Norm computation and analytic continuation of holomorphic discrete series

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1 Introduction: Holomorphic discrete series of $SU(1,1)$

Let $D := \{w \in \mathbb{C} : |w| < 1\}$. Then $\tilde{G} := \widetilde{SU}(1,1)$ (universal covering group) acts on $\mathcal{O}(D)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(w) := (cw+d)^{-\lambda} f\left(\frac{aw+b}{cw+d}\right)$$

where $\lambda \in \mathbb{C}$. This action preserves the sesquilinear form

$$\langle f, h\rangle_\lambda := \frac{\lambda-1}{\pi}\int_D f(w)\overline{h(w)}(1-|w|^2)^{\lambda-2}dw,$$

that is, $\langle \tau_\lambda(g)f, \tau_{\bar{\lambda}}(g)h\rangle_\lambda = \langle f, h\rangle_\lambda$ holds for any $g \in \tilde{G}$. When $\operatorname{Re}\lambda > 1$, this form converges for any polynomial f, h, and therefore τ_λ defines a unitary representation of $\tilde{G}$ when $\lambda > 1$. This is called the holomorphic discrete series representation.

On the other hand, if $\operatorname{Re}\lambda \le 1$, this integral diverges for any non-zero function f. However, when $\operatorname{Re}\lambda > 1$ and $f = \sum_{m=0}^\infty a_m w^m$, we can compute as

$$\|f\|_\lambda^2 = \sum_{m=0}^\infty \frac{m!}{(\lambda)_m}|a_m|^2 \qquad \text{where} \qquad (\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1),$$

because

$$\begin{aligned}
&\frac{\lambda-1}{\pi}\int_D w^m\bar{w}^n(1-|w|^2)^{\lambda-2}dw \\
=&\frac{\lambda-1}{\pi}\int_0^\infty\int_0^{2\pi} r^{m+n}e^{i\theta(m-n)}(1-r^2)^{\lambda-2}rd\theta dr \\
\overset{r^2=s}{=}&(\lambda-1)\delta_{mn}\int_0^1 s^m(1-s)^{\lambda-2}ds \\
=&\delta_{mn}(\lambda-1)B(m+1,\lambda-1) = \delta_{mn}\frac{\Gamma(m+1)\Gamma(\lambda)}{\Gamma(m+\lambda)} = \delta_{mn}\frac{m!}{(\lambda)_m}.
\end{aligned}$$

This expression is available even when $\operatorname{Re}\lambda \le 1$, and positive definite for $\lambda > 0$. Therefore τ_λ defines a unitary representation of $\tilde{G}$ when $\lambda > 0$.

This example shows that if once the norm is explicitly computed, we can treat the analytic continuation of the holomorphic discrete series representation.

2 Holomorphic discrete series of general Hermitian Lie group

Let G be a Hermitian simple Lie group, that is, its maximal compact subgroup K has a non-discrete center. We assume that G has a complexification $G^{\mathbb{C}}$. We denote their Lie algebras by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{g}^{\mathbb{C}}$. Then we can take $z \in \mathfrak{z}(\mathfrak{k})$ (the center of $\mathfrak{k}$) such that the eigenvalues of $ad(z)$ are $+\sqrt{-1}$, 0, $-\sqrt{-1}$. We denote the corresponding eigenspace decomposition by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-.$$

We denote $n = \dim_{\mathbb{C}} \mathfrak{p}^+$, $r = \operatorname{rank}_{\mathbb{R}} G$. Then there exists a domain $D \subset \mathfrak{p}^+$ such that the following diagram commutes.

$$\begin{tikzcd} G/K \arrow[r] \arrow[d, dashed, "\wr"] & G^{\mathbb{C}}/K^{\mathbb{C}}P^- \\ D \arrow[r, hook] & \mathfrak{p}^+ \arrow[u, "\exp"'] \end{tikzcd}$$

Let (τ, V) be a holomorphic representation of $K^{\mathbb{C}}$, and let χ be the character of $\tilde{K}^{\mathbb{C}}$ such that $\chi(\exp(-2t\sqrt{-1}z)) = e^{rt}$. Then we have the following isomorphism

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda})) \simeq \mathcal{O}(D, V)$$

where $\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda}))$ is the space of holomorphic sections, and $\mathcal{O}(D, V)$ is the space of V-valued holomorphic functions on D. $\tilde{G}$ acts on $\mathcal{O}(D, V)$ by the form

$$\tau_\lambda(g) f(w) = \chi(\mu(g^{-1}, w))^\lambda \tau(\mu(g^{-1}, w))^{-1} f(g^{-1}w)$$

for $g \in G$, $w \in D$, where $\mu : \tilde{G} \times D \to \tilde{K}^{\mathbb{C}}$ is a continuous map satisfying

$$\begin{aligned} \mu(gh, w) &= \mu(g, hw)\mu(h, w) & & (g, h \in \tilde{G},\ w \in D), \\ \mu(k, w) &= k & & (k \in \tilde{K},\ w \in D). \end{aligned}$$

