HARMONIC ANALYSIS ON COMPLEX SYMMETRIC SPACES

Simon Gindikin

Abstract. We discuss the harmonic analysis on complex semisimple Lie groups and, more general, on complex symmetric spaces from point of view of multidimensional complex analysis.

Harmonic analysis on complex semisimple Lie groups was constructed by E.Cartan and H.Weyl using 2 approaches: algebraic and analytic(transcendental). They permanently compared these methods. The algebraic way of E.Cartan uses Lie algebras. The transcendental method of H.Weyl is non direct: he had no appropriate analytic tools on complex groups G. Instead he works with their compact forms U and develops on them a version of the Peter-Weyl construction using the invariant integration. The crucial point is the famous unitary trick of H.Weyl. It turns out that irreducible representations of a group G and its maximal compact subgroup U coincide. The analytic nature of this a phenomenon was not clarified for a log time. A possible explanation is that U is the Riemannian manifold and invariant differential operators on U are elliptic. Irreducible representations are realized in joint eigenspaces of them. It turns out that the eigenfunctions not only analytic on U but extend holomorphically on the whole complex groups G.

Since the groups G are Stein manifolds and all spherical functions are holomorphic on G, it looks natural to develop harmonic analysis directly on G using tools of complex analysis (instead of real analysis on U). We discuss here this possibility and find some new possibilities already in classical situation (see also [Gi06]). It is natural to start the complex analysis on G from an integral Cauchy formula, hoping to have an universal analytical tool for harmonic analysis.

1. Cauchy integral formula on complex symmetric spaces. The objects of classical harmonic analysis are complex semisimple Lie groups but it is natural to work with more general objects - *complex symmetric spaces with complex groups of automorphisms*. They have the form $Z = G/H$ where G is a simply connected complex semisimple Lie group and H is its involutive subgroup corresponding to a holomorphic involution. The case of a complex semisimple Lie group G_1 is the special case corresponding to $G = G_1 \times G_1$ and a diagonal subgroup $H \cong G_1$. Let A be a maximal Abelian subgroup transversal to H and N be a corresponding unipotent subgroup such that $G^0 = HAN$ is a Zariski open set in G (complex Iwasawa decomposition); let Z^0 be the corresponding dense part of Z. Let $n = \dim Z$, $l = \dim A$ (the rank of Z). Let M be the centralizer of A in H. Then $F = G/AMN$ is a flag manifold. We call the homogeneous manifold $\Xi = G/MN$ *the horospherical manifold*. We have $\dim \Xi = n$. There is a natural fibering $\Xi \to F$ with the fibers A and a natural double fibering $Z \leftarrow G \to \Xi$. Correspondingly points $\zeta \in \Xi$ parameterize some submanifolds in Z which are called horospheres $E(\zeta)$ (projections on Z of MN-classes in G). They have codimension l. We have a dual family of submanifolds $S(z), z \in Z$, on Ξ, also of codimension l; we call them *pseudospheres*.

Let us take characters $\delta(a)$ of A and using the Iwasawa decomposition extend them to Z^0. Let us take those that holomorphically extend to all of Z and let $\Delta_j(z|\zeta_0), 1 \leq j \leq l$, be generators of this ring (let them correspond to $\delta_j(a)$). The corresponding characters can be connected with the generators of the lattice of weights dual to non multiple restricted roots. We connect these holomorphic functions with the "initial" horosphere $E(\zeta_0)$ corresponding to the unit element of A: they are constant on horospheres parallel to $E(\zeta_0)$. Acting by G we extend them up to holomorphic functions

$$\Delta_j(z|\zeta)$$

on $Z \times \Xi$. We call these functions *Sylvester's functions*: in the case when Z is the manifold of non degenerate symmetric matrices these functions are the principal minors. In the general case Z^0 is defined by the conditions $\Delta_j(z|\zeta) \neq 0$, a version of the classical Sylvester conditions. The horosphere $E(\zeta)$ is defined by the equations $\Delta_j(z|\zeta) = 1$. Holomorphic extensions of characters on Z have the form $\Delta^\mu(z|\zeta) = \prod \Delta_j(z|\zeta)^{\mu_j}, \mu \in \mathbb{Z}_+^l$. Let $\delta^\mu(a)$ be the corresponding characters on A.

