Hyperplane arrangements and characteristic polynomials.

Masahiko Yoshinaga (Hokkaido University)

概要：特性多項式は超平面配置の最も重要な不変量で、chamberの個数、有限体上の点の数えあげ、複素化した補集合のBetti数などと関係をもつ。また、特性多項式の組合せ論的性質は、対数的ベクトル場の加群の代数的構造によって統制されていると信じられている。本講では、ルート系に関係した配置の特性多項式のいくつかの計算方法を述べる。

§1. Characteristic polynomials

Let $\mathbb{K}$ be a field and $V=\mathbb{K}^{\ell}$. A hyperplane arrangement is a finite collection of (affine) hyperplanes $\mathcal{A}=\{H_1, H_2, \cdots, H_n\}$.

Notation: α_i : defining equation of H_i.

$$Q=\prod_{i=1}^{n}\alpha_i. \quad \left(Q^{-1}(0)=\bigcup_{i=1}^{n} H_i \right)$$

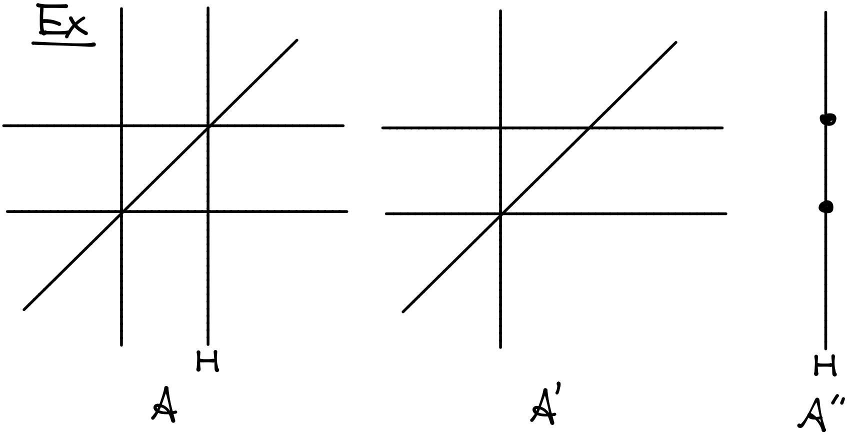
$$M(\mathcal{A}) := V \setminus \bigcup_{H\in\mathcal{A}} H.$$

<u>Def</u>
- $\mathcal{A}$ is central $\overset{\text{def}}{\iff}$ $\forall H\in\mathcal{A}\ (H \ni 0)$.
- $\mathcal{A}$ is essential $\overset{\text{def}}{\iff}$ $\exists H_{i_1}, \cdots, H_{i_\ell} \in \mathcal{A}$ s.t.

$$H_{i_1}\cap\cdots\cap H_{i_\ell}=\{pt\}$$

Def. (Triple) Fix a hyperplane $H \in \mathcal{A}$.
Set $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}' \cap H$. ($\mathcal{A}''$ is a hyperplane arrangement in an affine $(\ell-1)$ dim. space H. See figure below.)

Ex



Def (Characteristic polynomial) Let $\mathcal{A}$ be a hyperplane arrangement in V. Define $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ by the following recursive relations:

(i) $\chi(\phi, t) = t^{\dim V}$, and

(ii) $\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t)$.

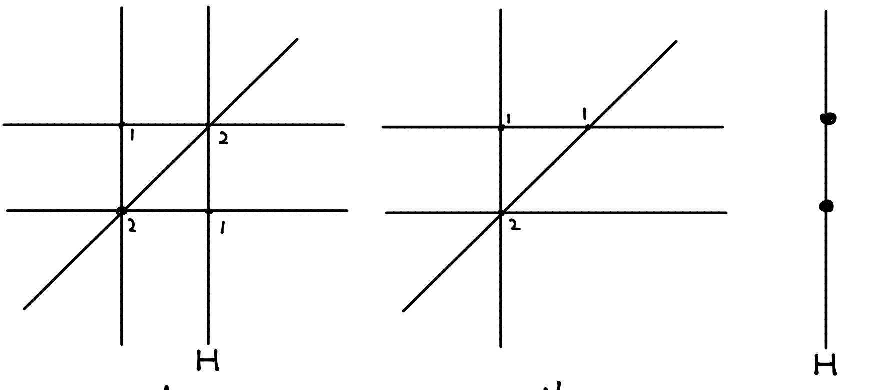
The above definition of $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ looks depending on the choice of $H \in \mathcal{A}$. However it is well-defined.

<u>Fact.</u> $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ is well defined.

<u>Ex.</u> For a line arrangement $\mathcal{A} = \{H_1, \cdots, H_n\}$ in $V = \mathbb{K}^2$, $\chi(\mathcal{A}, t)$ can be explicitly described as:

$$\chi(\mathcal{A}, t) = t^2 - |\mathcal{A}| \cdot t + \sum_{P:\ \text{intersection}} (|\mathcal{A}_P| - 1),$$

where $\mathcal{A}_P := \{H \in \mathcal{A} \mid H \ni P\}$.