This action preserves the following norm:

$$\|f\|_{\lambda,\tau}^2 := \frac{c_\lambda}{\pi^n} \int_D \left(\tau(B(w)^{-1}) f(w), f(w)\right)_\tau \chi(B(w))^{\lambda - p} dw \qquad (f \in \mathcal{O}(D, V))$$

where p is an integer determined from $\mathfrak{g}$, and $B : D \to \tilde{K}^{\mathbb{C}}$ is a continuous map satisfying

$$\mu(g, w) B(w) \mu(g, w)^* = B(gw) \qquad (g \in \tilde{G},\ w \in D)$$

where * is the inverse of the Cartan involution of $\tilde{K}^{\mathbb{C}}$. This norm can be zero for any holomorphic function f, but if λ is sufficiently large, this does not occur.

Example 2.1. *Let* $J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$, $J' = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$, *and set*

$$G = \left\{ g \in GL(2r, \mathbb{C}) : gJ{}^t g = J, gJ' = J'\bar{g} \right\} \simeq Sp(r, \mathbb{R}),$$

Then we have

$$D = \{ w \in \mathrm{Sym}(r, \mathbb{C}) : I - ww^* \textit{ is positive definite.} \},$$

and $\tilde{G}$ acts on $\mathcal{O}(D,V)$ by

$$\tau_\lambda\left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1}\right) f(w) := \det(Cw+D)^{-\lambda}\tau\left({}^t(Cw+D)\right) f\left((Aw+B)(Cw+D)^{-1}\right).$$

This preserves

$$\|f\|_{\lambda,\tau}^2 := \frac{c_\lambda}{\pi^n}\int_D \left(\tau((I-ww^*)^{-1})f(w), f(w)\right)_\tau \det(I-ww^*)^{\lambda-(r+1)}dw.$$

We define another norm on $\mathcal{P}(\mathfrak{p}^+,V)$.

$$\|f\|_{F,\tau}^2 := \frac{1}{\pi^n}\int_{\mathfrak{p}^+} |f(w)|_\tau^2 e^{-|w|^2}dw \qquad (f\in\mathcal{P}(\mathfrak{p}^+,V)),$$

where $|w|$ is a suitable K-invariant norm on $\mathfrak{p}^+$.

Let $W\subset\mathcal{P}(\mathfrak{p}^+,V)$ be an irreducible subspace under the K-action

$$\left(\hat{\tau}(k)f\right)(w) := \tau(k)f(k^{-1}w) \qquad (k\in K^{\mathbb{C}},\ f\in\mathcal{P}(\mathfrak{p}^+,V),\ w\in\mathfrak{p}^+).$$

Then since $\|\cdot\|_{\lambda,\tau}$ and $\|\cdot\|_{F,\tau}$ are both K-invariant, the ratio of two norms are constant on W. The goal of this talk is to calculate this ratio under some conditions.

Remark 2.2 (Known results)**.** *For several settings, this is already done by*

- *B. Ørsted (1980) for $G=SU(r,r)$, scalar type.*
- *J. Faraut and A. Korányi (1990) for G any Hermitian Lie group, scalar type.*
- *B. Ørsted and G. Zhang (1994, 1995) for $G=Sp(r,\mathbb{R})$, $V=(\mathbb{C}^r)^\vee$, $G=SU(r,r)$, $V=\mathbb{C}\boxtimes\mathbb{C}^r$, $G=SO^*(4r)$, $V=(\mathbb{C}^{2r})^\vee$.*
- *S. Hwang, Y. Liu and G. Zhang (2004) for $G=SU(n,1)$, $V=\bigwedge^p(\mathbb{C}^n)^\vee\boxtimes\mathbb{C}$, $\bigwedge^q\mathbb{C}^n\boxtimes\mathbb{C}$.*

3 Main result

We assume that G is of tube type. Then there exists an $e\in\mathfrak{p}^+$ such that

$$-[e,\vartheta e]\in\mathfrak{z}(\mathfrak{k}), \qquad -[e,[\vartheta e,e]]=2e,$$

where ϑ is the Cartan involution of $\mathfrak{g}^{\mathbb{C}}$. Using this, we define

$${}^cG := \mathrm{Int}(e^{\frac{\pi i}{4}(e-\vartheta e)})G, \qquad L := {}^cG\cap K^{\mathbb{C}}, \qquad K_L := L\cap K.$$

