Representations of a semi-direct product hypergroup

Satoshi Kawakami

Abstract

Let K be a compact hypergroup and K_0 a closed subhypergroup of K. For a state φ of the Banach $*$-algebra $M^b(K_0)$ the induced state $\mathrm{ind}_{K_0}^K\varphi$ of $M^b(K)$ is introduced. Applying the notion of a character mapping ch from the space $\mathrm{Rep}^{\mathrm{f}}(K)$ of finite representations of K into the space $M^1(\hat{K})$ of probability measures on the space $\hat{K}$ of equivalence classes of irreducible representations of K. For certain classes of hypergroups K it is shown that, whenever $\pi \in \mathrm{Rep}^{\mathrm{f}}(K_0)$, the formula $ch(\mathrm{ind}_{K_0}^K\pi) = \mathrm{ind}_{K_0}^K ch(\pi)$ holds.

Next we investigate hypergroup structures on distinguished dual objects related to a given hypergroup K, especially to a semi-direct product hypergroup $K = H \rtimes_\alpha G$ defined by an action α of a locally compact group G on a commutative hypergroup H. Typical dual objects are the sets of equivalence classes of irreducible representations of K, of infinite-dimensional irreducible representations of type I hypergroups K, and of quasi-equivalence classes of type II_1 factor representations of non-type I hypergroups K. The method of proof relies on the notion of a character of a representation of $K = H \rtimes_\alpha G$.

1. Preliminaries

For a locally compact space X we shall mainly consider the subspaces $C_c(X)$ and $C_0(X)$ of the space $C(X)$ of continuous functions on X which have compact support or vanish at infinity respectively. By $M(X)$, $M^b(X)$ and $M_c(X)$ we abbreviate the spaces of all (Radon) measures on X, the bounded measures and the measures with compact support on X respectively. Let $M^1(X)$ denote the set of probability measures on X and $M_c^1(X)$ its subset $M^1(X) \cap M_c(X)$. The symbol δ_x stands for the Dirac measures at $x \in X$.

A *hypergroup* $(K, *)$ is a locally compact (Hausdorff) space K together with a *convolution* $*$ in $M^b(K)$ such that $(M^b(K), *)$ becomes a Banach algebra and that the following properties are fulfilled.

(H1) The mapping

$$(\mu, \nu) \longmapsto \mu * \nu$$

from $M^b(K) \times M^b(K)$ into $M^b(K)$ is continuous with respect to the weak topology in $M^b(K)$.

(H2) For $x, y \in K$ the convolution $\delta_x * \delta_y$ belongs to $M_c^1(K)$.

(H3) There exists a *unit* element $e \in K$ with

$$\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$$

for all $x \in K$, and an *involution*

$$x \longmapsto x^-$$

in K such that

$$\delta_{x^-} * \delta_{y^-} = (\delta_y * \delta_x)^-$$

and

$$e \in \operatorname{supp}(\delta_x * \delta_y) \ \text{ if and only if } \ x = y^-$$

whenever $x, y \in K$.

(H4) The mapping

$$(x, y) \longmapsto \operatorname{supp}(\delta_x * \delta_y)$$

from $K \times K$ into the space $\mathcal{C}(K)$ of all compact subsets of K furnished with Michael topology is continuous.

A hypergroup $(K, *)$ is said to be *commutative* if the convolution $*$ is commutative.

Let $(K, *)$ and $(L, \circ)$ be two hypergroups with units e_K and e_L respectively. A continuous mapping $\varphi : K \to L$ is called a hypergroup *homomorphism* if $\varphi(e_K) = e_L$ and φ is the unique linear, weakly continuous extension from $M^b(K)$ to $M^b(L)$ such that

$$\delta_{\varphi(x)} = \varphi(\delta_x), \ \varphi(\delta_x^-) = \varphi(\delta_x)^- \text{ and } \delta_{\varphi(x)} \circ \delta_{\varphi(y)} = \varphi(\delta_x * \delta_y)$$

whenever $x, y \in K$. If $\varphi : K \to L$ is also a homeomorphism, it will be called an isomorphism from K onto L. An isomorphism from K onto K is called an automorphism of K. We denote by $\operatorname{Aut}(K)$ the set of all automorphisms of K. Then $\operatorname{Aut}(K)$ becomes a topological group equipped with the weak topology of $M^b(K)$. We call α an *action* of a locally compact group G on a hypergroup H if α is a continuous homomorphism from G into $\operatorname{Aut}(H)$.

