A characterization of nilpotent varieties of complex semisimple Lie algebras

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Introduction

A normal complex algebraic variety X is called a *symplectic variety* (cf. [Be]) if its regular locus X_{reg} admits a holomorphic symplectic 2-form ω such that it extends to a holomorphic 2-form on a resolution $f : \tilde{X} \to X$.

Affine symplectic varieties are constructed in various ways such as nilpotent orbit closures of a semisimple complex Lie algebra (cf. [CM]), Slodowy slices to nilpotent orbits (cf. [Sl]) or symplectic reductions of holomorphic symplectic manifolds with Hamiltonian actions. Usually these examples show up with $\mathbf{C}^*$-actions.

In this lecture we shall characterizes the nilpotent variety of a complex semisimple Lie algebra among affine symplectic varieties from a view point of algebraic geometry.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let N be the nilpotent variety of $\mathfrak{g}$. It is well known that N is an affine normal variety and its regular locus admits a holomorphic symplectic 2-form ω_{KK} called the Kostant-Kirilliv 2-form. Then (N, ω_{KK}) is an affine symplectic variety in our sense. Moreover, the scalar multiplication determines a $\mathbf{C}^*$-action on $\mathfrak{g}$ and it induces a $\mathbf{C}^*$-action also on N. The Kostant-Kirillov 2-form ω_{KK} has weight 1 with respect to this $\mathbf{C}^*$-action. The adjoint group G acts on $\mathfrak{g}$ and let $\mathfrak{g}//G$ be the GIT quotient of the G-action. Namely $\mathfrak{g} := \mathrm{Spec}\mathbf{C}[\mathfrak{g}]^G$, where $\mathbf{C}[\mathfrak{g}]^G$ is the G-invariant ring of the coordinate ring $\mathbf{C}[\mathfrak{g}]$ of $\mathfrak{g}$. By a theorem of Chevalley, $\mathbf{C}[\mathfrak{g}]^G$ is isomorphic to a polynomial ring $\mathbf{C}[f_1, ..., f_r]$ generated by algebraically independent G-invariant homogeneous polynomials f_i. Here r coincides with the rank of $\mathfrak{g}$. Let $\chi : \mathfrak{g} \to \mathfrak{g}//G = \mathbf{C}^r$ be the adjoint quotient map. Then $N = \chi^{-1}(0)$. In particular, N is a complete intersection of r homogeneous polynomials in the affine space $\mathfrak{g}$.

Our main theorem asserts that the converse holds true. More precisely, let (X, ω) be a $2n$-dimensional affine symplectic variety embedded in the affine space $\mathbf{C}^{2n+r}$ as a complete intersection of r homogeneous polynomials $f_i(z_1, ..., z_{2n+r}) = 0$ $(1 \leq i \leq r)$. The affine space $\mathbf{C}^{2n+r}$ has a standard $\mathbf{C}^*$-action with $wt(z_i) = 1$ for all i. It induces a $\mathbf{C}^*$-action on X. We assume that the symplectic form ω is homogeneous with respect to this $\mathbf{C}^*$-action. Namely, for some integer l, we have $t^*\omega = t^l \cdot \omega$ where $t \in \mathbf{C}^*$. The integer l is called the weight of ω and is denoted by $wt(\omega)$. When X is smooth, (X, ω) is isomorphic to $(\mathbf{C}^{2n}, \omega_0)$, where ω_0 is the standard symplectic 2-form $\Sigma dz_{2i-1} \wedge dz_{2i}$. In the remainder we restrict ourselves to the case when X is singular. Then we have:

Main Theorem. *There is a $\mathbf{C}^*$-equivariant isomorphism $(X, \omega) \cong (N, \omega_{KK})$ of symplectic varieties. Here N is the nilpotent variety of a complex semisimple Lie algebra $\mathfrak{g}$ and ω_{KK} is the Kostant-Kirillov 2-form.*

The proof consists of two steps. At first we prove that X coincides with a nilpotent orbit closure $\bar{O}$ of a semisimple complex Lie algebra $\mathfrak{g}$ (Theorem 2). Theorem 2 actually shows that X is the closure of a Richardson orbit O and $\bar{O}$ has a crepant resolution. We next prove in **6** that such a nilpotent orbit closure $\bar{O}$ must be the nilpotent variety N if it has complete intersection singularities.

A symplectic variety tends to have a large embedded codimension. The main theorem shows that the A_1 surface singularity is a unique homogeneous symplectic hypersurface. As is studied in [LNSV] we have some examples of quasihomogeneous symplectic hypersurfaces in higher dimensions.

The results of this note are concerned with symplectic varieties. However the proof of Theorem 2 is based on contact geometry. In particular, a structure theorem [KPSW] on contact projective manifolds plays a crucial role.

The results of this note have already been published in [Na].

1. Let X be a homogeneous symplectic variety of complete intersection defined in Introduction.

When X is smooth, the polynomials f_i are all linear forms; hence we may assume that $r = 0$ and $X = \mathbf{C}^{2n}$. We can write $\omega^n := \omega \wedge ... \wedge \omega = g \cdot dz_1 \wedge ... \wedge dz_{2n}$ with a nowhere vanishing homogeneous polynomial g. Since such a polynomial g must be a constant, we have $wt(\omega) = 2$. Now ω has a form $\Sigma a_{ij} dz_i \wedge dz_j$ with some constants a_{ij}. Then ω becomes the standard symplectic 2-form $\Sigma_{1 \leq i \leq n} dz_{2i-1} \wedge dz_{2i}$ after a suitable linear transformation of $\mathbf{C}^{2n}$.

From now on we consider the case when X is singular. Without loss of generality we may assume that $\deg(f_i) \geq 2$ for all i. In the remainder we put $a_i := \deg(f_i)$. By the adjunction formula (or the residue formula) we have

$$\omega^n = c \cdot \mathrm{Res}_X(dz_1 \wedge ... \wedge dz_{2n+r}/(f_1, ..., f_r))$$

with a nonzero constant c; hence

$$wt(\omega^n) = 2n + r - \Sigma a_i.$$

Since $wt(\omega^n) = n \cdot wt(\omega)$ and $wt(\omega) > 0$ (cf. [LNSV], Lemma 2.2), we have $\Sigma a_i = n + r$ and $wt(\omega) = 1$.

We next consider a resolution of the singular variety X. Since X is a normal Gorenstein singularity, its canonical divisor K_X is a Cartier divisor. A resolution $\pi : Y \to X$ is called *crepant* if $K_Y = \pi^* K_X$. A general symplectic variety does not have a crepant resolution, but our X has because it is of complete intersection:

Theorem 1. *X has a $\mathbf{C}^*$-equivariant crepant resolution $\pi : Y \to X$.*

Proof. Let us take a resolution $g : W \to X$ and apply the minimal model program to g ([BCHM]). We then finally get a $\mathbf{Q}$-factorial terminalisation $\pi : Y \to X$ of X. Namely Y has only $\mathbf{Q}$-factorial terminal singularities and $K_Y = \pi^* K_X$. We shall prove that Y is actually smooth.

The pullback $\pi^*\omega$ defines a symplectic structure on the regular part of Y. Let $f : Z \to Y$ be a resolution of Y. By the assumption $(\pi \circ f)^*\omega$ extends to a holomorphic 2-form on Z; hence Y is a symplectic variety. Then $\mathrm{Sing}(Y)$ has even codimension by Kaledin [Ka]. On the other hand, since Y has only terminal singularities, $\mathrm{Codim}_Y \mathrm{Sing}(Y) \geq 3$. Hence we have $\mathrm{Codim}_Y \mathrm{Sing}(Y) \geq 4$. Moreover the $\mathbf{C}^*$-action on X extends to a $\mathbf{C}^*$-action on Y (cf. [Na 1, Proposition A.7]). Note here that a symplectic variety has a natural Poisson structure and one can consider its Poisson deformation (cf. [Na 2]). Take a Poisson deformation Y_t of Y. Then the birational map $\pi : Y \to X$ also deforms to a birational map $\pi_t : Y_t \to X_t$, where X_t is a Poisson deformation of X. If we take the Poisson deformation Y_t general enough, then π_t is an isomorphism (cf. [Na 2, Theorem 5.5]). In particular, $Y_t = X_t$. Since X has only complete intersection singularities, so does Y_t. On the other hand, $\mathrm{Codim}_{Y_t} \mathrm{Sing}(Y_t) \geq 4$. By a result of Beauville [Be, Proposition

1.4], a symplectic singularity is a complete intersection singularity only if its singular locus has codimension ≤ 3. Therefore, Y_t must be smooth. Since Y has only $\mathbf{Q}$-factorial terminal singularities, any Poisson deformation of Y is locally trivial as a flat deformation by Proposition A.9 and Theorem 17 of [Na 1]. This means that Y is smooth. Q.E.D.

Here let us recall the notion of a contact structure. A complex manifold M of dimension $2n-1$ has a *contact structure* if there is an exact sequence of vector bundles on M:

$$0 \to D \to \Theta_M \stackrel{\eta}{\to} L \to 0,$$

where $\mathrm{rank}(D) = 2n-2$, L is a line bundle, and $D \times D \to L$, $(x, y) \to \eta([x, y])$ is a non-degenerate pairing. Notice that η can be regarded as a section of $\Omega^1_M \otimes L$. We call this twisted 1-form a *contact form* and call the line bundle L a *contact line bundle*.