We take a maximal abelian subalgebra $\mathfrak{h}\subset\mathfrak{k}$. We write the root system by

$$\Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}}) = \Delta_{\mathfrak{p}^+}\cup\Delta_{\mathfrak{k}^{\mathbb{C}}}\cup\Delta_{\mathfrak{p}^-}\subset(\mathfrak{h}^{\mathbb{C}})^\vee$$

in the usual fashion. We define the positive system $\Delta^+\subset\Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})$ such that $\Delta^+\supset\Delta_{\mathfrak{p}^+}$. Then there exist roots $\{\gamma_1,\dots,\gamma_r\}\subset\Delta_{\mathfrak{p}^+}$ such that, for a irreducible unitary representation $V_{-\mu}$ of K with lowest weight $-\mu\in(\mathfrak{h}^{\mathbb{C}})^\vee$, $V_{-\mu}$ has a K_L-invariant vector (K_L-spherical) if and only if $-\mu$ is of the form

$$-\mu = -(m_1\gamma_1+\cdots+m_r\gamma_r), \qquad m_j\in\mathbb{Z}, \quad m_1\ge\cdots\ge m_r.$$

Let $\mathfrak{a}_{\mathfrak{l}} := \sqrt{-1}\mathfrak{h} \cap \mathfrak{l}$. Then $(\mathfrak{a}_{\mathfrak{l}})^\vee$ is spanned by $\{\gamma_1|_{\mathfrak{a}_{\mathfrak{l}}}, \dots, \gamma_r|_{\mathfrak{a}_{\mathfrak{l}}}\}$.

For a representation V of $K^{\mathbb{C}}$, we denote by $\bar{V}$ the complex conjugate representation of V with respect to the real form L. For $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ and $\mathbf{m} = (m_1, \dots, m_r) \in (\mathbb{Z}_{\geq 0})^r$, we write

$$(\mathbf{s})_{\mathbf{m}} := \prod_{j=1}^{r} \left(s_j - \frac{d}{2}(j-1) \right)_{m_j}$$

where d is the integer satisfying $n = r + \frac{d}{2}r(r-1)$. Now we state the main theorem.

Theorem 3.1. *Let (τ, V) be an irreducible finite dimensional complex representation of $K^{\mathbb{C}}$ with restricted lowest weight $\left. -\frac{k_1}{2}\gamma_1 - \cdots - \frac{k_r}{2}\gamma_r \right|_{\mathfrak{a}_{\mathfrak{l}}}$. We assume $\tau|_{K_L}$ still remains irreducible. Let $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ be a K-irreducible subspace. If all the K_L-spherical irreducible subspaces in $W \otimes \bar{V}$ have the same lowest weight $-n_1\gamma_1 - \cdots - n_r\gamma_r$, then for any $f \in W$,*

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{(\lambda)_{\mathbf{k}}}{(\lambda)_{\mathbf{n}}} = \frac{1}{(\lambda + \mathbf{k})_{\mathbf{n}-\mathbf{k}}}$$

holds under the suitable choice of c_λ.

Example 3.2. *Let $G = Sp(r, \mathbb{R})$. Then we have $K = U(r)$, $L = GL(r, \mathbb{R})$, $K_L = O(r)$, and $\mathfrak{a}_{\mathfrak{l}} = \sqrt{-1}\mathfrak{h}$. We identify $(\sqrt{-1}\mathfrak{h})^\vee \simeq \mathbb{R}^r$ such that $-(m_1, \dots, m_r) \in \mathbb{R}^r$ is a lowest weight of a representation of K if and only if $m_1 \geq \cdots \geq m_r$, $m_j \in \mathbb{Z}$. Then we have $\gamma_j = 2\mathbf{e}_j$.*

We set $V := V_{-(\underbrace{1,\dots,1}_{k},0,\dots,0)} = \bigwedge^k (\mathbb{C}^r)^\vee$. Then this remains irreducible when restricted to K_L, and $\bar{V} \simeq V$ holds. We have

$$\begin{aligned} \mathcal{P}(\mathfrak{p}^+, V) &\simeq \bigoplus_{\substack{m_1 \geq \cdots \geq m_r \geq 0 \\ m_j \in \mathbb{Z}}} V_{-2\mathbf{m}} \otimes V_{-(\underbrace{1,\dots,1}_{k},0,\dots,0)} \\ &\simeq \bigoplus_{\substack{m_1 \geq \cdots \geq m_r \geq 0 \\ m_j \in \mathbb{Z}}} \bigoplus_{i_1 < \cdots < i_k} V_{-(2\mathbf{m}+\mathbf{e}_{i_1 \dots i_k})} \end{aligned}$$