Now, let $\mathcal{H}$ denote a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B}(\mathcal{H})$ be the Banach $*$-algebra of bounded linear operators on $\mathcal{H}$. We refer to U as a *representation* of K on $\mathcal{H}$ if U is a $*$-homomorphism from the Banach $*$-algebra $M^b(K)$ to $\mathcal{B}(\mathcal{H})$ such that $U(\delta_e) = 1$ and if for each $u, v \in \mathcal{H}$ the mapping

$$\mu \longmapsto \langle U(\mu)u, v \rangle$$

is continuous on $M^b(K)$. By $\operatorname{Rep}(K, \mathcal{H})$ we denote the set of classes of unitary equivalent representations of K on $\mathcal{H}$, by $\operatorname{Irr}(K, \mathcal{H})$ its subset of irreducible classes.

If the given hypergroup K is commutative, its dual $\widehat{K}$ can be in introduced as the set of all bounded continuous functions $\chi \neq 0$ on K satisfying

$$\chi(\delta_x^- * \delta_y) = \overline{\chi(x)}\chi(y)$$

for all $x, y \in K$. This set of characters of K becomes a locally compact space with respect to the topology of uniform convergence on compact sets, but generally fails to be a hypergroup.

Let $H := (H, \circ)$ be a hypergroup, G a locally compact group and α an action of G on H. Let $K = H \times G$ be the set product of H and G such that $M^b(K) = M^b(H) \otimes M^b(G)$ as Banach $*$-algebras. Then we define a convolution $*_\alpha$ in $M^b(K)$ by

$$\varepsilon_{(h_1,g_1)} *_\alpha \varepsilon_{(h_2,g_2)} := (\varepsilon_{h_1} \circ \varepsilon_{\alpha_{g_1}(h_2)}) \otimes \delta_{g_1 g_2}$$

with unit element

$$\varepsilon_{(e,e)} := \varepsilon_e \otimes \delta_e$$

where e denotes the unit element of H as well as that of G, and an involution by

$$(\mu \otimes \delta_g)^- := \alpha_g^{-1}(\mu^-) \otimes \delta_{g^{-1}} = \alpha_g^{-1}(\mu)^- \otimes \delta_{g^{-1}}$$

for all $\mu \in M^b(H)$ and $g \in G$ where the Dirac measures in $M^b(K)$ and $M^b(G)$ are denoted by ε and δ respectively. K will be written as $H \rtimes_\alpha G$.

Proposition $H \rtimes_\alpha G$ is a hypergroup.

Applying our imprimitivity theorem ([1]) we have the following.

Theorem (Mackey machine) Let $K = H \rtimes_\alpha G$ be a semi-direct product hypergroup defined by a smooth action α of a locally compact group G on a commutative hypergroup H. For any $\chi \in \widehat{H}$ and an irreducible representation τ of $G(\chi)$, $\pi^{(\chi,\tau)}$ is defined by $\mathrm{ind}_{H \rtimes_\alpha G(\chi)}^{H \rtimes_\alpha G}(\chi \otimes \tau)$.

Then the following statements hold:

(1) $\pi^{(\chi,\tau)}$ is an irreducible representation of K.

(2) All irreducible representations of K are obtained in this form.

(3) $\pi^{(\chi',\tau')}$ and $\pi^{(\chi,\tau)}$ are unitary equivalent if and only if $\mathrm{Orb}(\chi') = \mathrm{Orb}(\chi)$ and $\tau' \cong \tau$.

