LINEAR DIFFERENTIAL EQUATIONS ON THE RIEMANN SPHERE AND REPRESENTATIONS OF QUIVERS

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Abstract. In this note, we consider a generalization of additive Delinge-Simpson problems to non-Fuchisian differential equations.

1. Additive Deligne-Simpson problems

An additive Deligne-Simpson problem asks the existence of an irreducible Fuchsian differential equation on the Riemann sphere with prescribed local data. In this note, we shall consider analogous problems for differential equations on the Riemann sphere with unramified irregular singular points.

Firstly let us recall the classical case, i.e., the additive Deligne-Simpson problem for systems of linear Fuchsian differential equations (cf. [7]). A system of first order linear differential equations is called *Fuchsian* if it is of the form

$$\frac{d}{dx}Y=\sum_{i=1}^{p}\frac{A_i}{x-a_i}Y \quad (A_i\in M(n,\mathbb{C}),\ i=1,\dots,p).$$

Here we call each A_i the *residue matrix* at the singular point a_i. Also $A_0 = -\sum_{i=1}^{p} A_i$ is called the residue matrix at ∞. We say that $\frac{d}{dx}Y=\sum_{i=1}^{p}\frac{A_i}{x-a_i}Y$ is *irreducible* if $A_0,\dots,A_p$ have no nontrivial simultaneous invariant subspace of $\mathbb{C}^n$, i.e., if there exists $W \subsetneqq \mathbb{C}^n$ such that $A_iW\subset W$ for all $i=0,\dots,p$, then $W=\{0\}$.

Definition 1.1 (additive Deligne-Simpson problem for equations)**.** An *additive Deligne-Simpson problem for equations* consists of a collection of points $a_1,\dots,a_p$ in $\mathbb{C}$ and of conjugacy classes $C_0,C_1,\dots,C_p$ in $M(n,\mathbb{C})$. A *solution* of the additive Deligne-Simpson problem for equations is an *irreducible* Fuchsian differential equation

$$\frac{dY}{dx}=\sum_{i=1}^{p}\frac{A_i}{x-a_i}Y \quad (A_i\in M(n,\mathbb{C}),\ i=1,\dots,p)$$

whose residue matrices $A_i\in C_i$ for $i=0,\dots,p$.

We can reformulate this problem as a problem for matrices.

Definition 1.2 (additive Deligne-Simpson problem for matrices)**.** An *additive Deligne-Simpson problem for matrices* consists of a collection of conjugacy classes $C_0,C_1,\dots,C_p$ in $M(n,\mathbb{C})$. A *solution* of the additive Deligne-Simpson problem for matrices is a tuple of matrices $(A_i)_{0\le i\le p}\in\prod_{i=0}^{p}C_i$ satisfying

(1) $\sum_{i=0}^{p}A_i=0$,
(2) $(A_i)_{0\le i\le p}$ is irreducible, i.e., if there exists $W\subsetneqq\mathbb{C}^n$ such that $A_iW\subset W$ for all $i=0,\dots,p$, then $W=\{0\}$.

Obviously these two problems are equivalent to each other. Hence we simply call these two problems by the same name, additive Deligne-Simpson problem. The study of this problem is developed by V. Kostov who gives an necessary and sufficient condition of the existence of solutions under some generic conditions, see

[7]. After his study, a complete necessary and sufficient condition is given by W. Crawley-Boevey [3].

As a generalization of this problem, it seems to be natural to consider a similar problem for non-Fuchsian equations (see for example [1], [8]). Before formulating the generalized problem precisely, let us recall some facts of local theory around irregular singular points of differential equations with formal power series coefficients. Let $M(n,R)$ be the ring of $n\times n$ matrices with coefficients in a commutative ring R and $\mathrm{GL}(n,R)$ the group which consists of all invertible elements in $M(n,R)$ of the multiplication. Set $\mathbb{C}((x))=\{\sum_{i=r}^{\infty}c_ix^i\mid c_i\in\mathbb{C},\, r\in\mathbb{Z}\}$ and $\mathbb{C}[[x]]=\{\sum_{i=0}^{\infty}c_ix^i\mid c_i\in\mathbb{C}\}$. Let us attach to $C=\sum_{i=r}^{\infty}c_ix^i\in M(n,\mathbb{C}((x)))$ the integer $v(C)$ defined by $v(C)=\min\{i\mid c_i\neq 0\}$. We call this map $v\colon M(n,\mathbb{C}((x)))\to\mathbb{Z}$ the *valuation*. For $A\in M(n,\mathbb{C}((x)))$ and $X\in \mathrm{GL}(n,\mathbb{C}((x)))$ the *gauge transformation* of A by X is

