

The Gross-Prasad conjecture and local theta correspondence

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This note is a report on a joint work with Wee Teck Gan [9]. Details will appear elsewhere.

1 Restriction problem

In [10], [11], [6], [7], a restriction problem in the representation theory of classical groups was studied and a precise conjecture was formulated for this restriction problem. This so-called *Gross-Prasad* (GP) conjecture has generated much interest in recent years.

In this note, we shall focus on the restriction problem for unitary groups. Let F be a nonarchimedean local field of characteristic 0 and residue characteristic p , and let E be a quadratic field extension of F . Fix a trace zero element $\delta \in E^\times$. Given a positive integer n , there are precisely two isometry classes of n -dimensional (skew-)Hermitian spaces, which are distinguished from each other by their discriminants. Let V_n^+ be the n -dimensional Hermitian space with trivial discriminant and V_n^- the other n -dimensional Hermitian space. Similarly, let W_n^+ be the n -dimensional skew-Hermitian space with trivial discriminant and W_n^- the other n -dimensional skew-Hermitian space. For the GP conjecture, we consider the pair of spaces

$$V_n^+ \subset V_{n+1}^+ \quad \text{or} \quad W_n^+ = W_{n+1}^+$$

and the relevant pure inner form (other than itself)

$$V_n^- \subset V_{n+1}^- \quad \text{or} \quad W_n^- = W_{n+1}^-.$$

Thus, for $\epsilon = \pm 1$, we have the groups

$$G_n^\epsilon = \mathrm{U}(V_n^\epsilon) \times \mathrm{U}(V_{n+1}^\epsilon) \quad \text{or} \quad \mathrm{U}(W_n^\epsilon) \times \mathrm{U}(W_{n+1}^\epsilon)$$

and

$$H_n^\epsilon = \mathrm{U}(V_n^\epsilon) \quad \text{or} \quad \mathrm{U}(W_n^\epsilon),$$

and the embedding

$$\Delta : H_n^\epsilon \hookrightarrow G_n^\epsilon.$$

Let π be an irreducible smooth representation of G_n^ϵ . In the Hermitian case, one is interested in determining

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n^\epsilon}(\pi, \mathbb{C}).$$

We shall call this the *Bessel* case (B) of the GP conjecture. In the skew-Hermitian case, the restriction problem requires another piece of data: a Weil representation $\omega_{\psi, \chi, W_n^\epsilon}$, where ψ is a nontrivial additive

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character of F and χ is a character of $E^\times$ whose restriction to $F^\times$ is the quadratic character $\omega_{E/F}$ associated to E/F by local class field theory. Then one is interested in determining

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\Delta H_n^\epsilon}(\pi, \omega_{\psi, \chi, W_n^\epsilon}).$$

We shall call this the *Fourier-Jacobi* case (FJ) of the GP conjecture.

By surprisingly recent results of Aizenbud-Gourevitch-Rallis-Schiffmann [1] and Sun [17], it is known that the above Hom spaces have dimension at most 1. Thus the main issue is to determine when the Hom space is nonzero. In [6], an answer for this issue is formulated in the framework of the local Langlands correspondence, in its enhanced form due to Vogan [18] which takes into account all pure inner forms.

2 Local Langlands correspondence

Now suppose that ϕ is an L -parameter for the group G_n^+ . Then ϕ gives rise to a Vogan L -packet Π_ϕ consisting of certain irreducible smooth representations of G_n^+ and its (not necessarily relevant) pure inner forms. Moreover, if one fixes a Whittaker datum for G_n^+ , there is a natural bijection

$$\Pi_\phi \longleftrightarrow \operatorname{Irr}(S_\phi),$$

where S_ϕ is the component group associated to ϕ . Thus an irreducible smooth representation of G_n^+ (and its pure inner forms) is labelled by a pair (ϕ, η) , where ϕ is an L -parameter for G_n^+ and η is an irreducible character of S_ϕ . We shall write $\pi(\phi, \eta)$ for the corresponding irreducible smooth representation.