2. Induced representations of a compact hypergroup

Let K be a compact hypergroup and K_0 a closed subhypergroup of K. For a representation π_0 of K_0 with the representing Hilbert space $\mathcal{H}(\pi_0)$ we define the representation $\pi := \mathrm{ind}_{K_0}^K \pi_0$ induced by π_0 from K_0 to K by prescribing its representing Hilbert space as

$$\mathcal{H}(\pi) := \{\xi \in L^2(K, \mathcal{H}_0(\pi_0)) : \delta_h * \delta_x(\xi) = \pi_0(h)\xi(x) \text{ for all } h \in K_0,\ x \in K\}$$

and by setting

$$(\pi(k)\xi)(x) := (\delta_x * \delta_k)(\xi)$$

for all $\xi \in \mathcal{H}(\pi)$, $x, k \in K$. We note that π is indeed a representation of K.

Proposition For

$$\pi_0 \cong \pi_1 \oplus \cdots \oplus \pi_m$$

with $\pi_0, \pi_1, \cdots, \pi_m \in \mathrm{Rep}^{\mathrm{f}} K_0$ we have

$$\mathrm{ind}_{K_0}^K \pi_0 \cong \mathrm{ind}_{K_0}^K \pi_1 \oplus \cdots \oplus \mathrm{ind}_{K_0}^K \pi_m.$$

Now, let K be a second countable compact commutative hypergroup of strong type which implies that the dual space

$$\hat{K} = \{\chi_0, \chi_1, \cdots, \chi_n, \cdots\}$$

is a countably infinite discrete commutative hypergroup with unit χ_0. Let K_0 be a closed subhypergroup of K. Given a character τ of K_0. We consider the set

$$A(\tau) := \{\chi \in \hat{K} : \chi(h) = \tau(h) \text{ for all } h \in K_0\}.$$

Then the representation π induced by τ from K_0 to K is given by

Proposition

$$\pi = \mathrm{ind}_{K_0}^K \tau \cong \sum_{\chi \in A(\tau)}^{\oplus} \chi.$$

Here we remark some interesting phenomenon which differs from the group case.

Let K be a finite commutative hypergroup of strong type which is obtained as an extension hypergroup of $K_0 = \mathbb{Z}_p(2)$ by $L = \mathbb{Z}_q(2) \quad (p, q \in (0, 1])$ which means that $K/K_0 \cong L$. Then for any natural number $n = 1, 2, \cdots$, there exists a hypergroup K such that $|K| = 2 + n$ for sufficiently small p, q. In this case

$$\begin{aligned} \hat{K} &= \{\chi_0, \chi_1, \cdots, \chi_{n+1}\}, \\ \hat{K}_0 &= \{\tau_0, \tau_1\}, \\ A(\tau_0) &= K_0^{\perp} = \{\chi_0, \chi_1\}, \\ A(\tau_1) &= \{\chi_2, \chi_3, \cdots, \chi_{n+1}\} \end{aligned}$$

and

$$\mathrm{ind}_{K_0}^K \tau_1 = \chi_2 \oplus \chi_3 \oplus \cdots \oplus \chi_{n+1},$$

i.e. $\dim(\mathrm{ind}_{K_0}^K \tau_1) = n$.

Note that this arbitrary dimension of $\mathrm{ind}_{K_0}^K \tau_1$ cannot be achieved for an Abelian group K and a subgroup K_0, since in this case always $\dim(\mathrm{ind}_{K_0}^K \tau_1) = |K/K_0|$ holds.