A contact structure is naturally introduced in the following situation. Assume that M is a complex manifold and L is a line bundle on M. We put $(L^{-1})^{\times} := L^{-1} - (0 - section)$ and let $p : (L^{-1})^{\times} \to M$ be the projection map. As $(L^{-1})^{\times}$ is a $\mathbf{C}^*$-bundle, there is a natural $\mathbf{C}^*$-action on $(L^{-1})^{\times}$. The $\mathbf{C}^*$-action determines a vector field ζ on $(L^{-1})^{\times}$. Assume that $(L^{-1})^{\times}$ admits a holomorphic symplectic 2-form ω of weight 1 with respect to the $\mathbf{C}^*$-action. Then the 1-form $i_{\zeta}\omega$ on $(L^{-1})^{\times}$ has weight 1 and we can write $i_{\zeta}\omega = p^*\eta$ for $\eta \in \Gamma(M, \Omega^1_M \otimes L)$. Then this η determines a contact structure on M.

We can apply this construction to the projectivisation $\mathbf{P}(X) := X - \{0\}/\mathbf{C}^*$ of X. The $\mathbf{C}^*$-bundle $O_{\mathbf{P}(X)}(-1)^{\times} \to \mathbf{P}(X)$ induces a $\mathbf{C}^*$-bundle $O_{\mathbf{P}(X)_{reg}}(-1)^{\times} \to \mathbf{P}(X)_{reg}$. As X_{reg} is identified with $O_{\mathbf{P}(X)_{reg}}(-1)^{\times}$, there is a holomorphic symplectic 2-form ω of weight 1 on it. Then it determines a contact structure on $\mathbf{P}(X)_{reg}$ (cf. [LeB], [Na 3, Section 4]). More precisely there is an exact sequence of vector bundles on $\mathbf{P}(X)_{reg}$:

$$0 \to D \to \Theta_{\mathbf{P}(X)_{reg}} \stackrel{\eta}{\to} O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{reg}} \to 0,$$

where η is a contact 1-form.

2. We first claim that $\mathbf{P}(X)$ also has a crepant resolution [1]. Let L be a π-ample line bundle on Y. If necessary, replacing L by its suitable multiple, we may assume that L has a $\mathbf{C}^*$-linearisation (cf. [CG] Theorem 5.1.9). We

[1] This is a crucial conclusion obtained from the assumption $wt(z_i)$ are all 1.

put $A_m := \Gamma(Y, L^{\otimes m})$ for each $m \geq 0$. Note that each A_m has a grading determined by the $\mathbf{C}^*$-action. In particular, A_0 is the coordinate ring of X and $\mathbf{P}(X) = \mathrm{Proj}(A_0)$. Since A_m are graded A_0-modules, we can consider the associated coherent sheaves $\tilde{A}_m$ on $\mathbf{P}(X)$. Define $Z := \mathcal{P}roj_{\mathbf{P}(X)}(\oplus \tilde{A}_m)$. Then Z can be identified with $Y - \pi^{-1}(0)/\mathbf{C}^*$ and the projective morphism $\bar{\pi} : Z \to \mathbf{P}(X)$ can be identified with the natural map $Y - \pi^{-1}(0)/\mathbf{C}^* \to X - \{0\}/\mathbf{C}^*$ induced by the $\mathbf{C}^*$-equivariant resolution $\pi : Y \to X$. In particular, $\bar{\pi}$ is a birational map. Look at the commutative diagram

$$\begin{array}{ccc} Y - \pi^{-1}(0) & \longrightarrow & Y - \pi^{-1}(0)/\mathbf{C}^* \\ \downarrow & & \downarrow \\ X - \{0\} & \longrightarrow & X - \{0\}/\mathbf{C}^*. \end{array} \tag{1}$$

Pick a point $x := (z_1(x), ..., z_{2n+r}(x)) \in X - \{0\}$. We have $z_i(x) \neq 0$ for some i. Define $U_x := X \cap \{(z_1, ..., z_{2n+r}) \in \mathbf{C}^{2n+r}; z_i = z_i(x)\}$. Then U_x is isomorphically mapped onto a Zariski open subset of $\mathbf{P}(X)$ by the map $X - \{0\} \to \mathbf{P}(X)$. The map

$$\sigma_x : \mathbf{C}^* \times U_x \to X - \{0\}$$

sending $(t, x') \in \mathbf{C}^* \times U_x$ to $t \cdot x' \in X - \{0\}$ is an open immersion. We put $V_x := \pi^{-1}(U_x)$. Choose a point $y' \in V_x$ and put $x' := \pi(y')$. Denote by $O_{x'}$ (resp. $O_{y'}$) the $\mathbf{C}^*$-orbit of x' (resp. y').

Since $O_{x'}$ and $O_{y'}$ are both $\mathbf{C}^*$ orbits, there are natural surjections $\gamma_{x'} : \mathbf{C}^* \to O_{x'}$ $(t \to t \cdot x')$ and $\gamma_{y'} : \mathbf{C}^* \to O_{y'}$ $(t \to t \cdot y')$. Moreover $\gamma_{y'}$ factorizes $\gamma_{x'}$:

$$\mathbf{C}^* \stackrel{\gamma_{y'}}{\to} O_{y'} \to O_{x'}.$$

Since $wt(z_i) = 1$ for all i, we see that $\gamma_{x'}$ is an isomorphism; hence $\gamma_{y'}$ is also an isomorphism and $O_{y'} \cong O_{x'}$.

Let $T_{y'}V_x$ (resp. $T_{y'}O_{y'}$) be the tangent space of V_x (resp. $O_{y'}$) at y'. Then one has

$$T_{y'}V_x \cap T_{y'}O_{y'} = \{0\}.$$

In fact, the isomorphism $O_{y'} \to O_{x'}$ induces an isomorphism of the tangent spaces $T_{y'}O_{y'} \to T_{x'}O_{x'}$. This isomorphism induces an injection $T_{y'}V_x \cap T_{y'}O_{y'} \to T_{x'}U_x \cap T_{x'}O_{x'}$. Since $T_{x'}U_x \cap T_{x'}O_{x'} = \{0\}$ by the construction of U_x, we see that $T_{y'}V_x \cap T_{y'}O_{y'} = \{0\}$.

Let us consider the map

$$\sigma_{V_x} : \mathbf{C}^* \times V_x \to Y - \pi^{-1}(0).$$

This map induces a map of tangent spaces

$$T_{(t,y')}(\mathbf{C}^* \times V_x) \to T_{t\cdot y'}Y$$

for $(t, y') \in \mathbf{C}^* \times V_x$.

We claim that V_x is smooth at y' and this map of tangent spaces is an isomorphism. We first show the injectivity. We identify $T_{(t,y')}(\mathbf{C}^* \times \{y'\})$ with $T_t\mathbf{C}^*$ and identify $T_{(t,y')}(\{t\} \times V_x)$ with $T_{y'}V_x$. Then $T_{(t,y')}(\mathbf{C}^* \times V_x) = T_t\mathbf{C}^* \oplus T_{y'}V_x$. Assume that $(\alpha, \beta) \in T_t\mathbf{C}^* \oplus T_{y'}V_x$ is sent to zero by the map above. The map σ_{V_x} induces isomorphisms $\mathbf{C}^* \times \{y'\} \to O_{y'}$ and $\{t\} \times V_x \to t \cdot V_x$. Therefore $(\alpha, 0)$ is sent to an element of $T_{t\cdot y'}O_{y'}$ and $(0, \beta)$ is sent to an element of $T_{t\cdot y'}(t \cdot V_x)$. Since $T_{y'}V_x \cap T_{y'}O_{y'} = \{0\}$, we also have $T_{t\cdot y'}(t \cdot V_x) \cap T_{t\cdot y'}O_{y'} = \{0\}$ by the $\mathbf{C}^*$-action. This implies that $\alpha = \beta = 0$.

Note that $\dim Y = \dim V_x + 1$ and Y is smooth. If V_x is singular at y', then $\dim T_{y'}V_x > \dim V_x$; but then $\dim T_{(t,y')}(\mathbf{C}^* \times V_x) > \dim T_{t\cdot y'}Y$. This contradicts that the above map is an injection. Thus V_x must be smooth at y'. Moreover this implies that the map is an isomorphism.