$$X[A]=XAX^{-1}+\left(\frac{d}{dx}X\right)X^{-1}.$$

Definition 1.3 (Hukuhara-Turrittin-Levelt normal forms)**.** If an element $B\in M(n,\mathbb{C}((x)))$ is of the form

$$B=\mathrm{diag}(q_1(x^{-1})I_{n_1}+R_1x^{-1},\ldots,q_m(x^{-1})I_{n_m}+R_mx^{-1})$$

with $q_i(s)\in s^2\mathbb{C}[s]$ satisfying $q_i\neq q_j$ if $i\neq j$ and $R_i\in M(n_i,\mathbb{C})$, then B is called the *Hukuhara-Turrittin-Levelt normal form* or the *HTL normal form* shortly. Here I_m is the identity matrix of $M(m,\mathbb{C})$.

The theorem below is one of the most fundamental fact in the local theory of systems of linear differential equations with formal power series coefficients.

Theorem 1.4 (Hukuhara, Turrittin, Levelt, see [9] for example)**.** *Let A be an element in $M(n,\mathbb{C}((x)))$. Then there exists a field extension $\mathbb{C}((t))$ of $\mathbb{C}((x))$ with $t^q=x$, $q\in\mathbb{Z}_{>0}$ and $X\in\mathrm{GL}(n,\mathbb{C}((t)))$ such that A is reduced to the HTL normal form*

$$X[A]=\mathrm{diag}(q_1(t^{-1})I_{n_1}+R_1t^{-1},\ldots,q_m(t^{-1})I_{n_m}+R_mt^{-1}).$$

Now let us introduce truncated orbits which play the same role as the conjugacy classes C_i of residue matrices in the Deligne-Simpson problems of Fuchsian cases. Let us define $G_k=\mathrm{GL}(n,\mathbb{C}[[x]]/x^k\mathbb{C}[[x]])$, $k\geq 1$, which can be identified with

$$\left\{A_0+A_1x+\cdots+A_{k-1}x^{k-1}\;\middle|\;\begin{array}{l}A_0\in\mathrm{GL}(n,\mathbb{C}),\ A_i\in M(n,\mathbb{C}),\\ i=1,\ldots,k-1\end{array}\right\}.$$

Also define

$$\begin{aligned}\mathfrak{g}_k&=M(n,\mathbb{C}[[x]]/x^k\mathbb{C}[[x]])\\&=\left\{A_0+A_1x+\cdots+A_{k-1}x^{k-1}\;\middle|\;A_i\in M(n,\mathbb{C}),\ i=0,\ldots,k-1\right\}.\end{aligned}$$

The dual vector space $\mathfrak{g}_k^*=\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_k,\mathbb{C})$ is identified with

$$M(n,x^{-k}\mathbb{C}[[x]]/\mathbb{C}[[x]])=\left\{\frac{A_k}{x^k}+\cdots+\frac{A_1}{x}\;\middle|\;A_i\in M(n,\mathbb{C})\right\}$$

by the nondegenerate bilinear form $\mathfrak{g}_k\times\mathfrak{g}_k^*\ni(A,B)\mapsto \mathrm{pr}_{\mathrm{res}}(\mathrm{tr}(AB))\in\mathbb{C}$. Here $\mathrm{pr}_{\mathrm{res}}(\sum_{i=r}^{\infty}A_ix_i)=A_{-1}$.