Next we consider a non-commutative compact hypergroup, namely the semi-direct product hypergroup $K := H \rtimes_\alpha G$, where α is an action of a compact group

G on a compact commutative hypergroup H. For a representation π of K we denote the restrictions of π to H and G by ρ and τ respectively. We shall write $\pi = \rho \odot \tau$ expressing

$$\pi(h, g) := \rho(h)\tau(g)$$

for all $h \in H,\ g \in G$. The action $\hat{\alpha}$ of G on $\hat{H}$ induced by α is given by

$$\hat{\alpha}_g(\chi)(h) := \chi(\alpha_g^{-1}(h))$$

whenever $\chi \in \hat{H},\ g \in G,\ h \in H$. Let

$$G(\chi) := \{g \in G : \hat{\alpha}_g(\chi) = \chi\}$$

be the stabilizer of $\chi \in \hat{H}$ under the action $\hat{\alpha}$ of G on $\hat{H}$.

By an application of the Mackey machine, Any irreducible representation π of $K = H \rtimes_\alpha G$ is given by

$$\pi = \mathrm{ind}_{K_0}^{K}(\chi \odot \tau),$$

where $\chi \in \hat{H}, \tau = (\tau, \mathcal{H}(\tau)) \in \widehat{G(\chi)}$ and $K_0 = H \rtimes_\alpha G(\chi)$.

Moreover, the induced representation $\pi = \pi^{(\chi,\tau)} = \mathrm{ind}_{K_0}^{K}(\chi \odot \tau)$ is realized on the Hilbert space

$$\mathcal{H} := L^2(G(\chi) \setminus G, \mathcal{H}(\tau), \nu)$$

as follows : For $\xi \in L^2(G(\chi) \setminus G, \mathcal{H}(\tau), \nu)$,

$$(\pi(h, g)\xi)(x) = \chi(\alpha_{s(x)}(h))\tau(a(g, x))\xi(x \cdot g),$$

where

$$a(g, x)s(x \cdot g) = s(x)g$$

for some cross section $s : G(\chi) \setminus G \to G$, $h \in H,\ g \in G,\ x \in G(\chi) \setminus G$.

The induced representation $\pi^{(\chi,\tau)}$ is also written by $\pi^{(\chi,\tau)} = \rho^\chi \odot u^\tau$. Here we note that the representation u^τ of G and the representation ρ^χ of H are given by

$$u^\tau = \mathrm{ind}_{G(\chi)}^{G}\tau \quad \text{and} \quad \rho^\chi = \int_{O(\chi)}^{\oplus} \sigma\ \mu(d\sigma),$$

where μ is the $\hat{\alpha}$-invariant probability measure supporting the orbit $O(\chi)$ of χ.

3. Character formula for a compact commutative hypergroup

Let K be a second countably compact commutative hypergroup with countably infinite discrete dual hypergroup

$$\hat{K} = \{\chi_0, \chi_1, \cdots, \chi_n, \cdots\}.$$

For a character $\chi \in \hat{K}$ the weight $w(\chi)$ is given by

$$w(\chi) = \frac{1}{a_0(\chi)},$$

where $a_0(\chi) > 0$ and

$$\chi^{-}\chi = a_0(\chi)\chi_0 + a_1(\chi)\chi_1 + \cdots .$$

Definition The character $ch(\pi)$ of a representation $\pi \in \mathrm{Rep}^{\mathrm{f}}(K)$ is given by

$$ch(\pi) := \frac{w(\chi_1)}{w(\pi)}\chi_1 + \cdots + \frac{w(\chi_\ell)}{w(\pi)}\chi_\ell,$$

where

$$\pi \cong \chi_1 \oplus \cdots \oplus \chi_\ell$$

with $\chi_1, \cdots, \chi_\ell \in \hat{K}$, and

$$w(\pi) := w(\chi_1) + \cdots + w(\chi_\ell).$$

Now, let K_0 be a closed subhypergroup of K such that

$$|K/K_0| < \infty.$$

Definition For $\varphi \in S(M^b(K_0))$ the state induced by φ from K_0 to K is given as

$$\mathrm{ind}_{K_0}^{K}\varphi := \varphi \cdot 1_{K_0},$$

which means that

$$(\mathrm{ind}_{K_0}^{K}\varphi)(\mu) = \varphi(\mu)\mu(1_{K_0})$$

for all $\mu \in M^b(K_0)$.