We finally claim that σ_{V_x} is an open immersion. Assume that two points $(t_i, y_i) \in \mathbf{C}^* \times V_x$, $i = 1, 2$ are mapped to the same point of Y. Then y_1 and y_2 are contained in the same $\mathbf{C}^*$-orbit. Moreover $\pi(y_1) = \pi(y_2)$. (If $\pi(y_1) \neq \pi(y_2)$, then $\pi(y_1)$ and $\pi(y_2)$ must be contained in different $\mathbf{C}^*$-orbits because σ_{U_x} is an open immersion.) If $y_1 \neq y_2$, then the natural map $O_{y_1} \to O_{\pi(y_1)}$ of $\mathbf{C}^*$-orbits is not a bijection. This contradicts the previous observation. Thus $y_1 = y_2$. Then one has $t_1 = t_2$ because $\gamma_{y_1} : \mathbf{C}^* \to O_{y_1}$ ($t \to t \cdot y_1$) is an isomorphism. This shows that σ_{V_x} is an injection. Since $\mathbf{C}^* \times V_x$ and Y are both nonsingular and the map $T_{(t,y')}(\mathbf{C}^* \times V_x) \to T_{t\cdot y'}Y$ is an isomorphism, we see that σ_{V_x} is an open immersion.

Now the commutative diagram above is locally identified with

$$\begin{array}{ccc} \mathbf{C}^* \times V_x & \xrightarrow{p_2} & V_x \\ \downarrow & & \downarrow \\ \mathbf{C}^* \times U_x & \xrightarrow{p_2} & U_x. \end{array} \tag{2}$$

By the assumption $\mathbf{C}^* \times V_x \to \mathbf{C}^* \times U_x$ is a crepant resolution. This means that $V_x \to U_x$ is also a crepant resolution.

Therefore we get a crepant resolution $\bar{\pi} : Z \to \mathbf{P}(X)$ of $\mathbf{P}(X)$.

3. We next claim that Z is a contact projective manifold with the contact line bundle $\bar{\pi}^* O_{\mathbf{P}(X)}(1)$.

For simplicity we write L for $O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{reg}}$. The contact structure on $\mathbf{P}(X)_{reg}$ is expressed as a twisted 1-form $\eta \in \Gamma(\mathbf{P}(X)_{reg}, \Omega^1_{\mathbf{P}(X)_{reg}} \otimes L)$ such that $\eta \wedge (d\eta)^{n-1} \in O_{\mathbf{P}(X)_{reg}}$ is nowhere-vanishing. In our case L extends to the line bundle $O_{\mathbf{P}(X)}(1)$ on $\mathbf{P}(X)$. Let $i : \mathbf{P}(X)_{reg} \to \mathbf{P}(X)$ be the natural inclusion map. Since $\mathbf{P}(X)$ has only canonical singularities, we have $\bar{\pi}_* \Omega^1_Z \cong i_* \Omega^1_{\mathbf{P}(X)_{reg}}$ ([GKK]). Hence the pull-back $\bar{\pi}^*\eta$ is a section of $\Omega^1_Z \otimes \bar{\pi}^* O_{\mathbf{P}(X)}(1)$. Moreover, since $\bar{\pi}$ is a crepant resolution, $\bar{\pi}^*\eta \wedge (d\bar{\pi}^*\eta)^{n-1}$ is nowhere-vanishing.

Therefore we get a contact structure of Z with the contact line bundle $\bar{\pi}^* O_{\mathbf{P}(X)}(1)$.

4. When $n = 1$ we already know that $r = 1$ and $f = z_1^2 + z_2^2 + z_3^2$ after a suitable change of coordinates (cf. [LNSV], 3.1). Note that $Z = \mathbf{P}(X) = \mathbf{P}^1$ in this case. We assume that $n \geq 2$. Then $\mathrm{Codim}_X \mathrm{Sing}(X) = 2$ by [Be, Proposition 1.4]. Hence $\mathbf{P}(X)$ actually has singularities and $b_2(Z) \geq 2$. Note that K_Z is not nef because $K_Z = \bar{\pi}^* O_X(-n)$. Now we apply the following structure theorem of Kebekus, Peternell, Sommese and Wisniewski [KPSW].

Theorem. *Let Z be a contact projective manifold with a contact line bundle L. Assume that $b_2(Z) \geq 2$ and K_Z is not nef. Then Z is isomorphic to the projectivised cotangent bundle $\mathbf{P}(\Theta_M)$*[2] *of a projective manifold M of dimension n; moreover, $L \cong O_{\mathbf{P}(\Theta_M)}(1)$.*

Since the contact line bundle is $\bar{\pi}^* O_{\mathbf{P}(X)}(1)$ in our case, we have $\bar{\pi}^* O_{\mathbf{P}(X)}(1) \cong O_{\mathbf{P}(\Theta_M)}(1)$.

Let η_0 be the canonical contact structure on $\mathbf{P}(\Theta_M)$ induced by the canonical symplectic form on T^*M. Note here that an automorphism φ of the vector bundle Θ_M induces an automorphism of $Z := \mathbf{P}(\Theta_M)$, which is denoted by the same notation φ. Then Ω^1_Z and $O_{\mathbf{P}(\Theta_M)}(1)$ are both $\mathrm{Aut}(\Theta_M)$-linearlized. Then our contact form η can be written as $\eta = \varphi^*\eta_0$ for some $\varphi \in \mathrm{Aut}(\Theta_M)$ (cf. [KPSW], Proposition 2.14). We may assume that $\eta = \eta_0$ by composing φ with the initial identification $Z \cong \mathbf{P}(\Theta_M)$.

The embedding $X \to \mathbf{C}^{2n+r}$ induces an embedding $\mathbf{P}(X) \to \mathbf{P}^{2n+r-1}$.

[2]In this note we employ Grothendieck's notation for a projective space bundle. Namely $\mathbf{P}(\Theta_M) = T^*M - (0 - \mathit{section})/\mathbf{C}^*$.

Since $H^0(\mathbf{P}^{2n+r-1}, O_{\mathbf{P}^{2n+r-1}}(1)) \cong H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(1))$, the morphism $\bar{\pi}$ coincides with the one defined by the complete linear system $|O_{\mathbf{P}(\Theta_M)}(1)|$.

Lemma. $\chi(\mathbf{P}(X), O_{\mathbf{P}(X)}) = 1$.

Proof. We first claim that if $W \subset \mathbf{P}^m$ is a complete intersection of type $(d_1, ..., d_k)$, then $\chi(W, O_W(-i)) = 0$ for all $i > 0$ with $d_1 + ... + d_k + i < m + 1$. We prove this by the induction on k. Assume that this is true for $k - 1$. Let us take the complete intersection W' of type $(d_1, ..., d_{k-1})$ such that W is an element of $|O_{W'}(d_k)|$. By the exact sequence

$$0 \to O_{W'}(-i - d_k) \to O_{W'}(-i) \to O_W(-i) \to 0$$

we have $\chi(O_W(-i)) = \chi(O_{W'}(-i)) - \chi(O_{W'}(-i - d_k))$. Assume that $d_1 + ... + d_k + i < m + 1$. Then we have $d_1 + ... + d_{k-1} + (i + d_k) < m + 1$ and $d_1 + ... + d_{k-1} + i < m + 1$. By the induction assumption $\chi(O_{W'}(-i - d_k)) = \chi(O_{W'}(-i)) = 0$; hence $\chi(O_W(-i)) = 0$.

We next claim that $\chi(W, O_W) = 1$ if $d_1 + ... + d_k < m + 1$. This is also proved by the induction on k. We take the same W' as above. Then $\chi(O_W) = \chi(O_{W'}) - \chi(O_{W'}(-d_k))$. By the induction assumption $\chi(O_{W'}) = 1$. By the previous claim we have $\chi(O_{W'}(-d_k)) = 0$; hence $\chi(O_W) = 1$ as desired.

Let us return to the original situation. By the argument in **1** we have $\Sigma a_i < 2n + r$. Now one can apply the above claim to $\mathbf{P}(X) \subset \mathbf{P}^{2n+r-1}$. Q.E.D.

Since $\mathbf{P}(X)$ has only rational singularities, we have $\chi(Z, O_Z) = \chi(\mathbf{P}(X), O_{\mathbf{P}(X)}) = 1$. Let us consider the projection map $p : Z \to M$ of the projective space bundle. Since $R^i p_* O_Z = 0$ for $i > 0$, we have $\chi(Z, O_Z) = \chi(M, O_M)$. In particular, we see that $\chi(M, O_M) = 1$.

Here we recall a special case of the theorem of Demailly, Peternell and Schneider [DPS]

Theorem([DPS, Proposition on p.297]) : *Let M be a projective manifold with nef tangent bundle such that $\chi(M, O_M) \neq 0$. Then M is a Fano manifold. When $\dim M = 2$ or 3, M is a rational homogeneous space.*

In our case we have a much stronger condition. In fact, $O_{\mathbf{P}(\Theta_M)}(1)$ is the pull-back of a very ample line bundle by a birational morphism.

Proposition. *Let M be a Fano manifold. Assume that $|O_{\mathbf{P}(\Theta_M)}(1)|$ is free from base points. Then M is isomorphic to a rational homogeneous space,* *i.e.* $M \cong G/P$ *with a semisimple complex Lie group* G *and its parabolic subgroup* P.

