}{2}\delta_{c_3}, \\ \delta_{c_1} \circ \delta_{c_3} &= \frac{2pq}{1+p}\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{q-pq}{1+p}\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\ \delta_{c_2} \circ \delta_{c_3} &= \frac{1+p}{2}\delta_{c_1} + \frac{1-p}{2}\delta_{c_3}.\end{aligned}$$

Put $\widehat{\mathbb{Z}_{(p,q)}(4)} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$. Then the character table of $\mathbb{Z}_{(p,q)}(4)$ is

	c_0	c_1	c_2	c_3
χ_0	1	1	1	1
χ_1	1	$i\sqrt{pq}$	$-p$	$-i\sqrt{pq}$
χ_2	1	$-q$	1	$-q$
χ_3	1	$-i\sqrt{pq}$	$-p$	$i\sqrt{pq}$

It is easy to see that $\widehat{\mathbb{Z}_{(p,q)}(4)} \cong \mathbb{Z}_{(q,p)}(4)$ and $\mathbb{Z}_{(p,q)}(4)$ is interpreted as a (p, q) -deformation of $\mathbb{Z}_4$.

4. Deformations of non-abelian finite groups

Let α be an action of a finite group G on a finite hypergroup $H = (H, M^b(H), \circ, *)$. Then a semi-direct product hypergroup $S := H \rtimes_{\alpha} G$ is introduced in [5]. A convolution $\circ_{\alpha}$ in $M^b(S)$ is defined by

$$(\varepsilon_{h_1} \otimes \delta_{g_1}) \circ_{\alpha} (\varepsilon_{h_2} \otimes \delta_{g_2}) := (\varepsilon_{h_1} \circ \varepsilon_{\alpha_{g_1}(h_2)} \otimes \delta_{g_1 g_2}),$$

where ε and δ stand for Dirac measures in $M^b(H)$ and $M^b(G)$ respectively. Unit element is $\varepsilon_e \otimes \delta_e$. An involution $-$ is

$$(\mu \otimes \delta_g)^{-} := \alpha_g^{-1}(\mu^*) \otimes \delta_{g^{-1}}$$

for all $\mu \in M^b(H)$ and $g \in G$.

4.1 Deformation $S_q(3)$ of the symmetric group S_3

The symmetric group S_3 is a semi-direct product $\mathbb{Z}_3 \rtimes_{\alpha} \mathbb{Z}_2$ where α is an action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$.

Let α be an action of $\mathbb{Z}_2 = \{e, g\}$ on a hypergroup $\mathbb{Z}_q(3) = \{h_0, h_1, h_2\}$ ($0 < q \leq 1$) such that

$$\alpha_g(h_1) = h_2, \quad \alpha_g(h_2) = h_1.$$

Then we obtain a semi-direct product hypergroup

$$S_q(3) := \mathbb{Z}_q(3) \rtimes_{\alpha} \mathbb{Z}_2$$

which is a q -deformation of the symmetric group $S_3 = \mathbb{Z}_3 \rtimes_{\alpha} \mathbb{Z}_2$.

4.2 Deformation $D_{(p,q)}(4)$ of the dihedral group D_4

The dihedral group D_4 is written by a semi-direct product $\mathbb{Z}_4 \rtimes_{\alpha} \mathbb{Z}_2$.

Let $H = \mathbb{Z}_{(p,q)}(4) = \{h_0, h_1, h_2, h_3\}$ ($0 < p \leq 1, 0 < q \leq 1$) be the (p, q) -deformation of $\mathbb{Z}_4$ and α an action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_{(p,q)}(4)$ given by

$$\alpha_g(h_1) = h_3, \quad \alpha_g(h_2) = h_2, \quad \alpha_g(h_3) = h_1.$$

Then we obtain a semi-direct product hypergroup

$$D_{(p,q)}(4) := \mathbb{Z}_{(p,q)}(4) \rtimes_{\alpha} \mathbb{Z}_2.$$

Hence, we obtain a (p, q) -deformation $D_{(p,q)}(4)$ of the dihedral group D_4 .

4.3 Another deformation $W_q(4)$ of the dihedral group D_4

The dihedral group D_4 is also written by a semi-direct product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_{\beta} \mathbb{Z}_2$ where β is a flip action of $\mathbb{Z}_2$ on $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2) = \{(h_0, h_0), (h_0, h_1), (h_1, h_0), (h_1, h_1) ; h_0, h_1 \in \mathbb{Z}_q(2)\}$ be a q -deformation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let β be a flip action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$ given by

$$\beta_g((h_i, h_j)) = (h_j, h_i) \quad (i, j = 0 \text{ or } 1).$$

Then we obtain a semi-direct product hypergroup

$$W_q(4) := (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_{\beta} \mathbb{Z}_2.$$

The hypergroup $W_q(4)$ is another q -deformation of D_4 .