Theorem 3 Under the assumption that $|K/K_0| < \infty$ we have

$$ch(\mathrm{ind}_{K_0}^{K}\tau) = \mathrm{ind}_{K_0}^{K} ch(\tau) = ch(\tau) \cdot 1_{K_0}$$

whenever $\tau \in \mathrm{Rep}^{\mathrm{f}}(K_0)$.

4. Character formula for a compact semi-direct product hypergroup

We retain the notion of induced representation for a semi-direct product hypergroup $K = H \rtimes_\alpha G$, where H is a compact commutative hypergroup of strong type and G is an arbitrary compact group. For $\chi \in H$ and $\tau \in \widehat{G(\chi)}$,

$$\pi^{(\chi,\tau)} := \mathrm{ind}_{H \rtimes_\alpha G(\chi)}^{H \rtimes_\alpha G}(\chi \odot \tau)$$

is an irreducible representation of K. By the Mackey machine any irreducible representation π of K is obtained in the form

$$\pi = \pi^{(\chi,\tau)}$$

for some $\chi \in \hat{H}$ and $\tau \in \widehat{G(\chi)}$. Moreover we note that $\pi^{(\chi,\tau)} = \rho^{\chi} \odot u^{\tau}$, where $\rho^{\chi} \in \widehat{H^{\alpha}}$ and $u^{\tau} = \mathrm{ind}_{G(\chi)}^{G}\tau$.

We know that for a compact hypergroup K any irreducible representation of K is finite-dimensional. Therefore one can introduce the character $ch(\pi)$ of an irreducible representation π of K by

$$ch(\pi)(k) := \frac{1}{\dim \pi}\mathrm{tr}(\pi(k))$$

for all $k \in K$. We also define the *hypergroup dimension* $d(\pi)$ of $\pi = \pi^{(\chi,\tau)} \in \mathrm{Irr}(H \rtimes_{\alpha} G)$ by

$$d(\pi) := w(\chi) \dim \pi = w(\rho^{\chi}) \dim\tau,$$

where $w(\chi)$ and $w(\rho^{\chi}) = w(\chi)|O(\chi)|$ denote the weights of $\chi \in \hat{H}$ and $\rho^{\chi} \in \widehat{H^{\alpha}}$ respectively.

Any finite-dimensional representation π of K admits an irreducible decomposition

$$\pi \cong \sum_{j=1}^{\ell}{}^{\oplus} \pi^{j},$$

where $\pi^{j} := \pi^{(\chi_j,\tau_j)} \in \mathrm{Irr}(H \rtimes_{\alpha} G)$ for some $\chi_j \in \hat{H}$, $\tau_j \in \widehat{G(\chi_j)}$ for $j = 1, \cdots, \ell$.

Definition For $\pi \in \mathrm{Rep}^{\mathrm{f}}(K)$, the character $ch(\pi)$ is given by

$$ch(\pi) := \sum_{j=1}^{\ell} \frac{d(\pi^{j})}{d(\pi)} ch(\pi^{j}), \quad \text{where } d(\pi) = \sum_{j=1}^{\ell} d(\pi^{j}).$$

Now, let G_0 be a closed subgroup of G such that $|G_0 \setminus G| < \infty$, and let $G_0(\chi)$ be the stabilizer of χ in G_0. We abbreviate $H \rtimes_{\alpha} G_0$ by K_0.

Proposition For an irreducible representation $\pi_0 = \rho_0 \odot u_0$ of $K_0 = H \rtimes_{\alpha} G_0$ the character of the induced representation $\mathrm{ind}_{K_0}^{K}\pi_0$ of π_0 takes the form

$$\begin{aligned} ch(\mathrm{ind}_{K_0}^{K}\pi_0)(k) &= \int_G ch(\pi_0)(sks^{-1}) 1_{K_0}(sks^{-1})\omega_G(ds) \\ &= \int_G \rho_0(\alpha_s(h))u_0(sgs^{-1})) 1_{G_0}(sgs^{-1})\omega_G(ds) \\ &=: \mathrm{ind}_{K_0}^{K}(ch(\pi_0))(k) \end{aligned}$$

whenever $k = (h, g) \in K$.