Proof. The map $H^0(\mathbf{P}(\Theta_M), O_{\mathbf{P}(\Theta_M)}(1)) \otimes O_{\mathbf{P}(\Theta_M)} \xrightarrow{\beta} O_{\mathbf{P}(\Theta_M)}(1)$ is surjective. Let us consider the natural map

$$H^0(M, \Theta_M) \otimes O_M \xrightarrow{\alpha} \Theta_M.$$

We pull back α by the projection map $p : \mathbf{P}(\Theta_M) \to M$. Since $p_* O_{\mathbf{P}(\Theta_M)}(1) = \Theta_M$, $p^*\alpha$ factorizes β:

$$\beta : H^0(\mathbf{P}(\Theta_M), O_{\mathbf{P}(\Theta_M)}(1)) \otimes O_{\mathbf{P}(\Theta_M)} \xrightarrow{p^*\alpha} p^*\Theta_M \to O_{\mathbf{P}(\Theta_M)}(1).$$

Let $x \in M$ be an arbitrary point and restrict β to the fibre $p^{-1}(x) \cong \mathbf{P}^{n-1}$. Then we have

$$\beta(x) : H^0(\mathbf{P}(\Theta_M), O_{\mathbf{P}(\Theta_M)}(1)) \otimes O_{\mathbf{P}^{n-1}} \xrightarrow{p^*\alpha(x)} O_{\mathbf{P}^{n-1}}^{\oplus n} \to O_{\mathbf{P}^{n-1}}(1).$$

Note that $\beta(x)$ is also surjective. By taking the global sections $\beta(x)$ induces a map $\Gamma(\beta(x)) : H^0(\mathbf{P}(\Theta_M), O_{\mathbf{P}(\Theta_M)}(1)) \to H^0(\mathbf{P}^{n-1}, O_{\mathbf{P}^{n-1}}(1))$. If $\Gamma(\beta(x))$ is not surjective, then $\beta(x)$ cannot be surjective. Hence $\Gamma(\beta(x))$ must be surjective. This also shows that

$$\Gamma(p^*\alpha(x)) : H^0(\mathbf{P}(\Theta_M), O_{\mathbf{P}(\Theta_M)}(1)) \to H^0(\mathbf{P}^{n-1}, O_{\mathbf{P}^{n-1}}^{\oplus n})$$

is surjective. Since $\Gamma(p^*\alpha(x))$ can be identified with the map $H^0(M, \Theta_M) \otimes k(x) \xrightarrow{\alpha(x)} \Theta_M \otimes k(x)$, the map α is a surjection by Nakayama's lemma.

Let G be the neutral component of the automorphism group $\mathrm{Aut}(M)$ of M. Then G can be written as the extension of a complex torus T by a linear algebraic group L (cf. [Fu])

$$1 \to L \to G \to T \to 1.$$

Note that $q(M) = 0$ because M is a Fano manifold. If $\dim T > 0$, then $\dim \mathrm{Alb}(M) > 0$ by Theorem 5.5 of [Fu], which is a contradiction. Hence G is a linear algebraic group. As α is surjective, G acts transitively on M. Therefore $M \cong G/P$ for some parabolic subgroup P of G (cf. [Spr, 6.2]). Note that P always contains the radical $r(G)$ of G. Then $r(G)$ acts trivially on M; but, since G is the neutral component of $\mathrm{Aut}(M)$, G acts effectively on M. Hence $r(G) = \{1\}$ and G is semisimple. Q.E.D.

5. Assume that $n \geq 2$. Now M can be written as G/P with G a semisimple complex Lie group and P a parabolic subgroup of G. By the proof of the previous proposition we may assume that $G = \mathrm{Aut}^0(M)$. The cotangent bundle $T^*(G/P)$ of G/P has a natural Hamiltonian G-action and one can define the moment map $\mu : T^*(G/P) \to \mathfrak{g}^*$. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form. Then $\mathrm{Im}(\mu)$ coincides with the closure $\bar{O}$ of a nilpotent orbit $O \subset \mathfrak{g}$. The moment map induces a generically finite projective morphism of the projectivisations of $T^*(G/P)$ and $\bar{O}$:

$$\bar{\mu} : \mathbf{P}(\Theta_{G/P}) \to \mathbf{P}(\bar{O}).$$

Denote by $O_{\mathbf{P}(\bar{O})}(1)$ the restriction of the tautological line bundle $O_{\mathbf{P}(\mathfrak{g})}(1)$ of the projective space $\mathbf{P}(\mathfrak{g})$ to $\mathbf{P}(\bar{O})$. Then it can be checked that $O_{\mathbf{P}(\Theta_{G/P})}(1) = \bar{\mu}^* O_{\mathbf{P}(\bar{O})}(1)$ [3].

This means that $\bar{\pi} : \mathbf{P}(\Theta_{G/P}) \to \mathbf{P}(X)$ must be the Stein factorization of $\bar{\mu}$.

By looking at $\bar{\mu}$ we have an inequality

$$(1) \quad \dim \Gamma(\mathbf{P}(\Theta_{G/P}), O_{\mathbf{P}(\Theta_{G/P})}(1)) \geq \dim \Gamma(\mathbf{P}(\bar{O}), O_{\mathbf{P}(\bar{O})}(1)).$$

Let I be the ideal sheaf of $\mathbf{P}(\bar{O}) \subset \mathbf{P}(\mathfrak{g})$. There is an exact sequence

$$0 \to H^0(\mathbf{P}(\mathfrak{g}), O_{\mathbf{P}(\mathfrak{g})}(1) \otimes I) \to H^0(\mathbf{P}(\mathfrak{g}), O_{\mathbf{P}(\mathfrak{g})}(1)) \to H^0(\mathbf{P}(\bar{O}), O_{\mathbf{P}(\bar{O})}(1)).$$

Let $T_0\bar{O}$ be the tangent space of $\bar{O}$ at the origin $0 \in \bar{O}$. Let $\mathfrak{g} = \oplus \mathfrak{g}_i$ be the decomposition into the simple factors. The closure $\bar{O}$ is the product of nilpotent orbit closures $\bar{O}_i$ of $\mathfrak{g}_i$. Note that $T_0\bar{O} = \oplus T_0\bar{O}_i$. Each $T_0\bar{O}_i$ is a sub G_i-representation of the adjoint G_i-representation of $\mathfrak{g}_i$. Since $\mathfrak{g}_i$ is an irreducible G_i-representation, we have $T_0\bar{O}_i = \mathfrak{g}_i$. Hence $T_0\bar{O} = \mathfrak{g}$. This means that there is no hyperplane of $\mathfrak{g}$ containing $\bar{O}$; hence there is no hyperplane of $\mathbf{P}(\mathfrak{g})$ containing $\mathbf{P}(\bar{O})$. This shows that $H^0(\mathbf{P}(\mathfrak{g}), O_{\mathbf{P}(\mathfrak{g})}(1) \otimes I) = 0$. Since $h^0(\mathbf{P}(\mathfrak{g}), O_{\mathbf{P}(\mathfrak{g})}(1)) = \dim \mathfrak{g}$, we have an inequality

$$(2) \quad \dim \Gamma(\mathbf{P}(\bar{O}), O_{\mathbf{P}(\bar{O})}(1)) \geq \dim \mathfrak{g}.$$

[3] Let ω_{KK} be the Kostant-Kirillov 2-form on O. Then it gives a contact structure on $\mathbf{P}(O)$ with the contact line bundle $O_{\mathbf{P}(O)}(1)$. On the other hand, $\mu^*\omega_{KK}$ is a symplectic form on $T^*(G/P)$, which gives a contact structure on $\mathbf{P}(\Theta_{G/P})$ with the contact line bundle $\bar{\mu}^* O_{\mathbf{P}(\bar{O})}(1)$. Then we can apply [KPSW, Theorem 2.12] to conclude that $\bar{\mu}^* O_{\mathbf{P}(\bar{O})}(1) = O_{\mathbf{P}(\Theta_{G/P})}(1)$.

By (1) and (2) we have an inequality

$$\dim \Gamma(\mathbf{P}(\Theta_{G/P}), O_{\mathbf{P}(\Theta_{G/P})}(1)) \geq \dim \mathfrak{g}.$$

Since $\Gamma(\mathbf{P}(\Theta_{G/P}), O_{\mathbf{P}(\Theta_{G/P})}(1)) = \Gamma(G/P, \Theta_{G/P})$, this inequality is actually an equality. Hence $\bar{\pi}$ coincides with $\bar{\mu}$ and we have an isomorphism of polarised varieties $(\mathbf{P}(X), O_{\mathbf{P}(X)}(1)) \cong (\mathbf{P}(\bar{O}), O_{\mathbf{P}(\bar{O})}(1))$. As $X = \mathrm{Spec} \oplus_{m \geq 0} H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(m))$ and $\bar{O} = \mathrm{Spec} \oplus_{m \geq 0} H^0(\mathbf{P}(\bar{O}), O_{\mathbf{P}(\bar{O})}(m))$, this implies that $X = \bar{O}$.

Finally we give an intrinsic characterization of G. Notice that we have taken an isomorphism $Z \cong \mathbf{P}(\Theta_M)$ such that the contact structure corresponds to the canonical one induced by the canonical 2-form on T^*M. Then G acts on Z as contact automorphisms. Since $\bar{\pi}$ is G-equivariant, this also means that G acts on $\mathbf{P}(X)_{reg}$ as contact automorphisms. The G-action determines an embedding $\mathfrak{g} \subset H^0(\mathbf{P}(X)_{reg}, \Theta_{\mathbf{P}(X)_{reg}})$.