Let us consider differential operators with constant coefficients in logarithmic variables on A. They are defined by polynomial symbols: $P(D)\delta^\mu(a) = P(\mu)\delta^\mu(a)$. We transfer their action to functions $f(z)$ constant on orbits of N - parallel horospheres (denote it as $P(D_z|\zeta_0)$ and $P(D_z|\zeta)$ for other choices of unipotent subgroups).

Now we can write down the formula for *the Cauchy kernel*

$$K(w, z|\zeta) = W(D_z|\zeta)\left(\frac{1}{\prod_{1\leq j\leq l}(\Delta_j(w|\zeta) - \Delta_j(z|\zeta))}\right), \quad z, w \in Z, \zeta \in \Xi.$$

We apply here (for fixed ζ) the differential operator of order $n - l$ with the symbol

$$W(\mu) = \prod_{\alpha\in\Sigma_+} \frac{\langle \mu + \rho, \alpha\rangle}{\langle \rho, \alpha\rangle}$$

which is Weyl's polynomial for dimensions of irreducible representations, Σ_+ is a system of restricted roots and ρ is their half-sum. Its appearance is surprising since this formula is a result of purely analytic computations without any appeal to representations.

The kernel has unusual structure. If we apply a differential operator we obtain a combination of fractions whose denominators are monomials in $(\Delta_j(w|\zeta)-\Delta_j(z|\zeta))$ of degree $n - l + 1$ with relatively complicated coefficients. Their choice is defined by the condition that the integrand in the Cauchy formula must be a closed form. The representation with the differential operator is more compact and explicit and I do not think that it would be possible to obtain the applications below using an expression for the kernel as a sum. Of course for $l = 1$ there is only one factor and one term in the sum, so there is no difference in the 2 versions. It is remarkable that the operator with Weyl's symbols delivers the closed form.

We need also 2 remarkable differential forms. Let $\omega(z; dz)$ be the invariant holomorphic n-form on Z and $\lambda(z|\zeta; d\zeta)$ be the similar $(n-l)$-form on pseudospheres $S(z), z \in Z$, of Ξ which is holomorphic on z. Then *we have the Cauchy formula on* Z:

$$\int_\Gamma K(w, z|\zeta) f(z)\omega(z; dz) \wedge \lambda(z|\zeta; d\zeta) = c(\Gamma) f(w), \quad f \in \mathcal{O}(Z).$$

Here Γ is a cycle in $Z \times \Xi$ outside of the singularities of K. The integrand is the closed form and the coefficient $c(\Gamma)$ depends on the cohomology class of Γ (it has an explicit representation). It is crucial that in the Cauchy formula on complex symmetric spaces the codimension of cycles equal the rank.

2. The Cauchy-Fantappie formula for cycles of codimensions higher than 1. To obtain this Cauchy formula we need to develop some new tools in multidimensional complex analysis which are, probably, of broader interest. Leray [Le56,Le59] found the universal structure of multidimensional analogues of Cauchy formula - the Cauchy-Fantappie formula. We want to construct an analogue of the Cauchy-Fantappie formula for cycles of higher codimensions. As in Leray's case it is important to find an universal closed form convenient for specific computations. We suggest the form

$$\det\left[\xi_1, \xi_2, \ldots, \xi_l, (d\xi_1\partial/\partial p_1 + \cdots + d\xi_l\partial/\partial p_l)^{\{n-l\}}\right] \frac{1}{p_1 \cdots p_l}\bigg|_{p_j = \langle \xi_j, w-z\rangle} \cdot f(z) \wedge dz_1 \wedge \cdots \wedge dz_n.$$

Here $f(z)$ is a holomorphic function in a domain in $\mathbb{C}^n$; columns ξ_j are n-vectors, the last column repeats $n-l$ times, and we use the exterior product of 1-forms in computing the determinant (for this reason it is not zero). The determinant gives a differential operator on p of order $n-l$. After its application we make the substitution. The result is a holomorphic $(2n-l)$-form which is closed (this is not evident). For $l = 1$ we have Leray's form. In the general case the application of the operator gives a sum of negative monomials. Their coefficients can be directly computed from the condition of closed-ness (similar computations have appeared in other problems) but for our applications the special representation with a differential operator and a determinant is crucial.

Different specializations of the formula, roughly speaking, correspond to different choices of ξ_j as functions of Z on different cycles in the holomorphy domain of f avoiding the singularities (this can be very non trivial). It is interesting that we need basically all modifications of the Cauchy-Fantappie formula which were considered by Leray. For example, for the Cauchy formula in Ξ we need an analogue of Leray's construction where the function is reconstructed through a differential operator of boundary values. Also extremely important for us is an explicit form of the 2nd Cauchy-Fantappie formula which was found by Henkin and me [GH90] and its connection with the inversion of the Penrose transform which I found this year [Gi07].