where $\mathbf{e}_{i_1 \dots i_k}$ is the vector whose i_j-component is 1 and 0 otherwise, and $V_{-\nu} = \{0\}$ if ν is not dominant. $V_{-(2\mathbf{m}+\mathbf{e}_{i_1 \dots i_k})} \otimes \bar{V}$ is decomposed as

$$V_{-(2\mathbf{m}+\mathbf{e}_{i_1 \dots i_k})} \otimes \bar{V} \simeq \bigoplus_{j_1 < \cdots < j_k} V_{-(2\mathbf{m}+\mathbf{e}_{i_1 \dots i_k}+\mathbf{e}_{j_1 \dots j_k})},$$

but since $V_{-\nu}$ is K_L-spherical if and only if $-\nu \in (2\mathbb{Z})^r$, the only K_L-spherical subspace in $V_{-(2\mathbf{m}+\mathbf{e}_{i_1 \dots i_k})} \otimes \bar{V}$ is $V_{-2(\mathbf{m}+\mathbf{e}_{i_1 \dots i_k})}$. Therefore, for $f \in V_{-(2\mathbf{m}+\mathbf{e}_{i_1 \dots i_k})}$, we have

$$\frac{\|f\|^2_{\lambda,\tau_{\mathbf{e}_{12\dots k}}}}{\|f\|^2_{F,\tau_{\mathbf{e}_{12\dots k}}}} = \frac{(\lambda)_{\mathbf{e}_{12\dots k}}}{(\lambda)_{\mathbf{m}+\mathbf{e}_{i_1 \dots i_k}}} = \frac{1}{(\lambda + \mathbf{e}_{12\dots k})_{\mathbf{m}+\mathbf{e}_{i_1 \dots i_k}-\mathbf{e}_{12\dots k}}}.$$

Thus the norm

$$\|f\|^2_{\lambda,\tau_{\mathbf{e}_{12\dots k}}} = \sum_{\substack{m_1 \geq \cdots \geq m_r \geq 0 \\ m_j \in \mathbb{Z}}} \sum_{i_1 < \cdots < i_k} \frac{(\lambda)_{\mathbf{e}_{12\dots k}}}{(\lambda)_{\mathbf{m}+\mathbf{e}_{i_1 \dots i_k}}} \|f_{-(2\mathbf{m}+\mathbf{e}_{i_1 \dots i_k})}\|^2_{F,\tau_{\mathbf{e}_{12\dots k}}}$$

is continued meromorphically, and for $0 \le k \le r-1$ it becomes a unitary representation if and only if

$$\lambda \in \left\{ \frac{k}{2}, \frac{k+1}{2}, \dots, \frac{r-1}{2} \right\} \cup \left(\frac{r-1}{2}, \infty \right),$$

and when $\lambda = \frac{l}{2}$ $(l = k, \dots, r-1)$ the corresponding representation space is

$$\mathcal{H}_\lambda(D, V) = \sideset{}{^\oplus}\sum_{\substack{m_1 \ge \cdots m_l \ge 0 \\ m_{l+1} = \cdots m_r = 0 \\ m_j \in \mathbb{Z}}} \bigoplus_{1 \le i_1 < \cdots < i_k \le l} V_{-(2\mathbf{m} + \mathbf{e}_{i_1 i_2 \cdots i_k})}.$$

Remark 3.3. *Our result can be applied for*

- $G = Sp(r, \mathbb{R})$, $V = \bigwedge^k (\mathbb{C}^r)^\vee$.
- $G = SO^*(4r)$, $V = S^k(\mathbb{C}^{2r})^\vee$, $S^k(\mathbb{C}^{2r})$.
- $G = SU(r, r)$, $V = W^\vee \boxtimes \mathbb{C}$, $\mathbb{C} \boxtimes W$ *(W: any irreducible representation of $U(r)$).*
- $G = Spin_0(2, 2s)$, $V = \mathbb{C} \boxtimes V_{-(k, \dots, k, \pm k)}$ $(k \in \frac{1}{2}\mathbb{Z}_{\ge 0})$.
- $G = Spin_0(2, 2s+1)$, $V = \mathbb{C} \boxtimes V_{-\left(\frac{1}{2}, \dots, \frac{1}{2}\right)}$ *(spin representation).*