Theorem 4 For $\pi_0 \in \mathrm{Rep}^{\mathrm{f}}(K_0)$, where $K_0 = H \rtimes_\alpha G_0$,

$$ch(\mathrm{ind}_{K_0}^K \pi_0) = \mathrm{ind}_{K_0}^K ch(\pi_0)$$

holds.

5. Duals related to finite hypergroups

Let K be a finite hypergroup. For an irreducible representation π of K its character $ch(\pi)$ is given by

$$ch(\pi) := \frac{1}{\dim \pi} \mathrm{tr}(\pi(k))$$

for all $k \in K$. We consider the character set

$$\mathcal{K}(\hat{K}) := \{ch(\pi) : \pi \in \hat{K}\}.$$

We shall say, that the dual $\hat{K}$ of K *admits a hypergroup structure* if $\mathcal{K}(\hat{K})$ is a hypergroup with respect to the product of functions on K. Clearly, the dual $\hat{G}$ of a finite group G always admits a hypergroup structure.

Let α be an action of a finite Abelian group G on a finite commutative hypergroup H of strong type. Then a semi-direct product hypergroup $K = H \rtimes_\alpha G$ can be defined. Let

$$G(\chi) := \{g \in G : \hat{\alpha}_g(\chi) = \chi\}$$

be the stabilizer of $\chi \in \hat{H}$ under the action $\hat{\alpha}$. Now, let

$$\hat{H} := \{\chi_0, \chi_1, \cdots, \chi_n\},$$

where χ_0 denotes the trivial character of H.

Definition The action α is said to satisfy the *regularity condition* (or is called regular) provided

$$G(\chi_k) \supset G(\chi_i) \cap G(\chi_j)$$

for all $\chi_k \in \hat{H}$ such that

$$\chi_k \in \mathrm{supp}(\chi_i \chi_j) := \mathrm{supp}(\delta_{\chi_i} \hat{*} \delta_{\chi_j})$$

whenever $\chi_i, \chi_j \in \hat{H}$ and $\hat{*}$ symbolizes the convolution on $\hat{H}$, $k, i, j \in \{0, 1, \cdots, n\}$.

Theorem 5 The character set $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ of the semi-direct product hypergroup $H \rtimes_\alpha G$ is a commutative hypergroup if and only if the action α of G on H satisfies the regularity condition.

Examples Let $W_q(4) := (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_\beta \mathbb{Z}_2$, $D_q(4) := \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha \mathbb{Z}_2$, $Q_q(4) := \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2$.

1. $\widehat{W_q(4)}$ does not admit a hypergroup structure if $q \neq 1$, since the action α of $\mathbb{Z}_2$ on $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$ does not satisfy the regularity condition.
2. $\widehat{D_q(4)}$ and $\widehat{Q_q(4)}$ admit hypergroup structures in the sense that $\mathcal{K}(\widehat{D_q(4)})$ and $\mathcal{K}(\widehat{Q_q(4)})$ are hypergroups respectively. Moreover we see that

$$\mathcal{K}(\widehat{D_q(4)}) \cong \mathcal{K}(\widehat{Q_q(4)}),$$

although $D_q(4)$ is not isomorphic to $Q_q(4)$ as a hypergroup.

6. Duals related to hypergroups of non-type I

In this section we assume given a countably infinite discrete Abelian group G, a commutative hypergroup H of strong type and an action α of G on H. The semi-direct product hypergroup $K = H \rtimes_\alpha G$ is defined. For the subsequent discussion the following **Assumptions** are made:

(1) The action $\hat{\alpha}$ of G on $\hat{H}$ is free, i.e. $G(\chi) = \{e\}$ for all $\chi \neq \chi_0$.
(2) Every orbit in $\hat{H}$ under the action $\hat{\alpha}$ of G is relatively compact.