3. Holomorphic horospherical duality for (Z, Ξ). The first step in the construction of the complex analysis on Z corresponds to the classical harmonic analysis (finite dimensional representations). The central result is

The spaces of holomorphic functions $\mathcal{O}(Z)$ and $\mathcal{O}(\Xi)$ are isomorphic as G-modules.

On the algebraic level this result is known but in this analytic form it is surprising, both from from point of view of Lie groups and of complex analysis: we have an equivalency of reducible non unitary representations on non isomorphic homogeneous G-spaces, which are not biholomorphically equivalent (Ξ is not Stein and has a singular Stein extension). It is crucial that it is not an abstract isomorphism but that there are remarkable explicit intertwining operators. For $f \in \mathcal{O}(Z)$ we define a explicit holomorphic horospherical transform

$$\hat{f}(\zeta) = \int_{\gamma(\zeta)} \frac{f(z)}{\prod_{1\le j\le l}(1-\Delta_j(z|\zeta))}\omega(z;dz), \zeta \in \Xi.$$

Here γ is a cycle outside of the singularities of the kernel. It turns out that it is possible to take as such cycles some compact forms of Z (this involves non trivially the geometry of Z).

The inverse operator has the form

$$\check{F}(z) = c\int_{\sigma(z)} W(D)F(\zeta)\lambda(z|\zeta;d\zeta), \quad F \in \mathcal{O}(\Xi),$$

where $\sigma(z)$ is a real form of the pseudosphere $S(Z)$.

The proof that these operators are inverse is a direct corollary of the Cauchy formula. We need to take a cycle Γ with separated variables: it must be fibered over the cycle $\sigma(w) \subset S(w) \subset \Xi$ by cycles $\gamma(\zeta)$, whose projection on Ξ has the minimal dimension. The existence of such cycles is a non trivial geometrical fact.

These inversion formulas resemble Radon's inversion formula but in the holomorphic setting they include completely new elements. The analytic duality follows the geometrical duality of horospheres and pseudospheres on Z and Ξ. It is reminiscent of Martineau's duality [Ma62, Ma66], corresponding to the classical projective duality and the Penrose transform. However, in these constructions there are no dualities between spaces of holomorphic functions (rather, spaces of cohomology and holomorphic functions). I do not know other examples of singular integral operators between holomorphic functions which appear in this project. I believe they deserve consideration in a broader context.

The classical results of harmonic analysis are simple corollaries of this duality [Gi06]. On Ξ there is an action of A by "right multiplications" commuting with "left multiplications" by G. Decomposing $\mathcal{O}(\Xi)$ into invariant subspaces relative to this action of A, we obtain irreducible G-modules (this reduces to analysis for an Abelian compact group - maximal torus in A). The dual subspaces in $\mathcal{O}(Z)$ are irreducible spherical representations. Probably, it is possible to obtain our holomorphic duality starting from irreducible representations, but I do not see a natural way which does not include too much functional analysis.

4. Duality for compact symmetric spaces. The ultimate steps of this project are constructions of dual objects on the horospherical manifold Ξ to semisimple symmetric spaces - real forms of the Stein symmetric manifold Z. Let us start from a compact form X of Z. Let $\hat{X} \subset \Xi$ be the set of all ζ such that the horosphere $E(\zeta)$ does not intersect X. A priori this set could be empty but it turns out that it is a domain which also can be characterized as the set of all ζ such that $|\Delta(z|\zeta)| < 1$ for all j and $z \in X$. It is natural to interpret the domain $\hat{X}$ as *a geometrical dual* *object for the compact symmetric manifold X.* This is supported by the next fact on analytic duality:

There is a canonical G-isomorphism of the space $Hyp(X)$ of hyperfunctions on X and the space of holomorphic functions $\mathcal{O}(\hat{X})$.

The intertwining operators realizing this isomorphism extend the holomorphic isomorphism. Hyperfunctions Φ are functionals on holomorphic functions in a neighborhood of the compact X. Their horospherical transforms $\hat{\Phi}(\zeta)$ are values on kernels of the horospherical transform as functions of z. To construct the inverse operator we need to extend a functional from these kernels to all holomorphic functions in neighborhoods of X using the Cauchy formula. So it is again a direct consequence of the Cauchy formula. Let us remark that this duality is already new for the sphere S^n. There it turns out that the sphere has a geometrical dual object which is a complex manifold [Gi04a,05].

