Title : Holomorphic discrete series and Borel-de Siebenthal discrete series.

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Summary : Let G_0 be a simply connected noncompact real simple Lie group with maximal compact subgroup K_0. Assume that
(i) rank (G_0) = rank (K_0) so that G_0 has discrete series representations, and
(ii) the Riemannian symmetric space G_0/K_0 is *not* Hermitian.

Let $T_0 \subset K_0$ be a Cartan subgroup of K_0 as well as of G_0. Denote by $\mathfrak{g}, \mathfrak{k}$, and $\mathfrak{t}$, the complexifications of the Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0$ and $\mathfrak{t}_0$ of G_0, K_0 and T_0 respectively. Let Δ and $\Delta_{\mathfrak{k}}$ be the root systems of $(\mathfrak{g}, \mathfrak{t})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively. There exists a positive system Δ^+ of Δ, known as the Borel-de Siebenthal positive system [1] for which there is exactly one non-compact simple root, denoted ν. Let μ denote the highest root. Then the coefficient of ν in the highest root μ is 2 and we have a partition $\Delta = \cup_{-2\leq i\leq 2}\Delta_i$ where $\alpha \in \Delta$ belongs to Δ_i precisely when the coefficient of ν in α when expressed as a sum of simple roots is equal to i. Then $\Delta_{\mathfrak{k}} = \Delta_{-2} \cup \Delta_0 \cup \Delta_2$. Let $\Delta_{\mathfrak{k}}^+$ denote the positive system of $\Delta_{\mathfrak{k}}$ induced from Δ^+. Let G be the simply connected complexification of G_0. Denote by L_0, the centralizer in K_0 of the circle subgroup S_0 of T_0 determined by the fundamental weight ν^* corresponding to the noncompact simple root ν of the positive system Δ^+ and by $\bar{L}_0$ its image in G (under the homomorphism $p : G_0 \longrightarrow G$ defined by the inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$). Any $\bar{L}_0$-representation is regarded as an L_0-representation via p.

Let γ be the highest weight of an irreducible representation E_γ of $\bar{L}_0$ such that $\gamma + \rho_{\mathfrak{g}}$ is negative on $\Delta_1 \cup \Delta_2$. Here $\rho_{\mathfrak{g}}$ denotes half the sum of positive roots of $\mathfrak{g}$. The Borel-de Siebenthal discrete series of G_0, whose systematic study was carried out by Ørsted and Wolf [4], is a discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0 with Harish-Chandra parameter $\gamma + \rho_{\mathfrak{g}}$. The K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is described in terms of the Dolbeault cohomology as $\oplus_{m\geq 0} H^s(K_0/L_0; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$ where $s = \dim_{\mathbb{C}} K_0/L_0$, $\mathbb{E}_\gamma$ and $\mathbb{S}^m(\mathfrak{u}_{-1})$ denote the holomorphic vector bundles associated to the irreducible L_0-module E_γ, and the m-th symmetric power $S^m(\mathfrak{u}_{-1})$ of the irreducible L_0-module $\mathfrak{u}_{-1}$ respectively, where $\mathfrak{u}_{-1} = \oplus_{\alpha\in\Delta_{-1}}\mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha$ being the root space for $\alpha \in \Delta$ (see [4], [5]). Let K_1 be the simple factor of K_0 such that the complexification of its Lie algebra contains the root space $\mathfrak{g}_\mu$ corresponding to the highest root μ of $\mathfrak{g}$. The Borel-de Siebenthal discrete series is K_1-admissible (see [4], [2]). It turns out that $S_0 \subset K_1$ and $K_1/L_1 = K_0/L_0$ is an irreducible Hermitian symmetric space of the compact type where $L_1 = L_0 \cap K_1$. Let $K_0^* \subset G$ denote the dual of K_0 with respect to L_0 (denoting L_0 and $\bar{L}_0$ by the same notation L_0). Then K_0^*/L_0 is an irreducible bounded symmetric domain dual to K_0/L_0. Corresponding to the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0, we can associate a holomorphic discrete series representation $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of K_0^* with Harish-Chandra parameter $\gamma + \rho_{\mathfrak{k}}$ (see [6]). Here $\rho_{\mathfrak{k}}$ denotes half the sum of positive roots of $\mathfrak{k}$.

In this talk the following question will be addressed: Does there exist common L_0-types between the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ and the holomorphic

discrete series representation $\pi_{\gamma+\rho_{\mathfrak{k}}}$? This question can be settled completely in the quaternionic case, namely, when $\mathfrak{k}_1 \cong \mathfrak{su}(2)$ (see [6]). In the general case, affirmative answer is obtained under the following two hypotheses—(i) there exists a (non-constant) relative invariant for the prehomogeneous space $(L_0^{\mathbb{C}}, \mathfrak{u}_1)$, where $\mathfrak{u}_1$ is the representation of L_0 on the normal space at the identity coset for the (holomorphic) imbedding $K_0/L_0 \hookrightarrow G_0/L_0$, and, (ii) the longest element $w_{\mathfrak{k}}^0$ of the Weyl group of K_0 with respect to the positive system $\Delta_{\mathfrak{k}}^+$ normalizes L_0, that is, the bounded symmetric domain K_0^*/L_0 is of tube type (see [6]). The proof uses, among others, a decomposition theorem of Schmid [7] and Littelmann's path model [3].

References

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