Then an HTL normal form $B\in M(n,\mathbb{C}((x)))$ with $v(B)\geq -k$ can be seen as an element in $\mathfrak{g}_k^*$. Thus we can consider the G_k-orbit $\mathcal{O}_B=\left\{gBg^{-1}\in\mathfrak{g}_k^*\mid g\in G_k\right\}$ of B in $\mathfrak{g}_k^*$ called the *truncated orbit* of B. The integer k is called the *degree* of $\mathcal{O}_B$.

Now let us formulate generalization of Deligne-Simpson problems.

Definition 1.5 (generalized additive Deligne-Simpson problem for equations)**.** A *generalized additive Deligne-Simpson problem for equations* consists of a collection of points $a_1, \ldots, a_p$ in $\mathbb{C}$, of nonzero positive integers $k_0, \ldots, k_p$ and of HTL normal forms $B^{(i)} \in \mathfrak{g}^*_{k_i} \subset M(n, \mathbb{C}((x)))$ for $i = 0, \ldots, p$. A *solution* of the generalized additive Deligne-Simpson problem for equations is an *irreducible* differential equation

$$\frac{d}{dx}Y = \left(\sum_{i=1}^{p}\sum_{j=1}^{k_i}\frac{A_{i,j}}{(x-a_i)^j} + \sum_{2\leq j\leq k_0} A_{0,j}x^{j-2}\right)Y$$

satisfying that $A^{(i)}(x) \in \mathcal{O}_{B^{(i)}}$ for $i = 0, \ldots, p$. Here $A^{(i)}(x) = \sum_{j=1}^{k_i} A_{i,j}x^{-j}$ for $i = 0, \ldots, p$ and $A_{0,1} = -\sum_{i=1}^{p} A_{i,1}$. We say

$$\frac{d}{dx}Y = \left(\sum_{i=1}^{p}\sum_{j=1}^{k_i}\frac{A_{i,j}}{(x-a_i)^j} + \sum_{2\leq j\leq k_0} A_{0,j}x^{j-2}\right)Y$$

is *irreducible* if the collection $(A_{i,j})_{\substack{0\leq i\leq p\\ 1\leq j\leq k_i}}$ of coefficient matrices is irreducible.

This can be seen as a natural generalization of additive Deligne-Simpson problems which contains the original problems for Fuchsian equations as the special case $k_0 = \cdots = k_p = 1$. As we see in the Fuchsian case, we can reformulate the problem as follows.

Definition 1.6 (generalized additive Deligne-Simpson problem for matrices)**.** A *generalized additive Deligne-Simpson problem for matrices* consists of a collection of nonzero positive integers $k_0, k_1, \ldots, k_p$ and of HTL normal forms $B^{(i)} \in \mathfrak{g}^*_{k_i}$, $i = 0, \ldots, p$. A *solution* of the generalized additive Deligne-Simpson problem for matrices is an irreducible element in

$$\left\{ \left(\sum_{j=1}^{k_i} A_j^{(i)} x^{-j}\right)_{0\leq i\leq p} \in \prod_{i=0}^{p}\mathcal{O}_{B^{(i)}} \;\middle|\; \sum_{i=0}^{p} A_1^{(i)} = 0 \right\}.$$

Here $\left(\sum_{j=1}^{k_i} A_j^{(i)}x^{-j}\right)_{0\leq i\leq p}$ is said to be *irreducible* if the collection of matrices $(A_j^{(i)})_{\substack{0\leq i\leq p\\ 1\leq j\leq k_i}}$ is irreducible.

In the case $k_1 = \cdots = k_p = 1$ and $k_0 \leq 3$, P. Boalch obtains a necessary and sufficient condition for the existence of solutions of generalized additive Deligne-Simpson problems in [1] and for an arbitrary k_0, see [5].

2. A review of Crawley-Boevey's theorem of representations of quivers

The study of additive Deligne-Simpson problem for Fuchsian equations is developed by V. Kostov. After Kostov's study, W. Crawley-Boevey gave the complete answer of the additive Deligne-Simpson problem for Fuchsian equations by using his theory of representations of deformed preprojective algebras. Let us give a quick review of the statement of one of Crawley-Boevey's theorems for the representation theory of deformed preprojective algebras (see [2] for the detail).