Let us compare this duality with the unitary trick of H.Weyl. If we decompose $\mathcal{O}(\hat{X})$ relative to the right action of the compact form of A we obtain the same subspaces of holomorphic functions as for Ξ and correspondingly the same spherical irreducible representations on X. The analytic nature of this coincidence is that X is Riemannian and invariant differential operators are elliptic. Therefore all their eigenfunctions (in which irreducible representations are realized) are holomorphically extended to a complex neighborhood of X, which in this case coincides with all of Z. This is the reason why, on the level of irreducible representations, there is no difference between complex and compact groups. But in the more broad analytic picture the difference is essential and we operate instead on the set of irreducible representations with other geometrical dual objects.

5. Duality for Riemannian symmetric space of noncompact type. Let us connsider $X = G_{\mathbb{R}}/H_{\mathbb{R}}$, another real form of Z which is Riemannian of noncompact type. Here we have real forms of G, H such that $H_{\mathbb{R}}$ is maximal compact in $G_{\mathbb{R}}$. In the usual harmonic analysis on $X, L^2(X)$ is considered but we will work with holomorphic functions. The manifold X has a canonical Stein neighborhood which was defined in [AG90] and which I suggested to call *the complex crown of X*: $X \subset Crown(X) \subset Z$. We consider the $G_{\mathbb{R}}$-module $\mathcal{O}(Crown(X))$. The crucial geometrical fact is that the crown is convex relative to horospheres: the complement is the union of some horospheres $E(\zeta)$. This is a reformulation of the result [GK02]. *As a dual object we take the manifold $\hat{X}$ of all ζ such that the horospheres $E(\zeta)$ intersect the crown.* We define an analogue of the Penrose transform

$$\mathcal{P} : H^{(n-l)}(\hat{X}, \mathcal{O}) \to \mathcal{O}(Crown(X)).$$

We just integrate forms $\phi \wedge \mu(z(\zeta, d\zeta)$, where ϕ is a $\bar{\partial}$-closed $(n-l)$-form, on pseudospheres $S(z)$. Technically it is convenient to work with a holomorphic envelope of Ξ where the $S(z)$ are compactified. It is possible to construct also an explicit operator

$$\mathcal{Q} : \mathcal{O}(Crown(X)) \to H^{(n-l)}(\hat{X}, \mathcal{O})$$

such that *the operator $\mathcal{P}\mathcal{Q}$ is the unit operator. This implies that the Penrose transform $\mathcal{P}$ is surjective.*

The operator $\mathcal{P}$ has a relatively small kernel with an explicit description. The construction of the operator $\mathcal{Q}$ follows the concept of the holomorphic language for analytic cohomology [EGW95] and uses the Cauchy formula on Z in the form similar to the 2nd Cauchy-Fantappie formula (cf.[GH90]).

If we translate to irreducible representations, this duality corresponds to Helgason's conjecture which was proved in [KKMOOT]. Eigenspaces of invariant differential operators on X correspond in the dual picture to $A_{\mathbb{R}}$-invariant cohomology on $\hat{X}$ which are equivalent to cohomology on $F \setminus F_{\mathbb{R}}$ (the complement in the manifold of complex flags of the submanifold of real ones) with the coefficients in some line bundles. Then the operator $\mathcal{P}$ corresponds to the Poisson transform and $\mathcal{Q}$ to the operator of hyperfunction- boundary values. Only for a discrete set of eigenvalues does the Poisson transform have a kernel.

It was found in [GK02a] that all causal symmetric manifolds Y can be realized as an edge of the boundary of $Crown(X)$ for some X. In this way we have an interpretation of one multiplicity of maximal continuous spectrum for such Y, free of L^2-restrictions.

The final aim of this project is *the duality for pseudo Riemannian semisimple symmetric manifolds* $X = G_{\mathbb{R}}/H_{\mathbb{R}} \subset Z$. If $H_{\mathbb{R}}$ is not compact then there are some remarkable $G_{\mathbb{R}}$-invariant tubes in Z with the edge X which are, as a rule, non Stein.

The aim is to find appropriate geometrical dual objects for them in Ξ *such that an analytic duality connects some cohomology in dual manifolds.*

In a sense this is a refinement of the old program [GG77]. Some components of the final picture have already been constructed but we will not discuss them here.

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