On the other hand, by [LeB] the contact structure

$$\Theta_{\mathbf{P}(X)_{reg}} \xrightarrow{\eta} O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{reg}} \to 0$$

has a splitting (as $\mathbf{C}$-modules)

$$s : O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{reg}} \to \Theta_{\mathbf{P}(X)_{reg}}$$

so that the subspace

$$s(H^0(\mathbf{P}(X)_{reg}, O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{reg}}) \subset H^0(\mathbf{P}(X)_{reg}, \Theta_{\mathbf{P}(X)_{reg}})$$

is the infinitesimal contact automorphism group of $\mathbf{P}(X)_{reg}$. By the observation above it has the same dimension as $\dim \mathfrak{g}$. Hence $\mathfrak{g} \subset H^0(\mathbf{P}(X)_{reg}, \Theta_{\mathbf{P}(X)_{reg}})$ coincides with the infinitesimal contact automorphism group of $\mathbf{P}(X)_{reg}$ (or $\mathbf{P}(X)$) and G is the neutral component of the contact automorphism group of $\mathbf{P}(X)$.

We have thus proved:

Theorem 2. *Let X be a singular symplectic variety embedded in an affine space $\mathbf{C}^N$ as a complete intersection of homogeneous polynomials. Then X coincides with a nilpotent orbit closure $\bar{O}$ of a semisimple complex Lie algebra $\mathfrak{g}$.*

By the proof such an orbit O is a Richardson orbit and the Springer map $T^*(G/P) \to \bar{O}$ is a birational map.

A typical example of $\bar{O}$ is the nilpotent variety N of $\mathfrak{g}$. Let $\chi : \mathfrak{g} \to \mathfrak{g}//G = \mathbf{C}^r$ be the adjoint quotient map. Then $N = \chi^{-1}(0)$. In particular, N is a complete intersection of r homogeneous polynomials in $\mathfrak{g}$.

The following is the main theorem of this article.

Main Theorem. *Let (X, ω) be a singular symplectic variety embedded in an affine space $\mathbf{C}^N$ as a complete intersection of homogeneous polynomials. Assume that ω is also homogeneous. Then (X, ω) coincides with the nilpotent variety (N, ω_{KK}) of a semisimple complex Lie algebra $\mathfrak{g}$ together with the Kostant-Kirillov form ω_{KK}.*

6. In this section we prove that the nilpotent orbit closure $\bar{O}$ in Theorem 2 is actually the nilpotent variety N.

(6.1) Let $\mathbf{C}[x_1, ..., x_n]$ be a polynomial ring with n variables. For a homogeneous ideal I of $\mathbf{C}[x_1, ..., x_n]$, we put $R := \mathbf{C}[x_1, ..., x_n]/I$ and $d := \dim R$. Assume that I does not contain a non-zero homogeneous polynomial of degree 1. We denote by M the maximal ideal $(x_1, ..., x_n)$ of R.

Lemma. *The following are equivalent.*

(i) *The formal completion $\hat{R}$ along M is of complete intersection.*

(ii) *The ideal I is generated by $n - d$ homogeneous elements.*

Proof. Since it is clear that (ii) implies (i), we only have to prove that (i) implies (ii). The number of minimal generators of $\hat{I}$ equals $\dim_{\mathbf{C}}(I/IM)$ by Nakayama's lemma. The condition (i) then means that $\dim_{\mathbf{C}}(I/IM) = n-d$. One can take $n - d$ homogeneous elements f_1, ..., f_{n-d} from I such that $\bar{f}_1$, ..., $\bar{f}_{n-d} \in I/IM$ form a basis of I/IM. Then it can be checked that f_1, ..., f_{n-d} actually generate I (cf. the proof of Lemma (A.4) of [Na 1]). Q.E.D.

(6.2) Let R be the same as in (6.1) and put $X := \mathrm{Spec}(R)$. Assume that a reductive Lie group G acts on $\mathbf{C}^n = \mathrm{Spec}\mathbf{C}[x_1, ..., x_n]$ so that X is preserved by G. Moreover we assume that the G-action commutes with the $\mathbf{C}^*$-action on $\mathbf{C}^n$.

Lemma. *There are a G-representation V with $\dim V = n - d$ and a G-equivariant morphism $f : \mathbf{C}^n \to V$ of affine spaces such that $f^{-1}(0) = X$.*

Proof. Let I_k be the degree k part of the homogeneous ideal I. Since G respects the grading of $\mathbf{C}[x_1, ..., x_n]$, each I_k is a G-representation. Let k_1 be the minimal number such that $I_{k_1} \neq 0$. Let k_2 be the minimal number $k > k_1$ such that $I'_k := \mathrm{C}[x_1, ..., x_n]_{k-k_1} \cdot I_{k_1}$ does not coincide with I_k. Since I'_{k_2} is a G-subrepresentation of I_{k_2}, there is a G-subrepresentation I''_{k_2} of I_{k_2} such that $I_{k_2} = I'_{k_2} \oplus I''_{k_2}$. We next put $I'_k := \mathrm{C}[x_1, ..., x_n]_{k-k_2} \cdot I_{k_2}$ for $k > k_2$

and let k_3 be the minimal number k such that $I'_k \neq I_k$. Let I''_{k_3} be a G-subrepresentation of I_{k_3} such that $I_{k_3} = I'_{k_3} \oplus I''_{k_3}$. We repeat this process; then $I_{k_1} \oplus I''_{k_2} \oplus I''_{k_3} \oplus ...$ becomes a G-representation of dimension $n-d$. The V is its dual representation. Q.E.D.

(6.3) **Proposition**. *A nilpotent orbit closure $\bar{O}$ of an exceptional simple Lie algebra $\mathfrak{g}$ is of complete intersection if and only if $\bar{O} = N$.*

Proof. We put $m := \dim \mathfrak{g}$ and $2n := \dim \bar{O}$. Then $\bar{O}$ is an affine subvariety of $\mathbf{C}^m$ with codimension $r := m - 2n$. Assume that $\bar{O}$ is defined by r homogeneous polynomials f_i with $\deg(f_i) = a_i$. As remarked at the beginning of **1**, we have $\Sigma_{1 \leq i \leq r} a_i = n + r$. Since $a_i \geq 2$ for all i, we see that $\Sigma a_i \geq 2r$; thus $n \geq r$. In particular, $m = 2n + r \geq 3r$. Therefore we have

$$\mathrm{Codim}_{\mathfrak{g}} \bar{O} \leq 1/3 \cdot \dim \mathfrak{g}.$$

On the other hand, by the previous lemma there are a G-representation V with $\dim V = \mathrm{Codim}_{\mathfrak{g}} \bar{O}$ and a G-equivariant map $f : \mathfrak{g} \to V$ such that $f^{-1}(0) = \bar{O}$. There are very few (nontrivial) irreducible representations V of an exceptional simple Lie group G with $\dim V < \dim G$ (cf. [F-H], Exercise 24.52 (p.414, see also pp.531,532). These are:

G_2: $\dim \mathfrak{g} = 14$, $\dim V_{\omega_1} = 7$,

F_4: $\dim \mathfrak{g} = 52$, $\dim V_{\omega_4} = 26$,

E_6: $\dim \mathfrak{g} = 78$, $\dim V_{\omega_1} = \dim V_{\omega_6} = 27$,

E_7: $\dim \mathfrak{g} = 133$, $\dim V_{\omega_7} = 56$

Here we denote by V_{ω_i} the representations Γ_{ω_i} in [F-H]. As a consequence, we have no irreducible representation V with $\dim V \leq 1/3 \cdot \dim \mathfrak{g}$. Let us look at the G-equivariant map $f : \mathfrak{g} \to V$. Since there is no irreducible G-representation of dim $\leq 1/3 \cdot \dim \mathfrak{g}$, the G-representation V is a direct sum of trivial representations. This means that $\bar{O}$ is the common zeros of some invariant polynomials on $\mathfrak{g}$ (with respect to the adjoint representation). Notice that the nilpotent variety N of $\mathfrak{g}$ is the common zeros of *all* invariant polynomials on $\mathfrak{g}$. Since $\bar{O}$ is contained in N, we conclude that $\bar{O} = N$. Q.E.D.