Definition 2.1 (quivers)**.** A *quiver* $\mathsf{Q} = (\mathsf{Q}_0, \mathsf{Q}_1, s, t)$ is the quadruple consisting of Q_0, the set of *vertices*, and Q_1, the set of *arrows* connecting vertices in Q_0, and two maps $s, t : \mathsf{Q}_1 \to \mathsf{Q}_0$, which associate to each arrow $\rho \in \mathsf{Q}_1$ its *source* $s(\rho) \in \mathsf{Q}_0$ and its *target* $t(\rho) \in \mathsf{Q}_0$ respectively.

Definition 2.2 (representations of quivers)**.** Let Q be a finite quiver, i.e., Q_0 and Q_1 are finite sets. A *representation* M of Q is defined by the following data:

(1) To each vertex a in Q_0, a finite dimensional $\mathbb{C}$- vector space $M_a \cong \mathbb{C}^{m_a}$ is attached.
(2) To each arrow $\rho\colon a \to b$ in Q_1, a $\mathbb{C}$-linear map $\psi_\rho\colon M_a \to M_b$ is attached.

We denote the representation by $M = (M_a, \psi_\alpha)_{a\in\mathsf{Q}_0,\alpha\in\mathsf{Q}_1}$. The collection of integers defined by $\mathbf{dim}\, M = (\dim_{\mathbb{C}} M_a)_{a\in\mathsf{Q}_0}$ is called the *dimension vector* of M.

To each vector $\alpha \in \mathbb{Z}^{\mathsf{Q}_0}$, we attach the integers

$$q(\alpha) = \sum_{a\in\mathsf{Q}_0} \alpha_a^2 - \sum_{\rho\in\mathsf{Q}_1} \alpha_{s(\rho)}\alpha_{t(\rho)},$$
$$p(\alpha) = 1 - q(\alpha).$$

For a fixed vector $\alpha \in (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$, define

$$\mathrm{Rep}\,(\mathsf{Q}, \alpha) = \bigoplus_{\rho\in\mathsf{Q}_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(\rho)}}, \mathbb{C}^{\alpha_{t(\rho)}}).$$

More generally define

$$\mathrm{Rep}(\mathsf{Q}, V, \alpha) = \bigoplus_{\rho\in\mathsf{Q}_1} \mathrm{Hom}_{\mathbb{C}}(V_{s(\rho)}, V_{t(\rho)}),$$

where $V = (V_a)_{a\in\mathsf{Q}_0}$ is a collection of finite dimensional $\mathbb{C}$-vector spaces with $\dim_{\mathbb{C}} V_a = \alpha_a$. To each element $(\psi_\rho)_{\rho\in\mathsf{Q}_1} \in \mathrm{Rep}(\mathsf{Q}, V, \alpha)$, the representation $(V_a, \psi_\rho)_{a\in\mathsf{Q}_0,\rho\in\mathsf{Q}_1}$ associates. Thus we identify $(\psi_\rho)_{\rho\in\mathsf{Q}_1}$ with $(V_a, \psi_\rho)_{a\in\mathsf{Q}_0,\rho\in\mathsf{Q}_1}$ and vice versa.

The representation space $\mathrm{Rep}(\mathsf{Q}, V, \alpha)$ has an action of $\prod_{a\in\mathsf{Q}_0} \mathrm{GL}(V_a)$ as below. For $(\psi_\rho)_{\rho\in\mathsf{Q}_1} \in \mathrm{Rep}\,(\mathsf{Q}, V, \alpha)$ and $g = (g_a) \in \prod_{a\in\mathsf{Q}_0} \mathrm{GL}(V_a)$, then $g \cdot (\psi_\rho)_{\rho\in\mathsf{Q}_1} \in \mathrm{Rep}\,(\mathsf{Q}, V, \alpha)$ consists of $\psi'_\rho = g_{t(\rho)}\psi_\rho g_{s(\rho)}^{-1} \in \mathrm{Hom}_{\mathbb{C}}(V_{s(\rho)}, V_{t(\rho)})$.