4.4 Deformation $Q_q(4)$ of the quaternion group Q_4

The structure of the quaternion group $Q_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ is determined by

$$i^2 = j^2 = k^2 = -1, \quad ij = k.$$

Let α be an action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_4 = \{h_0, h_1, h_2, h_3\}$ such that

$$\alpha_g(h_1) = h_3, \quad \alpha_g(h_2) = h_2, \quad \alpha_g(h_3) = h_1.$$

Let c be a H -valued 2-cocycle of $\mathbb{Z}_2$ which is also given by

$$c(e, e) = c(e, g) = c(g, e) = h_0 \quad \text{and} \quad c(g, g) = h_2.$$

Then a twisted semi-direct product group $\mathbb{Z}_4 \rtimes_{\alpha}^c \mathbb{Z}_2$ is defined by the product

$$(h, g)(h', g') = (h\alpha_g(h')c(g, g'), gg')$$

for $h, h' \in \mathbb{Z}_4$ and $g, g' \in \mathbb{Z}_2$. The quaternion group Q_4 is isomorphic to $\mathbb{Z}_4 \rtimes_{\alpha}^c \mathbb{Z}_2$. Hence we interpret Q_4 as a twisted semi-direct product group $\mathbb{Z}_4 \rtimes_{\alpha}^c \mathbb{Z}_2$.

Let $H = \mathbb{Z}_{(1,q)}(4) = \{h_0, h_1, h_2, h_3\}$ be a q -deformation of $\mathbb{Z}_4$. Then, we obtain a twisted semi-direct product hypergroup

$$Q_q(4) := \mathbb{Z}_{(1,q)}(4) \rtimes_{\alpha}^c \mathbb{Z}_2.$$

The hypergroup $Q_q(4)$ is a q -deformation of the quaternion group $Q_4 = \mathbb{Z}_4 \rtimes_{\alpha}^c \mathbb{Z}_2$.

5. Deformations of finite hypergroups

In this section we discuss q -deformations of several kinds of finite hypergroups in a similar way to the case of finite groups.

5.1 Deformations of orbital hypergroups

(4) **Orbital hypergroup** Given an action α of a finite group G on a commutative hypergroup H , we obtain a orbit $O = \{\alpha_g(h) ; g \in G\}$ of $h \in H$ under the action α . Let $\{O_0, O_1, \dots, O_m\}$ be the set of all orbits in H . We denote an element c_j which is corresponding to each orbit O_j and put $H^{\alpha} = \{c_0, c_1, \dots, c_m\}$. Let $M^b(H)^{\alpha}$ denote the fixed point algebra of $M^b(H)$ under the action α , namely

$$M^b(H)^{\alpha} = \{\mu \in M^b(H) ; \alpha_g(\mu) = \mu \text{ for all } g \in G\}.$$

We note that $M^b(H)^{\alpha}$ is a $*$ -subalgebra of $M^b(H)$. For $c_j \in H^{\alpha}$, put

$$\delta_{c_j} = \frac{1}{|O_j|} \sum_{h \in O_j} \delta_h = \frac{1}{|G|} \sum_{g \in G} \alpha_g(\delta_h).$$

Then $\delta_{c_j} \in M^b(H)^{\alpha} \cap M^1(H)$. $\mathcal{K}^{\alpha}(H) = (H^{\alpha}, M^b(H)^{\alpha}, \circ, *)$ becomes a hypergroup which is called an orbital hypergroup of H by the action α .

Example 1 The orbital hypergroup $\mathcal{K}^{\alpha}(\mathbb{Z}_q(3)) = \{c_0, c_1\}$ is a q -deformation of $\mathcal{K}^{\alpha}(\mathbb{Z}_3)$.

The structure equations are

$$\delta_{c_1} \circ \delta_{c_1} = \frac{q}{2} \delta_{c_0} + \left(1 - \frac{q}{2}\right) \delta_{c_1}.$$

Example 2 The orbital hypergroup $\mathcal{K}^\alpha(\mathbb{Z}_{(p,q)}(4)) = \{c_0, c_1, c_2\}$ is a q -deformation of $\mathcal{K}^\alpha(\mathbb{Z}_4)$.