(6.4) Let G be a semisimple complex Lie group and let P be a parabolic subgroup of G. Let $O \subset \mathfrak{g}$ be the Richardson orbit for P. We assume that the closure $\bar{O}$ is normal and the Springer map $T^*(G/P) \to \bar{O}$ is birational. One can construct a flat deformation of $\bar{O}$ in the following way. Details can be found in [Na 4, Section 2]. Let $n(\mathfrak{p})$ (resp. $r(\mathfrak{p})$) be the nilradical (resp. solvable radical) of $\mathfrak{p}$. Let $\mathfrak{h} \subset \mathfrak{p}$ be a Cartan subalgebra of $\mathfrak{p}$ and define

$\mathfrak{k}(\mathfrak{p}) := \mathfrak{h} \cap r(\mathfrak{p})$. We then have $r(\mathfrak{p}) = \mathfrak{k}(\mathfrak{p}) \oplus n(\mathfrak{p})$. Notice that $\bar{O}$ is the G-orbit of $n(\mathfrak{p})$:

$$\bar{O} = G \cdot n(\mathfrak{p}).$$

Then $G \cdot r(\mathfrak{p})$ naturally contains $\bar{O}$. Restricting the adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{h}/W$ to $G \cdot r(\mathfrak{p})$, we have a map

$$\chi_{\mathfrak{p}} : G \cdot r(\mathfrak{p}) \to \mathfrak{h}/W.$$

Let $\nu : \mathcal{X} \to G \cdot r(\mathfrak{p})$ be the normalization map. Then the composition map $\mathcal{X} \to \mathfrak{h}/W$ factors through $\mathfrak{k}(\mathfrak{p})/W'$, where $W' \subset W$ is the stabilizer subgroup of $\mathfrak{k}(\mathfrak{p})$ as a set:

$$\chi^n_{\mathfrak{p}} : \mathcal{X} \to \mathfrak{k}(\mathfrak{p})/W'.$$

By [Na 4, Proposition 2.6] we have $(\chi^n_{\mathfrak{p}})^{-1}(0) = \bar{O}$ [4] and $\chi^n_{\mathfrak{p}}$ gives a flat deformation of $\bar{O}$. There is a natural $\mathbf{C}^*$-action on $\mathcal{X}$. If $(\chi^n_{\mathfrak{p}})^{-1}(0) = \bar{O}$ is of locally complete intersection, then all fibres $(\chi^n_{\mathfrak{p}})^{-1}(\bar{t})$ are also of locally complete intersection by the $\mathbf{C}^*$-action.

(6.5) A fibre of $\chi_{\mathfrak{p}}$ has been already studied in [Sl, 4.3]. For $t \in \mathfrak{h}$ define $Z_G(t) \subset G$ to be the centralizer of t in G; namely

$$Z_G(t) := \{g \in G; Ad_g(t) = t\}.$$

Similarly define $Z_{\mathfrak{g}}(t) \subset \mathfrak{g}$ to be the centralizer of t in $\mathfrak{g}$. Note that $Z_{\mathfrak{g}}(t)$ is a reductive Lie algebra. Then $\mathfrak{p}_t := \mathfrak{p} \cap Z_{\mathfrak{g}}(t)$ is a parabolic subalgebra of $Z_{\mathfrak{g}}(t)$. Let $O_t \subset Z_{\mathfrak{g}}(t)$ be the Richardson orbit for $\mathfrak{p}_t$. Take an element $\bar{\bar{t}}$ from the image of the map $\mathfrak{k}(\mathfrak{p}) \to \mathfrak{h}/W$. Then the fibre $\chi_{\mathfrak{p}}^{-1}(\bar{\bar{t}})$ can be described as follows. Let $\{t_1, ..., t_n\}$ be the inverse image of $\bar{\bar{t}}$ by the map $\mathfrak{k}(\mathfrak{p}) \to \mathfrak{h}/W$. Then one has

$$\chi_{\mathfrak{p}}^{-1}(\bar{\bar{t}}) = \bigcup_{1 \leq i \leq n} \rho_i(G \times^{Z_G(t_i)} (t_i + \bar{O}_{t_i})),$$

where $\rho_i : G \times^{Z_G(t_i)} (t_i + \bar{O}_{t_i}) \to G \cdot r(\mathfrak{p})$ is a map defined by $\rho_i([g, t_i + x]) = Ad_g(t_i + x)$. As remarked in [Sl, p.56, Remark], $\chi_{\mathfrak{p}}^{-1}(\bar{\bar{t}})$ is not necessarily irreducible. However a fibre of $\chi^n_{\mathfrak{p}}$ is always irreducible and normal. Consider the Brieskorn-Slodowy diagram ([Na 4, p.728 (2)]):

$$\begin{array}{ccc} G \times^P r(\mathfrak{p}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathfrak{k}(\mathfrak{p}) & \longrightarrow & \mathfrak{k}(\mathfrak{p})/W' \end{array} \qquad (3)$$

[4] Notice that we assume that $\bar{O}$ is normal.

Here $G \times^P r(\mathfrak{p})$ gives a simultnaneous resolution of the flat family $\mathcal{X} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$. Take an element t from $\mathfrak{k}(\mathfrak{p})$. The fibre of the map $G \times^P r(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$ over t is $G \times^P (t + n(\mathfrak{p}))$. Notice that

$$G\times^P(t+n(\mathfrak{p})) = G\times^P(P\times^{P_t}(t+n(\mathfrak{p}_t))) = G\times^{P_t}(t+n(\mathfrak{p}_t)) = G\times^{Z_G(t)}(Z_G(t)\times^{P_t}(t+n(\mathfrak{p}_t))).$$

Let $\bar{t} \in \mathfrak{k}(\mathfrak{p})/W'$ be the image of t by the map $\mathfrak{k}(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})/W'$. Then the map

$$G \times^P (t + n(\mathfrak{p})) \to \mathcal{X}_{\bar{t}}$$

coincides with the map

$$G \times^{Z_G(t)} (Z_G(t) \times^{P_t} (t + n(\mathfrak{p}_t))) \to G \times^{Z_G(t)} (t + \tilde{O}_t),$$

where $\tilde{O}_t$ is the normalization of the orbit closure $\bar{O}_t$. In particular, one has

$$(\chi^n_{\mathfrak{p}})^{-1}(\bar{t}) = G \times^{Z_G(t)} (t + \tilde{O}_t) \cong G \times^{Z_G(t)} \tilde{O}_t.$$

Note that $(\chi^n_{\mathfrak{p}})^{-1}(\bar{t})$ is locally the product of $G/Z_G(t)$ and $\tilde{O}_t$. If the central fibre $(\chi^n_{\mathfrak{p}})^{-1}(0)$ is locally of complete intersection, then $\tilde{O}_t$ is locally of complete intersection.

(6.6) Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Φ be the root system for $\mathfrak{g}$. Choose a base Δ of Φ. Recall that every parabolic subgroup of G is conjugate to a standard parabolic subgroup P_I for a subset I of Δ. We denote by $L(P_I)$ the Levi subgroup of P_I containing H. For example, if $I = \emptyset$, then P_I is a Borel subgroup and $L(P_I)$ is nothing but the maximal torus H of G. In the remainder we assume that P is a standard one P_I. One has

$$\mathfrak{k}(\mathfrak{p}_I) = \{h \in \mathfrak{h}; \alpha(h) = 0, \forall \alpha \in I\}.$$

Define

$$\mathfrak{k}(\mathfrak{p}_I)^{reg} := \{h \in \mathfrak{k}(\mathfrak{p}_I); \alpha(h) \neq 0, \forall \alpha \in \Phi - \Phi_I\},$$

where Φ_I is the root subsystem of Φ generated by I. Choose $\beta \in \Delta - I$ and consider the larger parabolic subgroup $P_{I\cup\{\beta\}}$. Then $\mathfrak{k}(\mathfrak{p}_{I\cup\{\beta\}})$ is naturally contained in $\mathfrak{k}(\mathfrak{p}_I)$. We take an element t_β from $\mathfrak{k}(\mathfrak{p}_{I\cup\{\beta\}})^{reg}$. Notice that $Z_G(t_\beta) = L(P_{I\cup\{\beta\}})$. Moreover $P_I \cap Z_G(t_\beta)$ is a parabolic subgroup of $Z_G(t_\beta)$, which determines a Richardson orbit O_{t_β} of $Z_{\mathfrak{g}}(t_\beta)$. We then have

$$(\chi^n_{\mathfrak{p}_I})^{-1}(\bar{t}_\beta) \cong G \times^{Z_G(t_\beta)} \tilde{O}_{t_\beta}.$$

(6.7) *Example.* Let P_I be the standard parabolic subgroup of $SL(5)$ determined by the following marked Dynkin diagram, where the white vertices are simple roots belonging to I:

$\bullet$ —— $\circ$ —— $\bullet$ —— $\circ$

We have two black vertices. Take the 1-st black vertex as β. Then the semisimple reduction $[Z_{\mathfrak{g}}(t_\beta), Z_{\mathfrak{g}}(t_\beta)]$ is of type $A_2 \times A_1$. Moreover O_{t_β} is the Richardson orbit of the first A_2 for the parabolic subalgebra corresponding to

$\bullet$ —— $\circ$

Next take the 2-nd black vertex as β. Then $[Z_{\mathfrak{g}}(t_\beta), Z_{\mathfrak{g}}(t_\beta)]$ is of type A_3. The orbit O_{t_β} is the Richardson of A_3 for the parabolic subalgebra corresponding to