4 Proof for $G = Sp(r, \mathbb{R})$ case

In this section we prove the theorem for $G = Sp(r, \mathbb{R})$ case. In this case we have

$$K = U(r), \quad L = GL(r, \mathbb{R}), \quad K_L = O(r), \quad \mathfrak{p}^+ = \mathrm{Sym}(r, \mathbb{C}),$$
$$D = \{ w \in \mathrm{Sym}(r, \mathbb{C}) : I - ww^* \text{ is positive definite} \}.$$

Let (τ, V) be an irreducible unitary representation of K, and let $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ be a K-irreducible subspace under the action

$$(\hat{\tau}(k) f)(w) := \tau(k) f(k^{-1} w {}^t k^{-1}) \qquad (k \in K^{\mathbb{C}},\ f \in \mathcal{P}(\mathfrak{p}^+, V),\ w \in \mathfrak{p}^+).$$

For $f \in W$ we compute the ratio

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{\dfrac{c_\lambda}{\pi^n} \displaystyle\int_D \big(\tau((I - ww^*)^{-1}) f(w), f(w) \big)_\tau \det(I - ww^*)^{\lambda - (r+1)} dw}{\dfrac{1}{\pi^n} \displaystyle\int_{\mathfrak{p}^+} |f(w)|^2_\tau e^{-\mathrm{tr}(ww^*)} dw}$$

under the condition

- V has the lowest weight $-\mathbf{k} = (-k_1, \dots, -k_r)$.
- $V|_{K_L} = V|_{O(r)}$ still remains irreducible.
- All the $K_L = O(r)$-spherical irreducible subspaces in $W \otimes \bar{V} \simeq W \otimes V$ have the same lowest weight $-2\mathbf{n} = (-2n_1, \dots, -2n_r)$.

We set $\|f\|_{\lambda,\tau}^2/\|f\|_{F,\tau}^2 =: R_{W,\lambda}$, and compute this ratio.

As a preparation, we set

$$
{}^tB := \left\{ \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ b_{r1} & b_{r2} & \cdots & b_{rr} \end{pmatrix} \in GL(r,\mathbb{R}) : b_{jj} > 0,\ j = 1, \dots, r \right\},
$$

$$
\Omega := \{ x \in \mathrm{Sym}(r,\mathbb{R}) : \text{positive definite} \}
$$

Then we have a bijection ${}^tB \to \Omega$, $b \mapsto b{}^tb$. Also, for $x \in \Omega$ and $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, we set

$$
\Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1-s_2}\Delta_2(x)^{s_2-s_3}\cdots\Delta_{r-1}(x)^{s_{r-1}-s_r}\Delta_r(x)^{s_r}
$$

where $\Delta_k(x) := \det((x_{ij})_{1\le i,j\le k})$. For example, for $b = (b_{ij}) \in {}^tB$ we have $\Delta_{\mathbf{s}}(b{}^tb) = b_{11}^{2s_1}b_{22}^{2s_2}\cdots b_{rr}^{2s_r}$. Also we set

$$
\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\operatorname{tr}(x)}\Delta_{\mathbf{s}}(x)\det(x)^{-\frac{r+1}{2}}dy.
$$

This integral converges if $\operatorname{Re} s_j > \frac{1}{2}(j-1)$, and we have

$$
\begin{aligned}
\Gamma_\Omega(\mathbf{s}) &= 2^{\frac{n+r}{2}}\int_{{}^tB} e^{-\operatorname{tr}(b{}^tb)}\Delta_{\mathbf{s}}(b{}^tb)\det(b{}^tb)^{-\frac{r+1}{2}}b_{11}^r b_{22}^{r-1}\cdots b_{rr}db \\
&= 2^{\frac{n+r}{2}}\int_{{}^tB} e^{-\sum_{i\ge j}b_{ij}^2}b_{11}^{2s_1}b_{22}^{2s_2}\cdots b_{rr}^{2s_r}(b_{11}b_{22}\cdots b_{rr})^{-(r+1)}b_{11}^r b_{22}^{r-1}\cdots b_{rr}db \\
&= 2^{\frac{n+r}{2}}\prod_{j=1}^r\int_0^\infty e^{-b_{jj}^2}b_{jj}^{2s_j-j}db_{jj}\prod_{i\ge j}\int_{-\infty}^\infty e^{-b_{ij}^2}db_{ij} \\
&= (2\pi)^{\frac{n-r}{2}}\prod_{j=1}^r\Gamma\left(s_j - \frac{j-1}{2}\right),
\end{aligned}
$$

and therefore we have

$$
\frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})} = \prod_{j=1}^r\frac{\Gamma\left(s_j+m_j-\frac{j-1}{2}\right)}{\Gamma\left(s_j-\frac{j-1}{2}\right)} = \prod_{j=1}^r\left(s_j-\frac{j-1}{2}\right)_{m_j} = (\mathbf{s})_{\mathbf{m}}.
$$