Let $M = (M_a, \psi_\rho^M)_{a\in\mathsf{Q}_0,\rho\in\mathsf{Q}_1}$ and $N = (N_a, \psi_\rho^N)_{a\in\mathsf{Q}_0,\rho\in\mathsf{Q}_1}$ be representations of a quiver Q. Then N is called the *subrepresentation* of M if we have the following:

(1) There exists a direct sum decomposition $M_a = N_a \oplus N'_a$ for each $a \in \mathsf{Q}_0$.
(2) For each $\rho\colon a \to b \in \mathsf{Q}_1$, the equality $\psi_\rho^M|_{N_a} = \psi_\rho^N$ holds.

In this case we denote $N \subset M$. Moreover if

3. for each $\rho\colon a \to b \in \mathsf{Q}_1$, we have $\psi_\rho^M|_{N'_a} \subset N'_b$,

then we say M has a *direct sum decomposition* $M = N \oplus N'$ where N' is the representation $(N'_a, \psi_\rho^M|_{N'_a})_{a\in\mathsf{Q}_0,\rho\in\mathsf{Q}_1}$.

The representation M is said to be *irreducible* if M has no subrepresentations other than M and $\{0\}$. Here $\{0\}$ is the representation of Q which consists of zero vector spaces and zero linear maps. On the other hand if any direct sum decomposition $M = N \oplus N'$ satisfies either $N = \{0\}$ or $N' = \{0\}$, then M is said to be *indecomposable*.

Let us recall the double of a quiver Q.

Definition 2.3 (double of a quiver)**.** Let $\mathsf{Q} = (\mathsf{Q}_0, \mathsf{Q}_1)$ be a finite quiver. Then the *double quiver* $\overline{\mathsf{Q}}$ of Q is the quiver obtained by adjoining the reverse arrow $\rho^*\colon b \to a$ of each arrow $\rho\colon a \to b$. Namely $\overline{\mathsf{Q}} = (\overline{\mathsf{Q}}_0 = \mathsf{Q}_0, \overline{\mathsf{Q}}_1 = \mathsf{Q}_1 \cup \mathsf{Q}_1^*)$ where $\mathsf{Q}_1^* = \{\rho^*\colon t(\rho) \to s(\rho) \mid \rho \in \mathsf{Q}_1\}$.

The *moment map* $\mu_\alpha\colon \mathrm{Rep}(\overline{\mathsf{Q}}, \alpha) \to \prod_{a\in\mathsf{Q}_0} M(\alpha_a, \mathbb{C})$ is the map whose images $(\mu_\alpha(x)_a)_{a\in\mathsf{Q}_0}$ are defined by

$$\mu_\alpha(x)_a = \sum_{\substack{\rho\in\mathsf{Q}_1\\ t(\rho)=a}} \psi_\rho\psi_{\rho^*} - \sum_{\substack{\rho\in\mathsf{Q}_1\\ s(\rho)=a}} \psi_{\rho^*}\psi_\rho,$$

for $x = (x_a, \psi_\rho)_{a\in\mathsf{Q}_0,\rho\in\overline{\mathsf{Q}}_1} \in \mathrm{Rep}\,(\overline{\mathsf{Q}}, \alpha)$.

Let Q be a finite quiver and $\overline{\mathsf{Q}}$ the double of Q. Let us fix a vector $\alpha \in (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$ and a collection of complex numbers $\lambda = (\lambda_a) \in \mathbb{C}^{\mathsf{Q}_0}$. Then define the subspace of $\mathrm{Rep}\,(\overline{\mathsf{Q}}, \alpha)$ by

$$\mathrm{Rep}\,(\overline{\mathsf{Q}}, \alpha)_\lambda = \{x \in \mathrm{Rep}\,(\overline{\mathsf{Q}}, \alpha) \mid \mu_\alpha(x)_a = \lambda_a I_{\alpha_a} \text{ for all } a \in \mathsf{Q}_0\}.$$

In [2] Crawley-Boevey studies irreducible representations in $\mathrm{Rep}\,(\overline{\mathsf{Q}}, \alpha)_\lambda$ and obtains a necessary and sufficient condition for the existence of irreducible representations. In order to give the statement of the theorem, let us recall the definition of the root system of a quiver Q (cf. [6]).