The structure equations are

$$\begin{aligned} \delta_{c_1} \circ \delta_{c_1} &= p\delta_{c_0} + (1-p)\delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_2} = \delta_{c_2}, \\ \delta_{c_2} \circ \delta_{c_2} &= \frac{pq}{1+p}\delta_{c_0} + \frac{q}{1+p}\delta_{c_1} + (1-q)\delta_{c_2}. \end{aligned}$$

5.2 Deformations of character hypergroups

(5) **Character hypergroup of semi-direct product hypergroup** Let $S = H \rtimes_\alpha G$ be a semi-direct product hypergroup defined by an action α of a finite abelian group G on a finite commutative hypergroup H (Refer to [5]). $\hat{S} = \widehat{H \rtimes_\alpha G}$ is the set of all equivalence classes of irreducible representations of S . For $(\pi, \mathcal{H}(\pi)) \in \hat{S}$, the character $ch(\pi)$ of π is defined by

$$ch(\pi)((h, g)) = \frac{1}{\dim \pi} \text{tr}(\pi(h, g))$$

where $(h, g) \in H \rtimes_\alpha G$ and tr is the trace of $B(\mathcal{H}(\pi))$ (Refer to [K,Y]). Put $\mathcal{K}(\hat{S}) = \{ch(\pi) ; \pi \in \hat{S}\}$.

Proposition 5.1 ([7]) If the action α satisfies the regularity condition, then $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ becomes a commutative hypergroup by the product of functions on $S = H \rtimes_\alpha G$.

This hypergroup is called a character hypergroup of the semi-direct product hypergroup $S = H \rtimes_\alpha G$.

Example 3 The character hypergroup $\mathcal{K}(\widehat{S_q(3)})$ of $S_q(3) = \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2$ is a q -deformation of $\mathcal{K}(\widehat{S_3})$.

Remark we note that $\mathcal{K}(\widehat{S_q(3)}) = \mathbb{Z}_2 \vee \mathbb{Z}_q(2)$.

Example 4 The character hypergroup $\mathcal{K}(\widehat{D_{(p,q)}(4)})$ of $D_{(p,q)}(4) = \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2$ is a q -deformation of $\mathcal{K}(\widehat{D_4})$.

Example 5 The character hypergroup $\mathcal{K}(\widehat{Q_q(4)})$ of $Q_q(4) = \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2$ is a q -deformation of $\mathcal{K}(\widehat{D_4})$.

Remark We see that $\mathcal{K}(\widehat{Q_q(4)}) = \mathcal{K}(\widehat{D_{(1,q)}(4)})$ although $Q_q(4)$ is not isomorphic to $D_{(1,q)}(4)$.

5.3 Deformations of conjugacy class hypergroups

(6) **Generalized conjugacy class hypergroup** Let $H \rtimes_{\alpha} G$ be a semi-direct product hypergroup. Then there exists the canonical conditional expectation E from $M^b(S)$ onto the center $Z(M^b(H \rtimes_{\alpha} G))$ of $M^b(H \rtimes_{\alpha} G)$. Put

$$\mathcal{K}(H \rtimes_{\alpha} G) = \{E(\delta_{(h,g)}) ; (h,g) \in H \rtimes_{\alpha} G\}.$$

Proposition 5.2 ([6]) If the action α satisfies the regularity condition, then $\mathcal{K}(H \rtimes_{\alpha} G)$ becomes a commutative hypergroup with the convolution in $Z(M^b(H \rtimes_{\alpha} G)) = M^b(\mathcal{K}(H \rtimes_{\alpha} G))$. Moreover $\hat{\mathcal{K}}(H \rtimes_{\alpha} G) \cong \widehat{\mathcal{K}(H \rtimes_{\alpha} G)}$ holds.

We call $\mathcal{K}(H \rtimes_{\alpha} G)$ a generalized conjugacy class hypergroup of $H \rtimes_{\alpha} G$.

Example 6 The conjugacy class hypergroup $\mathcal{K}(S_q(3))$ of $S_q(3)$ is a q -deformation of $\mathcal{K}(S_3)$.

Remark We note that $\mathcal{K}(S_q(3)) = \mathbb{Z}_q(2) \vee \mathbb{Z}_2$.

Example 7 Conjugacy class hypergroup $\mathcal{K}(D_{(p,q)}(4))$ of $D_{(p,q)}(4)$ is a (p, q) -deformation of $\mathcal{K}(D_4)$.

Example 8 The conjugacy class hypergroup $\mathcal{K}(Q_q(4))$ of $Q_q(4)$ is a q -deformation of $\mathcal{K}(Q_4)$.

Remark We note that $\mathcal{K}(D_{(1,q)}) \cong \mathcal{K}(Q_q(4))$.

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