Now we start the proof. Let $K_W(z,w) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V))$ be the reproducing kernel of W with respect to $\|\cdot\|_{F,\tau}$. Then the ratio $R_{W,\lambda}$ is computed as

$$
R_{W,\lambda} = \frac{c_\lambda\displaystyle\int_D \mathrm{Tr}_V\left(\tau((I-ww^*)^{-1})K_W(w,w)\right)\det(I-ww^*)^{\lambda-(r+1)}dw}{\displaystyle\int_{\mathfrak{p}^+}\mathrm{Tr}_V(K_W(w,w))e^{-\operatorname{tr}(ww^*)}dw}.
$$

To proceed the computation, we prove the following lemma.

Lemma 4.1. *For any integrable function f on $\mathfrak{p}^+$, we have*

$$
\int_{\mathfrak{p}^+} f(w)dw = c\int_\Omega\int_K f(kx^{\frac{1}{2}}{}^tk)dkdx
$$

where c is a constant.

Proof. By the integral formulas

$$\int_{\mathfrak{p}^+} f(w)dw = c' \int_K \int_0^\infty \cdots \int_0^\infty f(ka{}^t\!k) \prod_{i<j} |a_i^2 - a_j^2| \prod_{j=1}^r a_j da_1 \cdots da_r dk,$$
$$\int_\Omega f(x)dx = c'' \int_{K_L} \int_0^\infty \cdots \int_0^\infty f(k' t {}^t\!k') \prod_{i<j} |t_i - t_j| dt_1 \cdots dt_r dk'$$

where $a = \mathrm{diag}(a_1, \ldots, a_r)$, $t = \mathrm{diag}(t_1, \ldots, t_r)$, we can deduce the lemma by putting $a_j^2 = t_j$. □

By this lemma, this is equal to

$$R_{W,\lambda} = \frac{c_\lambda \int_{\Omega\cap(I-\Omega)} \int_K \mathrm{Tr}_V \left(\tau((I - kxk^*)^{-1}) K_W(kx^{\frac{1}{2}}{}^t\!k, kx^{\frac{1}{2}}{}^t\!k) \right) \det(I-x)^{\lambda-(r+1)} dkdx}{\int_\Omega \int_K \mathrm{Tr}_V \left(K_W(kx^{\frac{1}{2}}{}^t\!k, kx^{\frac{1}{2}}{}^t\!k) \right) e^{-\mathrm{tr}(x)} dkdx}.$$

Since the reproducing kernel satisfies

$$K_W(kz{}^t\!k, k^{*-1} w \bar{k}^{-1}) = \tau(k) K_W(z,w) \tau(k^{-1}) \qquad (z, w \in \mathfrak{p}^+,\ k \in K^{\mathbb{C}}),$$

we have, for $k \in K$ and $x \in \Omega$,

$$\begin{aligned} K_W(kx^{\frac{1}{2}}{}^t\!k, kx^{\frac{1}{2}}{}^t\!k) &= \tau(k) K_W(x^{-\frac{1}{4}} x x^{-\frac{1}{4}}, x^{\frac{1}{4}} I x^{\frac{1}{4}}) \tau(k^{-1}) \\ &= \tau(k) \tau(x^{-\frac{1}{4}}) K_W(x, e) \tau(x^{\frac{1}{4}}) \tau(k^{-1}). \end{aligned}$$

Therefore we have

$$\mathrm{Tr}_V \left(K_W(kx^{\frac{1}{2}}, kx^{\frac{1}{2}}) \right) = \mathrm{Tr}_V(K_W(x, I)),$$
$$\mathrm{Tr}_V \left(\tau((I - kxk^*)^{-1}) K_W(kx^{\frac{1}{2}}{}^t\!k, kx^{\frac{1}{2}}{}^t\!k) \right) = \mathrm{Tr}_V \left(\tau((I-x)^{-1}) K_W(x, I) \right).$$

Thus we get

$$R_{W,\lambda} = \frac{c_\lambda \int_{\Omega\cap(I-\Omega)} \mathrm{Tr}_V \left(\tau((I-x)^{-1}) K_W(x, I) \right) \det(I-x)^{\lambda-(r+1)} dkdx}{\int_\Omega \mathrm{Tr}_V (K_W(x, I)) e^{-\mathrm{tr}(x)} dkdx}.$$