Let Q be a finite quiver. The *Euler form* is

$$\langle \alpha, \beta \rangle = \sum_{a \in \mathsf{Q}_0} \alpha_a \beta_a - \sum_{\rho \in \mathsf{Q}_1} \alpha_{s(\rho)} \beta_{t(\rho)}$$

for $\alpha, \beta \in \mathbb{Z}^{\mathsf{Q}_0}$ and the *symmetric bilinear form* is

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

Each element $\epsilon_a \in \mathbb{Z}^{\mathsf{Q}_0}$ $(a \in \mathsf{Q}_0)$ is called the *fundamental root* if $(\epsilon_a)_a = 1$, $(\epsilon_a)_b = 0$, $(b \in \mathsf{Q}_0 \backslash \{a\})$ and moreover there is no loop at the vertex a. Denote by Π the set of fundamental roots. For a fundamental root ϵ_a define the *fundamental reflection* s_a by

$$s_a(\alpha) = \alpha - (\alpha, \epsilon_a)\epsilon_a \text{ for } \alpha \in \mathbb{Z}^{\mathsf{Q}_0}.$$

The group $W \subset \mathrm{Aut}\,\mathbb{Z}^{\mathsf{Q}_0}$ generated by all fundamental reflections is called *Weyl group* of the quiver Q. Note that the bilinear form $(\ ,\ )$ is W-invariant. Similarly we can define the reflection $r_a \colon \mathbb{C}^{\mathsf{Q}_0} \to \mathbb{C}^{\mathsf{Q}_0}$ by

$$r_a(\lambda)_b = \lambda_b - (\epsilon_a, \epsilon_b)\lambda_a$$

for $\lambda \in \mathbb{C}^{\mathsf{Q}_0}$ and $a, b \in \mathsf{Q}_0$. Define the set of *real roots* by

$$\Delta^{\mathrm{re}} = \bigcup_{w \in W} w(\Pi).$$

For an element $\alpha = (\alpha_a)_{a \in \mathsf{Q}_0} \in \mathbb{Z}^{\mathsf{Q}_0}$ the *support* of α is the set of ϵ_a such that $\alpha_a \neq 0$, and denoted by $\mathrm{supp}\,(\alpha)$. We say the support of α is *connected* if the subquiver consisting of the set of vertices a satisfying $\epsilon_a \in \mathrm{supp}\,(\alpha)$ and all arrows joining these vertices, is connected. Define the *fundamental set* $F \subset \mathbb{Z}^{\mathsf{Q}_0}$ by

$$F = \left\{\alpha \in (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0} \backslash \{0\} \mid (\alpha, \epsilon) \leq 0 \text{ for all } \epsilon \in \Pi, \text{ support of } \alpha \text{ is connected}\right\}.$$

Then define the set of *imaginary roots* by

$$\Delta^{\mathrm{im}} = \bigcup_{w \in W} w(F \cup -F).$$

Then the *root system* is defined by

$$\Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}.$$

An element $\alpha \in \Delta \cap (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$ is called *positive root* and we denote by Δ^+ the set of positive roots.

Now we are ready to see Crawley-Boevey's theorem. For a fixed $\lambda = (\lambda_a) \in \mathbb{C}^{\mathsf{Q}_0}$, let us consider the set Σ_λ of positive roots α satisfying the following;

(1) $\lambda \cdot \alpha = \sum_{a \in \mathsf{Q}_0} \lambda_a \alpha_a = 0$,
(2) if there exists a decomposition $\alpha = \beta_1 + \beta_2 + \cdots$, with $\beta_i \in \Delta^+$ and $\lambda \cdot \beta_i = 0$, then $p(\alpha) > p(\beta_1) + p(\beta_2) + \cdots$.