We note that in this case the action $\hat{\alpha}$ of G on $\hat{H}$ is non-smooth.

Under these assumptions $K = H \rtimes_\alpha G$ has a type II_1 factor representation and represents a hypergroup of non-type I.

Given a type II_1 factor representation π of K we introduce the character $ch(\pi)$ of π by

$$ch(\pi)(k) := \tau(\pi(k))$$

for all $k \in K$, where τ denotes the unique trace of the type II_1 factor $\pi(K)''$.

The dual object to be considered in this section will be the set $\widehat{K^{\mathrm{II}_1}}$ of quasi-equivalence classes of type II_1 factor representations of the (non-type I) hypergroup K. We are interested in studying the character set

$$\mathcal{K}(\widehat{K^{\mathrm{II}_1}}) := \{ch(\pi) : \pi \in \widehat{K^{\mathrm{II}_1}}\} \cup \{1_H\}.$$

For a type II_1 factor representation π of the hypergroup $K = H \times_\alpha G$, the von Neumann algebra $\pi(H)''$ generated by $\pi(H)$ is commutative, hence isomorphic to $L^\infty(O, \mu)$ for $O \in \Gamma$, $O \neq O_0$ where O is the closure of some orbit of $\chi \in \hat{H}$ under the action α of G, and an $\hat{\alpha}$-invariant ergodic probability measure $\mu =: \mu^\pi$ with $\mathrm{supp}(\mu^\pi) = O$.

Lemma Let π be a type II_1 factor representation of the hypergroup $K = H \rtimes_\alpha G$ with representing Hilbert space $\mathcal{H}$ such that $\mu^\pi = \mu^{\pi^O}$. Then π is quasi-equivalent to the canonical type II_1 factor representation π^O.

Finally, let L be a compact Abelian group and G a countably infinite discrete subgroup of L which is dense in L. Suppose that the action $\hat{\alpha}$ of L on $\hat{H}$ is free.

Lemma Under the assumption just stated any G-invariant ergodic probability measure μ with $\mathrm{supp}(\mu) = \overline{\mathrm{Orb}_G(\chi)} = \mathrm{Orb}_L(\chi)$ exists and it is unique.

Theorem 6 Keeping the assumptions preceding Lemma and considering the semi-direct product hypergroup $K = H \rtimes_\alpha G$ the character set $\mathcal{K}(\widehat{K^{\mathrm{II}_1}})$ becomes a commutative hypergroup isomorphic to the orbital hypergroup $\mathcal{K}^L(\hat{H})$ of $\hat{H}$ under the action $\hat{\alpha}$ of L.

Example (Discrete Mautner group) Let $K = \mathbb{C} \rtimes_\alpha \mathbb{Z} \subset \mathbb{C} \rtimes_\alpha \mathbb{T}$. Then

$$\Gamma(\hat{\mathbb{C}}) = \{O^\lambda : \lambda \in \mathbb{R}_+\}$$

where $O^\lambda = \{z \in \mathbb{C} : |z| = \lambda\}$. For each $\lambda \in \mathbb{R}_+$ a type II_1 factor representation π^λ is defined and

$$\pi^\lambda(K)'' = L^\infty(O^\lambda) \rtimes_\alpha \mathbb{Z}$$

is a type II_1 factor whenever $\lambda \neq 0$. Moreover,

$$ch(\pi^\lambda)(z, n) = J_0(\lambda|z|) \cdot 1_{\{0\}}(n)$$

for all $z \in \mathbb{C}$, $n \in \mathbb{Z}$, and $\mathcal{K}(\widehat{K^{\mathrm{II}_1}})$ is isomorphic to the Bessel-Kingman hypergroup $BK(J_0)$ of order 0.

References

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