More explicitly, ϕ is of the form

$$\phi = \phi^\diamond \times \phi^\heartsuit,$$

where $\phi^\diamond$ and $\phi^\heartsuit$ are conjugate self-dual representations of the Weil-Deligne group WD_E of E of the appropriate dimension and sign. We may decompose $\phi^\diamond$ and $\phi^\heartsuit$ into direct sums

$$\phi^\diamond = \bigoplus_i m_i \phi_i^\diamond \quad \text{and} \quad \phi^\heartsuit = \bigoplus_j l_j \phi_j^\heartsuit$$

with pairwise inequivalent irreducible representations of WD_E . Then

$$S_\phi = S_{\phi^\diamond} \times S_{\phi^\heartsuit},$$

where

$$S_{\phi^\diamond} = \prod_i (\mathbb{Z}/2\mathbb{Z}) a_i \quad \text{and} \quad S_{\phi^\heartsuit} = \prod_j (\mathbb{Z}/2\mathbb{Z}) b_j$$

with some bases $\{a_i\}$ and $\{b_j\}$, and the products range over all i and j such that $\phi_i^\diamond$ and $\phi_j^\heartsuit$ are conjugate self-dual with the same sign as $\phi^\diamond$ and $\phi^\heartsuit$ respectively.

3 Gross-Prasad conjecture

To state the GP conjecture, we need a distinguished character of S_ϕ depending on ϵ -factors. We consider the Bessel and Fourier-Jacobi cases separately.

- **Bessel case.** We fix a nontrivial character ψ^E of E/F which determines the local Langlands correspondence for the even unitary group in $G_n^\epsilon = \mathrm{U}(V_n^\epsilon) \times \mathrm{U}(V_{n+1}^\epsilon)$. We set $\psi_{-2}^E(x) = \psi^E(-2x)$ and define:

$$\begin{cases} \eta^\spadesuit(a_i) = \epsilon(\frac{1}{2}, \phi_i^\diamond \otimes \phi^\heartsuit, \psi_{-2}^E); \\ \eta^\spadesuit(b_j) = \epsilon(\frac{1}{2}, \phi^\diamond \otimes \phi_j^\heartsuit, \psi_{-2}^E). \end{cases}$$

- **Fourier-Jacobi case.** We need to fix a nontrivial character ψ of F and a character χ of $E^\times$ with $\chi|_{F^\times} = \omega_{E/F}$ to specify the Weil representation $\omega_{\psi, \chi, W_n^\epsilon}$ of $\mathrm{U}(W_n^\epsilon)$. The recipe for the distinguished character $\eta^\clubsuit$ of S_ϕ depends on the parity of $n = \dim W_n^\epsilon$.

- If n is odd, define

$$\begin{cases} \eta^\clubsuit(a_i) = \epsilon(\frac{1}{2}, \phi_i^\diamond \otimes \phi^\heartsuit \otimes \chi^{-1}, \psi_2^E); \\ \eta^\clubsuit(b_j) = \epsilon(\frac{1}{2}, \phi^\diamond \otimes \phi_j^\heartsuit \otimes \chi^{-1}, \psi_2^E), \end{cases}$$

where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

- If n is even, the fixed character ψ is used to fix the local Langlands correspondence for $\mathrm{U}(W_n^\epsilon)$. We set

$$\begin{cases} \eta^\clubsuit(a_i) = \epsilon(\frac{1}{2}, \phi_i^\diamond \otimes \phi^\heartsuit \otimes \chi^{-1}, \psi^E); \\ \eta^\clubsuit(b_j) = \epsilon(\frac{1}{2}, \phi^\diamond \otimes \phi_j^\heartsuit \otimes \chi^{-1}, \psi^E), \end{cases}$$

where the ϵ -factors are defined using any nontrivial additive character ψ^E of E/F .

With this short preparation, the GP conjecture can be stated as follows.