$\circ$ —— $\bullet$ —— $\circ$

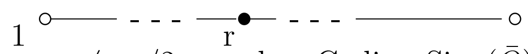
Let $O \subset sl(5)$ be the Richardson orbit for P_I. Assume that $\bar{O}$ is locally of complete intersection. Then $\tilde{O}_t$ is locally of complete intersection for any $t \in \mathfrak{k}(\mathfrak{p}_I)$ by (6.5). As above we take the 1-st black vertex as β and consider the corresponding O_{t_β}. It is then easily checked that $\mathrm{Codim}_{\tilde{O}_{t_\beta}}\mathrm{Sing}(\tilde{O}_{t_\beta}) = 4$. By [Be, Proposition 1.4] $\tilde{O}_{t_\beta}$ is not locally of complete intersection. This is absurd. The second choice of β also leads us to a contradiction. In this case $\mathrm{Sing}(\tilde{O}_{t_\beta})$ has codimension 2 in $\tilde{O}_{t_\beta}$ and Beauville's proposition cannot be used. Instead we use the previous lemma. First notice that every nilpotent orbit closure in $sl(m)$ is normal; hence $\tilde{O}_{t_\beta} = \bar{O}_{t_\beta}$. By a direct calculation one has $\dim \bar{O}_{t_\beta} = 8$ and $\dim sl(4) = 15$. Suppose that $\bar{O}_{t_\beta}$ is locally of complete intersection. As proved in **5**, $T_0\bar{O}_{t_\beta} = sl(4)$; one can apply Lemma (6.1) to the embedding $\bar{O}_{t_\beta} \subset sl(4)$. Then $\bar{O}_{t_\beta}$ is defined as the common zeros of 7 homogeneous polynomials f_i $(1 \leq i \leq 7)$. We put $a_i := \deg(f_i)$. By the argument at the beginning of **1** we have $a_1 + ... + a_7 = 11$. On the other hand, since $a_i \geq 2$ for all i, we have $a_1 + ... + a_7 \geq 14$. This is a contradiction.

(6.8) We are now going to prove that when $\mathfrak{g}$ is a classical simple Lie algebra, the nilpotent orbit closure $\bar{O}$ in Theorem 2 is actually the nilpotent variety N. We employ the following strategy. We shall derive a contradiction assuming that $\bar{O}$ in Theorem 2 is not the nilpotent variety. First we construct a flat deformation of $\bar{O}$: $\chi^n_{\mathfrak{p}} : \mathcal{X} \to \mathfrak{k}(\mathfrak{p})/W'$ as in (6.4). The parabolic subalgebra $\mathfrak{p}$ corresponds to a marked Dynkin diagram for $\mathfrak{g}$. As demonstrated in (6.7), we take a suitable simple root β and the corresponding element $t_\beta \in \mathfrak{k}(\mathfrak{p})$ (cf. (6.6)). We next consider the fibre $(\chi^n_{\mathfrak{p}})^{-1}(t_\beta)$. Then this fibre is isomorphic to $G \times^{Z_G(t_\beta)} \tilde{O}_{t_\beta}$. If $\bar{O}$ is of complete intersection, then $\tilde{O}_{t_\beta}$ is also of complete intersection. But O_{t_β} is a Richardson orbit in a classical simple Lie algebra which is smaller than $\mathfrak{g}$. Moreover the corresponding parabolic subalgebra (= the polarization of O_{t_β}) is a maximal parabolic subalgebra. Finally we derive a contradiction in such a case.

We first treat the case $\mathfrak{g}$ is of type A.

Proposition. *A nilpotent orbit closure $\bar{O}$ of $sl(m)$ has complete intersection singularities if and only if $\bar{O} = N$.*

Proof. Note that every nilpotent orbit O of $\mathfrak{g} := sl(m)$ is a Richardson orbit and its closure is normal. Moreover the Springer map $T^*(G/P) \to \bar{O}$ is birational. As remarked just above, we only have to prove that $\bar{O}$ does not have complete intersection singularities when P is a *maximal* parabolic subgroup of $SL(m)$ with $m \geq 3$. Namely P corresponds to to a marked Dynkin diagram with only one black vertex:



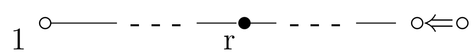
When $r \neq m/2$, one has $\mathrm{Codim}_{\bar{O}}\mathrm{Sing}(\bar{O}) \geq 4$. Then $\bar{O}$ does not have complete intersection singularities by [Be, Proposition 1.4]. Assume that $\bar{O}$ has complete intersection singularities when $r = m/2$. By a direct calculation we have $\dim \bar{O} = 2r^2$ and $\dim sl(m) = 4r^2 - 1$. By Lemma (6.1) $\bar{O}$ is a subvariety of $\mathbf{C}^{4r^2-1}$ defined as the complete intersection of $2r^2 - 1$ homogeneous polynomials f_i. We put $a_i := \deg(f_i)$. As discussed at the beginning of **1**, $\Sigma a_i = r^2 + (2r^2 - 1)$. On the other hand, since $a_i \geq 2$, we have $\Sigma a_i \geq 2(2r^2 - 1)$. Combining these inequalities we get

$$r^2 \geq 2r^2 - 1,$$

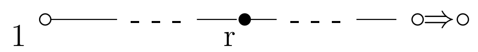
which implies that $r = 1$ and then $m = 2$. This contradicts the first assumption that $m \geq 3$. Q.E.D.

(6.9) Let G be $Sp(2n)$ or $SO(n)$ and let P_I be a maximal parabolic subgroup. Namely P is the standard parabolic subgroup corresponding to one of the following Dynkin diagram.

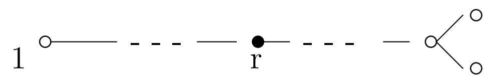
C_n



$B_{[n/2]}$



$D_{n/2}$



Let $O \subset \mathfrak{g}$ be the Richardson orbit for P_I. We shall prove that $\bar{O}$ is not a homogeneous symplectic variety of complete intersection. When $G = Sp(2n)$, the parabolic subgroup P_I is the stabilizer group of an isotropic flag of type $(r, 2n-r, r)$. Let $Gr_{iso}(r, 2n)$ be the isotropic Grassmann variety parametrizing such flags. Then

$$\dim Gr_{iso}(r, 2n) = \dim Gr(r, 2n) - 1/2 \cdot r(r-1) = r(2n-r) - 1/2 \cdot r(r-1).$$

Since $\dim \bar{O} = 2 \dim Gr_{iso}(r, 2n)$, we have $\dim \bar{O} = 2r(2n-r) - r(r-1)$. On the other hand, $\dim sp(2n) = 2n^2 + n$, hence $\mathrm{Codim}_{sp(2n)} \bar{O} = 2n^2 + n - 4rn + 3r^2 - r$. Assume that $\bar{O}$ is of complete intersection in $sp(2n)$. Let f_i be the defining equations of $\bar{O}$ and put $a_i := \deg(f_i)$. Then $\Sigma a_i = 1/2 \cdot \dim \bar{O} + \mathrm{Codim}_{sp(2n)} \bar{O}$ by **1**. Since $a_i \geq 2$ for all i, we have $(3r-2n-1)(3r-2n) \leq 0$. The only possibilities are following two cases:

(i) $n = 3k$ for some integer k and $r = 2k$.

(ii) $n = 3k+1$ for some integer k and $r = 2k+1$.

In both cases $a_i = 2$ for all i (i.e. $\dim V = 1/3 \cdot \dim sp(2n)$.) In the first case $O = O_{[3^{2k}]}$ (i.e the nilpotent orbit consisting of the matrices of Jordan type $(3, ..., 3)$ ($2k$ Jordan blocks of size 3). In the second case $O = O_{[3^{2k}, 2]}$. Assume that $O_{[3^{2k}]} \subset sp(6k)$ is of complete intersection. By the calculation above we have $\mathrm{codim}_{sp(6k)} \bar{O} = 6k^2 + k$. By Lemma (6.2) there are a G-representation V of dim $6k^2 + k$ and a G-equivariant map $f : sp(6k) \to V$ such that $f^{-1}(0) = \bar{O}$. By the construction of V (cf. Lemma (6.2)), the dual representation V^* coincides with I_2 because $a_i = 2$ for all i. But there is only one (adjoint) invariant quadratic polynomial on $sp(6k)$ up to constant. Hence V contains one and only one trivial representation as a direct factor. Since an irreducible representation of $sp(6k)$ with $\dim \leq 1/3 \cdot \dim sp(6k)$ is a trivial representation or a standard representation (cf. [F-H], p.531, (24.52)), V is a direct sum of a trivial representation and a finite number of standard representations.

Let us consider the first case (i). Notice that, in this case, $\dim V = 1 + (6k^2 + k - 1)$. If $k \geq 2$, then $6k$ does not divide $6k^2 + k - 1$, which is a contradiction. When $k = 1$, one has $\dim V = 7$ and V may possibly be a direct sum of the 6-dimensional standard representation and the trivial representation. Since $a_i = 2$ for all i, these irreducible factors must be contained in $\mathrm{Sym}^2(sp(6k)^*)$ the 2-nd symmetric product of the dual representation of the adjoint one. By the Killing form $\mathrm{Sym}^2(sp(6)^*) \cong \mathrm{Sym}^2(sp(6))$ as $Sp(6)$-representations. It is easily checked that $\mathrm{Sym}^2(sp(6))$ does not contain the standard representation as a direct factor. Hence we have a contradiction also in this case.

In the second case (ii) we have $\dim V = 1 + (6k^2 + 5k)$. Noticing that the standard representation has dimension $6k + 2$, we write $6k^2 + 5k = k(6k+2)+3k$; hence $6k+2$ does not divide $6k^2+5k$. This is a contradiction.