Now we set

$$B_{W,\lambda} := \int_{\Omega\cap(I-\Omega)} \mathrm{Tr}_V \left(\tau((I-x)^{-1}) K_W(x, I) \right) \det(I-x)^{\lambda-(r+1)} dkdx,$$
$$\Gamma_W := \int_\Omega \mathrm{Tr}_V (K_W(x, I)) e^{-\mathrm{tr}(x)} dkdx$$

so that $R_{W,\lambda} = c_\lambda B_{W,\lambda}/W_W$. Now, we regard $K_W(x, I) \in \mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V))$ as a function of x. We define the action $\tilde{\tau}$ of $K^{\mathbb{C}}$ on $\mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V))$ by

$$(\tilde{\tau}(k)F)(x) := \tau(k) F(k^{-1} x {}^t\!k^{-1}) \tau({}^t\!k) \quad (k \in K^{\mathbb{C}},\ F \in \mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V)),\ x \in \mathfrak{p}^+).$$

Then $K_W(x, I)$ is K_L-invariant under $\tilde{\tau}$. Now we use the following isomorphism

$$(\tilde{\tau}, \mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V))) \simeq (\hat{\tau} \otimes \bar{\tau}, \mathcal{P}(\mathfrak{p}^+, V) \otimes \bar{V}),$$

where $(\bar{\tau}, \bar{V})$ is the complex conjugate of (τ, V) as a representation of L. In this case $L = GL(r, \mathbb{R})$, and we have $(\bar{\tau}, \bar{V}) \simeq (\tau, V)$. Then under this isomorphism $K_W(x, I)$ sits in $W \otimes V$. Since all the K_L-spherical component in this tensor product has the lowest weight $-2\mathbf{n}$, we conclude that $K_W(x, I)$ sits in $V_{-2\mathbf{n}}$, that is, there exists a function $F \in \mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V))$ such that, for any $b = (b_{ij})_{1 \le j \le i \le r} \in {}^tB$,

$$(\tilde{\tau}(b)F)(x) = b_{11}^{-2n_1} b_{22}^{-2n_2} \cdots b_{rr}^{-2n_r} F(x) = \Delta_{\mathbf{n}}(b^{-1}{}^t b^{-1}) F(x),$$

and

$$\int_{K_L} (\tilde{\tau}(k)F)(x) dk = K_W(x, I).$$

By using F, we rewrite $B_{W,\lambda}$ and Γ_W.

$$B_{W,\lambda} = \int_{\Omega \cap (I - \Omega)} \mathrm{Tr}_V \left(\tau((I - x)^{-1}) F(x) \right) \det(I - x)^{\lambda - (r+1)} dx,$$

$$\Gamma_W = \int_{\Omega} \mathrm{Tr}_V (F(x)) e^{-\operatorname{tr}(x)} dx.$$

From now we calculate $B_{W,\lambda}$. For $y \in \Omega$, we set

$$I(y) := \int_{\Omega \cap (y - \Omega)} \mathrm{Tr}_V \left(\tau((y - x)^{-1}) F(x) \right) \det(y - x)^{\lambda - (r+1)} dx$$

so that $I(e) = B_{W,\lambda}$. We take $b \in {}^tB$ such that $y = b{}^t b$, and set $x = bz{}^t b$. Then

$$\begin{aligned}
I(y) &= \int_{\Omega \cap (I - \Omega)} \mathrm{Tr}_V \left(\tau((b(I - z){}^t b)^{-1}) F(bz{}^t b) \right) \det(b(I - z){}^t b)^{\lambda - (r+1)} \det(b{}^t b)^{\frac{r+1}{2}} dz \\
&= \int_{\Omega \cap (I - \Omega)} \mathrm{Tr}_V \left(\tau((I - z)^{-1}) \tau(b^{-1}) F(bz{}^t b) \tau({}^t b^{-1}) \right) \det(I - z)^{\lambda - (r+1)} \det(b{}^t b)^{\lambda - \frac{r+1}{2}} dz \\
&= \int_{\Omega \cap (I - \Omega)} \mathrm{Tr}_V \left(\tau((I - z)^{-1}) F(z) \right) \Delta_{\mathbf{n}}(b{}^t b) \det(I - z)^{\lambda - (r+1)} \det(b{}^t b)^{\lambda - \frac{r+1}{2}} dz \\
&= I(e) \Delta_{\mathbf{n}}(y) \det(y)^{\lambda - \frac{r+1}{2}} = B_{W,\lambda} \Delta_{\lambda + \mathbf{n}}(y) \det(y)^{-\frac{r+1}{2}}.
\end{aligned}$$