Theorem 2.4 (Crawley-Boevey. Theorem 1.2 in [2])**.** *Let Q be a finite quiver and $\overline{\mathsf{Q}}$ the double of Q. Let us fix a dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$ and $\lambda = (\lambda_a) \in \mathbb{C}^{\mathsf{Q}_0}$. Then there exists an irreducible representation in* $\mathrm{Rep}\,(\overline{\mathsf{Q}}, \alpha)_\lambda$ *if and only if* $\alpha \in \Sigma_\lambda$.

3. Additive Deligne-Simpson problems for Fuchsian equations

In [3] Crawley-Boevey gives a correspondence between irreducible Fuchsian equations with prescribed conjugacy classes of residue matrices and irreducible representations in $\mathrm{Rep}(\overline{\mathsf{Q}},\alpha)_\lambda$ with suitable Q, α and λ. Then Theorem 2.4 can be applied to determine the existence of solutions of additive Deligne-Simpson problems. Let us give a review of this result.

Let C be a conjugacy class of $M(n,\mathbb{C})$. Then there exist complex numbers $\xi_1,\ldots,\xi_d$ such that

$$\prod_{i=1}^{d}(A-\xi_i I_n)=0 \tag{1}$$

for all $A\in C$. The minimal polynomial of C is an example of the above equation. Moreover note that $m_i=\mathrm{rank}\prod_{j=1}^{i}(A-\xi_j I_n)$, $i=1,\ldots,d$ are independent of the choice of $A\in C$. Conversely, if $B\in M(n,\mathbb{C})$ satisfies $\mathrm{rank}\prod_{j=1}^{i}(B-\xi_j I_n)=m_i$ for all $i=1,\ldots,d$, then $B\in C$. This observation leads us to the following correspondence between the elements in C and some representations of a quiver.

Proposition 3.1 (Crawley-Boevey [3], cf. Lemma 4.5 in [5])**.** *Let us fix a conjugacy class C of $M(n,\mathbb{C})$ and choose $\xi_1,\ldots,\xi_d\in\mathbb{C}$ so that the equation* (1) *holds for all $A\in C$. Put $m_k=\mathrm{rank}\prod_{i=1}^{k}(A-\xi_i I_n)$ for $k=1,\ldots,d-1$ and $A\in C$, also put $m_0=n$. Define a quiver Q as below.*

$$\underset{0}{\bigcirc}\xleftarrow{\ \rho_1\ }\underset{1}{\bigcirc}\xleftarrow{\ \rho_2\ }\quad\cdots\cdots\quad\xleftarrow{\ \rho_{d-1}\ }\underset{d-1}{\bigcirc}$$

Put $\mathbf{m}=(m_i)_{i\in\mathsf{Q}_0}\in(\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$ and define

$$Z=\left\{M=(M_a,\psi_\rho)\in\mathrm{Rep}\,(\overline{\mathsf{Q}},\mathbf{m})\;\middle|\;\begin{array}{l}\mu_{\mathbf{m}}(M)_i=(\xi_i-\xi_{i+1})I_{m_i}\ \textit{for all}\ i=1,\ldots,d-1,\\ \psi_\rho:\ \textit{injective},\ \psi_{\rho^*}:\ \textit{surjective for all}\ \rho\in\mathsf{Q}_1,\rho^*\in\mathsf{Q}_1^*\end{array}\right\}.$$

Then

$$\Phi_\xi\colon\{A\in C\}\longrightarrow Z/\prod_{i=1}^{d-1}\mathrm{GL}(m_i,\mathbb{C})$$

defined as below is a bijection. Let us define $\Phi_\xi(A)=(M_a,\psi_\rho)$ as follows:

$$M_0=\mathbb{C}^n,\qquad M_k=\mathrm{Im}\prod_{i=1}^{k}(A-\xi_i I_n)\ \textit{for all}\ k=1,\ldots,d-1,$$
$$\psi_{\rho_i}:M_{i+1}\hookrightarrow M_i\ :\ \textit{inclusion},\qquad \psi_{\rho_i^*}=(A-\xi_i)|_{M_{i-1}}.$$

The inverse map is given by $(M_a,\psi_\rho)_{a\in\mathsf{Q}_0,\rho\in\overline{\mathsf{Q}}_1}\mapsto\psi_{\rho_1}\psi_{\rho_1^*}+\xi_1$.