Assume that $G = SO(n)$ and $\bar{O}$ has complete intersection singularities. Since $a_i \geq 2$ for all i, the equality

$$\Sigma a_i = 1/2 \cdot \dim \bar{O} + \mathrm{Codim}_{so(n)} \bar{O}$$

implies that $(3r - n)(3r - n + 1) \leq 0$. There are two possibilities:

(i) $n = 3k$ for some integer k, $r = k$ and $O = O_{[3^k]}$.

(ii) $n = 3k + 1$ for some integer k, $r = k$ and $O = O_{[3^k,1]}$.

In both cases $a_i = 2$ for all i (i.e. $\dim V = 1/3 \cdot \dim so(n)$). We can again use Lemma (6.2) to have a G-equivariant map $f : so(n) \to V$. Put $\mathfrak{g} = so(n)$ with $n = 3k$ or $n = 3k + 1$. Then $\dim V$ is respectively $1/2 \cdot (3k^2 - k)$ or $1/2 \cdot (3k^2 + k)$. Note that an irreducible representation of $\mathfrak{g}$ with dim $\leq 1/3 \cdot \dim \mathfrak{g}$ is a trivial representation or a standard representation (cf. [F-H], p.531, (24.52): Note that, when $\mathfrak{g}$ is of D_4, two more different irreducible representations exist, but the D_4 case is not contained in the case (i) or the case (ii).). Since there is only one (adjoint) invariant quadratic polynomial on $so(n)$ up to constant, V is a direct sum of a trivial representation and a finite number of standard representations. By writing $k = 2l$ or $k = 2l + 1$ according as k is even or odd, one can easily check that $\dim V - 1$ is not divided by n in both cases; hence we have a contradiction.

(6.10) Let $\mathfrak{g}$ be a complex simple Lie algebra of type B, C or D. Let O be the Richardson orbit of $\mathfrak{g}$ for a parabolic subgroup P of G. Assume that the Springer map $s : T^*(G/P) \to \bar{O}$ is birational.

Proposition *The closure $\bar{O}$ of such an orbit is of complete intersection if and only if $\bar{O} = N$.*

Proof. We only have to deal with a Richadson orbit for a standard parabolic subgroup P_I. If the Dynkin diagram corresponding to P_I has only one black vertex, then we have already checked that $\bar{O}$ is not of complete intersection. Assume that there are more than one black vertices, but at least one vertex is a white vertex. Take a white vertex w on the leftmost position. Note that if the Dynkin diagram is of type B or C, it is unique, but if the Dynkin diagram is of type D, the choice of such a vertex might have two possibilities.

If there is a black vertex b left adjacent to w, then take the simple root β corresponding to b and apply (6.6). Then the problem is reduced to the case where the Dynkin diagram is of type A and has only one black vertex with $r = 1$, or the Dynkin diagram is a smaller one of the same type as $\mathfrak{g}$ and has only one black vertex with $r = 1$. In each case $\bar{O}_{t_\beta}$ is normal; we only have to check this in the second case. There is a nilpotent orbit $O'_{t_\beta} \subset \bar{O}_{t_\beta}$ such that $\mathrm{Codim}_{\bar{O}_{t_\beta}} \bar{O}'_{t_\beta} = 2$. One can check that $\mathrm{Sing}(\bar{O}_{t_\beta}, O'_{t_\beta})$ is of type a or of type g in the list of [K-P, p.551]. By Theorem 1, (b) of [K-P] we see that $\bar{O}_{t_\beta}$ is normal. Moreover, in each case, $\bar{O}_{t_\beta}$ is not of complete intersection (cf. (6.8), (6.9)). By the argument in (6.5), the original nilpotent orbit closure $\bar{O}$ is not of complete intersection.

Assume that there is no black vertex left adjacent to w. By the definition of w this means that w is on the leftmost position on the diagram. In this case we consider the maximal connected Dynkin subdiagram $\mathcal{D}$ containing w whose vertices are all white. Let w' be a vertex on the rightest position of $\mathcal{D}$. Let b be a black vertex right adjacent to w'. We take the simple root β corresponding to b and apply (6.6). Then the problem is reduced to the case where the Dynkin diagram is of type A and has only one black vertex. Then $\bar{O}_{t_\beta}$ is normal and is not of complete intersection (cf. (6.8)). By the argument in (6.5), the original nilpotent orbit closure $\bar{O}$ is not of complete intersection. Q.E.D.

(6.11) Let O be a Richardson orbit of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \oplus_{1 \le i \le m} \mathfrak{g}_i$ be the decomposition into the simple factors. Then we have $\bar{O} = \bar{O}_1 \times ... \times \bar{O}_m$ where each O_i is a Richadson orbit of $\mathfrak{g}_i$. If the Springer map $T^*(G/P) \to \bar{O}$ is birational, then each Springer map $T^*(G_i/P_i) \to \bar{O}_i$ is birational. Assume that $\bar{O}$ is of complete intersection. Then each $\bar{O}_i$ is also of complete intersection. By (6.3), (6.8) and (6.10) each $\bar{O}_i$ coincides with the nilpotent variety N_i of $\mathfrak{g}_i$. Then $\bar{O}$ is the nilpotent variety N of $\mathfrak{g}$.

7. Remarks

(1) What happens in Main theorem if we do not assume ω is homogeneous ? The author does not know the answer, but the following example would

be instructive. Let $X \subset \mathbf{C}^5$ be a hypersurface defined by $z_1^2 + z_2^2 + z_3^2 = 0$, where $(z_1, ..., z_5)$ are coordinates of $\mathbf{C}^5$. Note that $X = S \times \mathbf{C}^2$, where $S \subset \mathbf{C}^3$ is a hypersurface defined by $f := z_1^2 + z_2^2 + z_3^2 = 0$. We put $\omega_S := \mathrm{Res}(dz_1 \wedge dz_2 \wedge dz_3/f)$ and $\omega_{\mathbf{C}^2} := dz_4 \wedge dz_5$. Define $\omega := \omega_S + \omega_{\mathbf{C}^2}$. Then (X, ω) is an affine symplectic variety. But ω is not homogeneous because $wt(\omega_S) = 1$ and $wt(\omega_{\mathbf{C}^2}) = 2$. Note that $\omega \wedge \omega$ is a holomorphic volume form on X of weight 3. Let us prove that there is no homogeneous symplectic 2-form on X. Assume that such a form Ω exists. Then $\Omega \wedge \Omega$ is a holomorphic volume form on X of an even weight, say $2m$. Then one can write $\Omega \wedge \Omega = g \cdot \omega \wedge \omega$ with a nowhere vanishing function g of *nonzero* weight. But such g does not exists; hence one gets a contradiction.

(2) Let X be an affine symplectic variety in $\mathbf{C}^N$ defined by a homogeneous ideal I (not necessarily of complete intersection) where I contains no nonzero linear form. Denote by R the coordinate ring of X. By the assumption R is graded: $R = \oplus_{n \geq 0} R_n$. Assume that $wt(\omega) = 1$. Then ω induces a Poisson structure on R of weight -1. In particular, it induces a Lie algebra structure on R_1

$$[\cdot, \cdot] : R_1 \times R_1 \to R_1.$$

Let us call this Lie algebra $\mathfrak{g}$. Since $R_1 = T_0^* X$, we have $\dim \mathfrak{g} = N$. The natural surjection $\oplus \mathrm{Sym}^i(R_1) \to R$ induces a closed embedding $X \to \mathfrak{g}^*$. To prove that $\mathfrak{g}$ is semisimple, it seems that one needs some geometric arguments as in **1 - 5**. When $\mathfrak{g}$ is semisimple, $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ by the Killing form. This is nothing but the closed embedding $X \to \mathfrak{g}$ of Main theorem, where X is identified with the nilpotent variety N.

(3) Let X be the same as in (2). Then $\mathbf{P}(X)$ admits a contact structure with the contact line bundle $O_{\mathbf{P}(X)}(1)$ in the sense of **1**. Let G be the contact automorphism group of $\mathbf{P}(X)_{reg}$. The Lie algebra $\mathfrak{g}$ is contained in $H^0(\mathbf{P}(X), \Theta_{\mathbf{P}(X)})$ and the map $H^0(\mathbf{P}(X), \Theta_{\mathbf{P}(X)}) \xrightarrow{\eta} H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(1))$ induces an isomorphism $\mathfrak{g} \cong H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(1))$ by [Be 2], Proposition 1.1. In general we only know that $\dim \mathfrak{g} \geq N$. The closed embedding $\mathbf{P}(X) \to \mathbf{P}(\mathfrak{g}^*)$ is a G-equivariant map. By a similar argument to [Be 2], Section 1, the G-action on $\mathbf{P}(X)$ lifts to a G-action on X. Moreover the above embedding lifts to a G-equivariant closed embedding $X \to \mathfrak{g}^*$. By this embedding X is identified with a coadjoint orbit closure of $\mathfrak{g}^*$. In particular, G acts transitively on X_{reg}. But we do not know when G is semisimple.

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