$\mathcal{A}$ $\qquad$ $\mathcal{A}'$ $\qquad$ $\mathcal{A}''$

$\chi(\mathcal{A}, t) = t^2 - 5t + 6$ $\qquad$ $\chi(\mathcal{A}', t) = t^2 - 4t + 4$ $\qquad$ $\chi(\mathcal{A}'', t) = t - 2$

§2. Several Properties of $\chi(\mathcal{A}, t)$

Thm.1 (Crapo-Rota 1970) Let $\mathcal{A}$ be a hyperplane arrangement in $V = \mathbb{F}_q^{\ell}$. (q : prime power) Then,

$$|M(\mathcal{A})| = \chi(\mathcal{A}, q).$$

Thm.2 (Zaslavsky 1975) Let $\mathcal{A}$ be a hyperplane arrangement in $V = \mathbb{R}^{\ell}$. (Assume $\mathcal{A}$: essential.)

Then (i) $(-1)^{\ell}\chi(\mathcal{A}, -1) = \#\{\text{chambers}\}$

(ii) $(-1)^{\ell}\chi(\mathcal{A}, 1) = \#\{\text{bounded chambers}\}$

Thm3. (Orlik-Solomon 1980) Let $\mathcal{A}$ be a hyperplane arrangement in $V = \mathbb{C}^{\ell}$. Then the Poincaré polynomial $P(M(\mathcal{A}), t) := \sum_{i \geq 0} b_i(M(\mathcal{A}))t^i$ is expressed as

$$P(M(\mathcal{A}), t) = (-t)^{\ell} \cdot \chi(\mathcal{A}, -\frac{1}{t}).$$

In particular, the coefficients of $\chi(\mathcal{A}, t)$ are Betti numbers of $M(\mathcal{A})$.

§3. Finite "field" method

Thm 1 is useful also for the purpose of computing the characteristic polynomial $\chi(\mathcal{A}, t)$.

<u>Thm.4</u> (Athanasiadis 1996) Let $\mathcal{A}$ be an arrangement defined over $\mathbb{Z}$, i.e., each linear equation α_H $(H \in \mathcal{A})$ has integer coefficients. Then there exist finitely many primes $p_1, \cdots, p_k$ s.t. for any integer $M>0$ with $p_i \nmid M$ $(\forall i)$, the equality

$$\left| (\mathbb{Z}/M\mathbb{Z})^\ell \setminus \bigcup_{H\in\mathcal{A}} \overline{H} \right| = \chi(\mathcal{A}, M)$$

holds, where $\overline{H} = \{\overline{x} \in (\mathbb{Z}/M\mathbb{Z})^\ell \mid \alpha_H(x) \equiv 0 \bmod M\}$.

<u>Ex.</u> Let $H_{ij} = \{(x_1, \cdots, x_\ell) \mid x_i - x_j = 0\}$ and $\mathcal{A} := \{H_{ij} \mid 1 \le i < j \le \ell\}$. Since $M(\mathcal{A} \bmod M)$ is

$$M(\mathcal{A} \bmod M) = \{(x_1, \cdots, x_\ell) \in (\mathbb{Z}/M\mathbb{Z})^\ell \mid x_i \neq x_j \ \forall i<j\},$$

we have $|M(\mathcal{A} \bmod M)| = M \cdot (M-1) \cdots (M-\ell+1)$.
Hence $\chi(\mathcal{A}, t) = t(t-1) \cdots (t-\ell+1)$.

Athanasiadis systematically apply Thm 4, and computing many characteristic polynomials.

Let $V=\mathbb{R}^\ell$ be a real vector space with an inner product $(\, ,\,)$. Let $V \supset \Phi$ be an irreducible root system. We fix a positive system $\Phi^+ \subseteq \Phi$. Denote the exponents of Φ by $e_1, \cdots, e_\ell$ and the Coxeter number by h. Thanks to the inne product, we may consider Φ^+ the set of linear forms on V.

For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, define

$$H_{\alpha,k} := \{ x \in V \mid \alpha(x) = k \}.$$

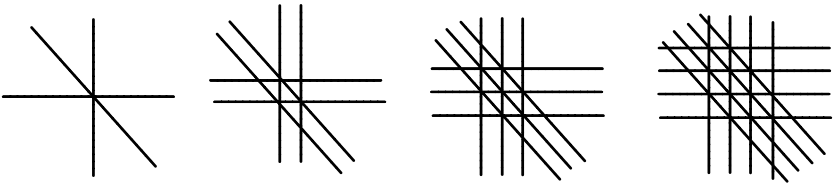
$$\mathcal{A}_\Phi^{[a,b]} := \{ H_{\alpha,k} \mid \alpha \in \Phi^+,\ a \le k \le b \}.$$

<u>Thm.5</u> (Athanasiadis 2004) Let $m \in \mathbb{Z}_{\ge 0}$. Then

$$\chi(\mathcal{A}_\Phi^{[-m,m]}, t) = \prod_{i=1}^{\ell} (t - e_i - mh).$$

Athanasiadis proved Thm 5. by using Thm. 4 and techniques counting (coweight) lattice points in some regions.