- (B) Given a *tempered* L -parameter ϕ for $G_n^+ = \mathrm{U}(V_n^+) \times \mathrm{U}(V_{n+1}^+)$ and a representation $\pi(\phi, \eta) \in \Pi_\phi$ of a relevant pure inner form G_n^ϵ ,

$$\mathrm{Hom}_{\Delta H_n^\epsilon}(\pi(\phi, \eta), \mathbb{C}) \neq 0 \iff \eta = \eta^\spadesuit.$$

- (FJ) Given a *tempered* L -parameter ϕ for $G_n^+ = \mathrm{U}(W_n^+) \times \mathrm{U}(W_{n+1}^+)$ and a representation $\pi(\phi, \eta) \in \Pi_\phi$ of a relevant pure inner form G_n^ϵ ,

$$\mathrm{Hom}_{\Delta H_n^\epsilon}(\pi(\phi, \eta), \omega_{\psi, \chi, W_n^\epsilon}) \neq 0 \iff \eta = \eta^\clubsuit.$$

In a stunning series of papers [20], [21], [22], [23], Waldspurger has established the Bessel case of the GP conjecture for tempered L -parameters ϕ in the case of special orthogonal groups; the case of general generic L -parameters is then dealt with by Mœglin-Waldspurger [13]. Beuzart-Plessis [3], [4], [5] has since extended Waldspurger's techniques to settle the Bessel case of the GP conjecture for unitary groups in the tempered case.

4 Main result

We establish the Fourier-Jacobi case of the GP conjecture, as well as two conjectures of D. Prasad concerning local theta correspondence in the (almost) equal rank case, under the following assumptions:

- We assume the local Langlands correspondence and Arthur's multiplicity formula for unitary groups. By the recent work of Arthur [2] and Mok [14], this is now known for quasi-split unitary groups, conditional on the stabilization of the twisted trace formula.
- We use the work of Beuzart-Plessis [3], [4], [5] on the Bessel case of the GP conjecture. This also relies on the local Langlands correspondence for unitary groups.
- We further assume that $p \neq 2$ throughout this paper. This assumption is necessary in order to use the Howe duality, which was proved by Waldspurger [19] for $p \neq 2$ but is not known for $p = 2$.

Let us describe the main idea of the proof. The Bessel and Fourier-Jacobi cases of the GP conjecture are related by the local theta correspondence. More precisely, put $V_n = V_n^\epsilon$ and $W_n = W_n^{\epsilon'}$, and so on. Then there is a see-saw diagram

$$\begin{array}{ccc}
 \mathrm{U}(W_n) \times \mathrm{U}(W_n) & & \mathrm{U}(V_{n+1}) \\
 | & \searrow & | \\
 \mathrm{U}(W_n) & & \mathrm{U}(V_n) \times \mathrm{U}(V_1)
 \end{array}$$

and the associated see-saw identity reads:

$$\mathrm{Hom}_{\mathrm{U}(W_n)}(\Theta_{\psi, \chi, V_n, W_n}(\sigma) \otimes \omega_{\psi, \chi, V_1, W_n}, \pi) \cong \mathrm{Hom}_{\mathrm{U}(V_n)}(\Theta_{\psi, \chi, V_{n+1}, W_n}(\pi), \sigma)$$

for irreducible smooth representations π of $\mathrm{U}(W_n)$ and σ of $\mathrm{U}(V_n)$. Hence the left-hand side of the see-saw identity concerns the Fourier-Jacobi case (FJ) whereas the right-hand side concerns the Bessel case (B). It is thus apparent that precise knowledge of the local theta correspondence for unitary groups of (almost) equal rank will give the precise relation of (FJ) to (B).