Now we calculate $\int_{\Omega} I(y) e^{-\operatorname{tr}(y)} dy$ by two ways.

$$\int_{\Omega} I(y) e^{-\operatorname{tr}(y)} dy = B_{W,\lambda} \int_{\Omega} e^{-\operatorname{tr}(y)} \Delta_{\lambda + \mathbf{n}}(y) \det(y)^{-\frac{r+1}{2}} dy = B_{W,\lambda} \Gamma_{\Omega}(\lambda + \mathbf{n}),$$

$$\begin{aligned}
\int_{\Omega} I(y) e^{-\operatorname{tr}(y)} dy &= \iint_{x \in \Omega, y - x \in \Omega} e^{-\operatorname{tr}(y)} \mathrm{Tr}_V \left(\tau((y - x)^{-1}) F(x) \right) \det(y - x)^{\lambda - (r+1)} dx dy \\
&= \iint_{x \in \Omega, z \in \Omega} e^{-\operatorname{tr}(x + z)} \mathrm{Tr}_V \left(\tau(z^{-1}) F(x) \right) \det(z)^{\lambda - (r+1)} dx dz \\
&= \mathrm{Tr}_V \left(\int_{\Omega} e^{-\operatorname{tr}(x)} F(x) dx \int_{\Omega} e^{-\operatorname{tr}(z)} \tau(z^{-1}) \det(z)^{\lambda - (r+1)} dz \right).
\end{aligned}$$

Since $\int_{\Omega} e^{-\operatorname{tr}(z)} \tau(z^{-1}) \det(z)^{\lambda - (r+1)} dz$ commutes with K_L-action and $V|_{K_L}$ is irreducible, this is the scalar operator on V. Moreover, for a lowest weight vector $v \in V$ and $z = b{}^t b \in \Omega$ ($b \in {}^tB$), we have

$$\begin{aligned}
&\left(\tau(z^{-1}) v, v \right)_{\tau} = \left(\tau({}^t b^{-1} b^{-1}) v, v \right)_{\tau} = |\tau(b^{-1}) v|_{\tau}^2 \\
=& (b_{11}^{k_1} \ldots b_{rr}^{k_r})^2 |v|_{\tau}^2 = \Delta_{\mathbf{k}}(b{}^t b) |v|_{\tau}^2 = \Delta_{\mathbf{k}}(z) |v|_{\tau}^2,
\end{aligned}$$

and

$$\left(\int_\Omega e^{-\operatorname{tr}(z)}\tau(z^{-1})\det(z)^{\lambda-(r+1)}dzv, v\right)_\tau$$
$$= \int_\Omega e^{-\operatorname{tr}(z)}\Delta_{\mathbf{k}}(z)\det(z)^{\lambda-(r+1)}dz|v|_\tau^2 = \Gamma_\Omega\left(\lambda+\mathbf{k}-\frac{r+1}{2}\right)|v|_\tau^2.$$

Therefore $\int_\Omega e^{-\operatorname{tr}(z)}\tau(z^{-1})\det(z)^{\lambda-(r+1)}dz = \Gamma_\Omega\left(\lambda+\mathbf{k}-\frac{r+1}{2}\right)\mathrm{id}_V$, and

$$B_{W,\lambda} = \frac{\Gamma_\Omega\left(\lambda+\mathbf{k}-\frac{r+1}{2}\right)}{\Gamma_\Omega(\lambda+\mathbf{n})}\operatorname{Tr}_V\left(\int_\Omega e^{-\operatorname{tr}(x)}F(x)dx\right) = \frac{\Gamma_\Omega\left(\lambda+\mathbf{k}-\frac{r+1}{2}\right)}{\Gamma_\Omega(\lambda+\mathbf{n})}\Gamma_W.$$

Thus we have

$$R_{W,\lambda} = c_\lambda\frac{\Gamma_\Omega\left(\lambda+\mathbf{k}-\frac{r+1}{2}\right)}{\Gamma_\Omega(\lambda+\mathbf{n})}.$$

Now we determine $c_\lambda := \frac{\Gamma_\Omega(\lambda+\mathbf{k})}{\Gamma_\Omega\left(\lambda+\mathbf{k}-\frac{r+1}{2}\right)}$. Then we have

$$R_{W,\lambda} = \frac{\Gamma_\Omega(\lambda+\mathbf{k})}{\Gamma_\Omega(\lambda+\mathbf{n})} = \frac{\Gamma_\Omega(\lambda+\mathbf{k})}{\Gamma_\Omega(\lambda)}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda+\mathbf{n})} = \frac{(\lambda)_{\mathbf{k}}}{(\lambda)_{\mathbf{n}}}.$$

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