<u>Ex</u> $\Phi = A_2$. ($e_1 = 1$, $e_2 = 2$. $h = 3$)



$\mathcal{A}_\Phi^{[0,0]}$ $\quad$ $\mathcal{A}_\Phi^{[0,1]}$ $\quad$ $\mathcal{A}_\Phi^{[-1,1]}$ $\quad$ $\mathcal{A}_\Phi^{[-1,2]}$

$\chi = (t-1)(t-2)$ $\quad$ $(t-3)^2$ $\quad$ $(t-4)(t-5)$ $\quad$ $(t-6)^2$

<u>Thm. 6</u> (Y. 2004)

$$\chi(\mathcal{A}_\Phi^{[1-m,m]}, t) = (t - mh)^\ell.$$

The proof of Thm. 6 relies on algebro-geometric characterizations of free arrangements.

<u>Problem</u> Give a combinatorial proof for Thm. 6.

For other parameters $a, b \in \mathbb{Z}$, $\chi(\mathcal{A}_\Phi^{[a,b]}, t)$ does not factor as in Thm 5 and Thm 6. However there are several mysterious conjectures.

<u>Conj</u> (Postnikov–Stanley)

(i) Suppose $0 \le a \le b$. Then $\mathcal{A} = \mathcal{A}_\Phi^{[-a,b]}$ satisfies

$$\chi(\mathcal{A}, (a+b+1)h - t) = (-1)^\ell \chi(\mathcal{A}, t)$$

(ii) Suppose $0 \le a < b$. Then all roots of the equation $\chi(\mathcal{A}_\Phi^{[-a,b]}, t) = 0$ have the same real part

$$\mathrm{Re}\, t = \frac{(a+b+1)h}{2}.$$

§4. Free arrangements

Fix a system of coordinate $(x_1, \cdots, x_\ell)$ of $V \cong \mathbb{C}^\ell$.

$S := \mathbb{C}[x_1, \cdots, x_\ell]$ is the polynomial ring.

$\mathrm{Der}_V := \bigoplus_{i=1}^{\ell} S \cdot \partial_i = \{ \sum_{i=1}^{\ell} f_i \cdot \partial_i \mid f_i \in S \}$ is the free module of polynomial vector fields.

Let $\mathcal{A}$ be a central arrangement.

<u>Def.</u> (i) $D(\mathcal{A}) := \{\delta \in \mathrm{Der}_V \mid \delta\alpha_H \in (\alpha_H) \ \forall H \in \mathcal{A}\}$ is called the module of logarithmic vector fields.

(ii) $\mathcal{A}$ is said to be <u>free</u> with exponents $(e_1, \cdots, e_\ell)$ if there exist $\delta_1, \cdots, \delta_\ell \in D(\mathcal{A})$ s.t. $D(\mathcal{A}) = S\cdot\delta_1 \oplus \cdots \oplus S\cdot\delta_\ell$ and $\delta_i = \sum_j f_{ij}\partial_j$ has homogeneous coefficients f_{ij} with $\deg f_{ij} = e_i$.

<u>Thm. 7</u> ① Let $\delta_1, \cdots, \delta_\ell \in D(\mathcal{A})$. Then $\delta_1, \cdots, \delta_\ell$ form a free basis of $D(\mathcal{A})$ iff $\det(f_{ij}) \doteq Q$. (Saito's criterion)

② If $\mathcal{A}$ is free with exponents $(e_1, \cdots, e_\ell)$, then $\chi(\mathcal{A}, t) = (t-e_1)(t-e_2)\cdots(t-e_\ell)$. (Terao's factorization Thm)

<u>Ex.</u> $Q = \prod_{1\le i<j\le\ell}(x_i - x_j)$. Let $\mathcal{A}$ be the arrangement defined by $Q=0$. Set $\delta_k := \sum_{i=1}^{\ell} x_i^k \partial_i$. Then by Vandermonde's formula, $\delta_0, \delta_1, \cdots, \delta_{\ell-1}$ form a basis.

In particular, $\mathcal{A}$ is free with exponents $0, 1, \cdots, \ell-1$.

The arrangement $\mathcal{A}_{\Phi}^{[a,b]}$ is not central. So it does not make sense to discuss freeness. However if we consider the coning $c\mathcal{A}_{\Phi}^{[a,b]}$, it becomes central.

<u>Thm. 8</u> (Y. 2004) Let $m \geq 0$

① $c\mathcal{A}_{\Phi}^{[-m,m]}$ is free with exponents $(1, e_1+mh, \cdots, e_\ell+mh)$.

② $c\mathcal{A}_{\Phi}^{[1-m,m]}$ is free with exponents $(1, \underbrace{mh, mh, \cdots, mh}_{\ell})$

Then Thm 8 + Thm 7② imply Thm 5 and Thm 6.

<u>Reference</u>:

- Orlik-Terao. Arrangements of hyperplanes.
- Athanasiadis, Bull. London. Math. Soc (2004)
- R. Stanley, An introduction to hyperplane arrangements (on Web)
- Yoshinaga. Invent. Math. (2004)