More precisely, one would need to know:

- (Θ) For irreducible tempered representations π and σ , the theta lifts $\Theta_{\psi, \chi, V_{n+1}, W_n}(\pi)$ and $\Theta_{\psi, \chi, V_n, W_n}(\sigma)$ are irreducible (if nonzero).
- (P1) If σ has parameter (ϕ, η) and $\Theta_{\psi, \chi, V_n, W_n}(\sigma)$ has parameter (ϕ', η') , then (ϕ', η') can be precisely described in terms of (ϕ, η) .
- (P2) Likewise, if π has parameter (ϕ, η) and $\Theta_{\psi, \chi, V_{n+1}, W_n}(\pi)$ has parameter (ϕ', η') , then (ϕ', η') can be precisely described in terms of (ϕ, η) .

In fact, in [15], [16], D. Prasad has formulated precise conjectures regarding (P1) and (P2) for the theta correspondence for $\mathrm{U}(V_n) \times \mathrm{U}(W_n)$ and $\mathrm{U}(V_{n+1}) \times \mathrm{U}(W_n)$ respectively. We shall also denote by (weak P1) the part of the conjecture (P1) concerning only the correspondence of L -parameters $\phi \mapsto \phi'$; likewise we have (weak P2). Then in our earlier paper [8], we have shown:

Proposition 4.1. *The statements (Θ), (weak P1) and (weak P2) hold.*

Given Proposition 4.1, the main work is to determine how η' depends on (ϕ, η) in (P1) and (P2). In fact, the precise determination of η' in (P1) is a very subtle issue, as it depends on certain ϵ -factors. In the case of (P2), the dependence of η' on (ϕ, η) is more simplistic.

More explicitly, choose appropriate conjugate self-dual characters χ_V and χ_W of $E^\times$ to fix the Weil representation and appropriate Whittaker data (depending on ψ and δ) to fix the local Langlands correspondence. Then the part of the conjectures (P1) and (P2) concerning the correspondence of $(\phi, \eta) \mapsto \eta'$ states the following.

(P1) Let ϕ be an L -parameter for $U(V_n)$ and let $\eta \in \text{Irr}(S_\phi)$. Suppose that

$$S_\phi = \prod_i (\mathbb{Z}/2\mathbb{Z})a_i.$$

Then there is a natural bijection $S_\phi \rightarrow S_{\phi'}$ and

$$\eta'(a_i)/\eta(a_i) = \epsilon(\frac{1}{2}, \phi_i \otimes \chi_W^{-1}, \psi_2^E),$$

where

$$\psi_2^E(x) = \psi(\text{Tr}_{E/F}(\delta x)).$$

(P2) Let ϕ be an L -parameter for $U(W_n)$ and let $\eta \in \text{Irr}(S_\phi)$. Then there is a natural injection $S_\phi \rightarrow S_{\phi'}$ and η' is the unique irreducible character of $S_{\phi'}$ such that

$$\eta'|_{S_\phi} = \eta.$$

Our first observation is:

Proposition 4.2. *Assume (B) and (P2). Then (FJ) and (P1) follow.*

In view of Proposition 4.2 and the work of Beuzart-Plessis [3], [4], [5], it remains to show the statement (P2), and our main result is:

Theorem 4.3. *The conjecture (P2), and hence (FJ) and (P1), holds.*

The proof of (P2) proceeds by the following steps:

- First, by our results in [8], the non-tempered case can be reduced to the tempered case on smaller unitary groups.
- Next, we show that the tempered case can be reduced to the square-integrable case on smaller unitary groups. This is achieved by a nontrivial extension of the techniques in [12].
- Finally, we show the square-integrable case by a global argument. More precisely, we shall globalize an irreducible square-integrable representation π of $U(W_n)$ to an irreducible cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ such that
 - Π_v is not square-integrable for all places outside the place of interest, so that (P2) is known for Π_v outside the place of interest;
 - Π has tempered A -parameter whose global component group is equal to the local component group of the L -parameter of π ;
 - Π has nonzero global theta lift to a unitary group which globalizes $U(V_{n+1})$.

The desired result then follows for the place of interest by applying Arthur's multiplicity formula for the automorphic discrete spectrum, which can be viewed as a sort of product formula.

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