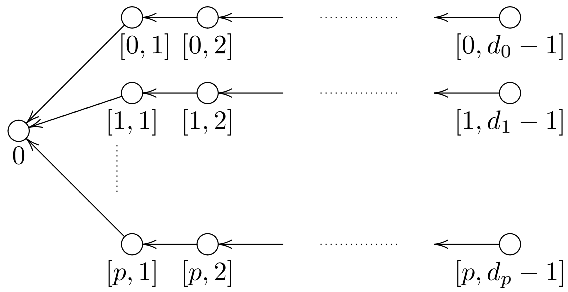
Furthermore for any $M=(M_a,\psi_\rho)_{a\in\mathsf{Q}_0,\rho\in\overline{\mathsf{Q}}_1}\in Z$ and any subspace $S\subset\mathbb{C}^n$ invariant under $\psi_{\rho_1}\psi_{\rho_1^*}+\xi_1$, there exists a subrepresentation N of M such that $N=0$ (resp. $N=M$) if and only if $S=0$ (resp. $S=\mathbb{C}^n$).

By using this correspondence, we have the following.

Theorem 3.2 (Crawley-Boevey [3])**.** *Let $C_0,\ldots,C_p$ be conjugacy classes of $M(n,\mathbb{C})$. For $i=0,\ldots,p$, choose $\xi_{[i,1]},\ldots,\xi_{[i,d_i]}\in\mathbb{C}$ so that*

$$\prod_{j=1}^{d_i}(A^{(i)}-\xi_{[i,j]}I_n)=0$$

for all $A^{(i)} \in C_i$. *Let* $\xi = (\{\xi_{[i,1]}, \ldots, \xi_{[i,d_i]}\})_{0\le i\le p}$ *be the collection of ordered sets* $\{\xi_{[i,1]}, \ldots, \xi_{[i,d_i]}\}$. *Set* $m_0 = n$ *and* $m_{[i,j]} = \operatorname{rank} \prod_{k=1}^{j}(A^{(i)} - \xi_{[i,k]}I_n)$ *for* $j = 1, \ldots, d_i - 1$. *Consider the following quiver* Q.



Define $\alpha = (\alpha_a)_{a\in\mathsf{Q}_0} \in (\mathbb{Z}_{\ge 0})^{\mathsf{Q}_0}$ *by* $\alpha_0 = m_0$ *and* $\alpha_{[i,j]} = m_{[i,j]}$ *for* $i = 0, \ldots, p$, $j = 1, \ldots, d_i - 1$. *Also define* $\lambda = (\lambda_a)_{a\in\mathsf{Q}_0} \in \mathbb{C}^{\mathsf{Q}_0}$ *by* $\lambda_0 = -\sum_{i=0}^{p}\xi_{[i,1]}$ *and* $\lambda_{[i,j]} = \xi_{[i,j]} - \xi_{[i,j+1]}$ *for* $i = 0, \ldots, p$, $j = 1, \ldots, d_i - 1$.

Then there exists a bijection

$$\Phi_\xi \colon \left\{ (A^{(i)})_{0\le i\le p} \in \prod_{i=0}^{p} C_i \;\middle|\; \begin{array}{c} \sum_{i=0}^{p} A^{(i)} = 0, \\ (A^{(i)})_{0\le i\le p} \textit{ is irreducible} \end{array} \right\} / \mathrm{GL}(n,\mathbb{C})$$
$$\to \left\{ M \in \mathrm{Rep}\,(\overline{\mathsf{Q}}, \alpha)_\lambda \mid M \textit{ is irreducible} \right\} / \prod_{a\in\mathsf{Q}_0} \mathrm{GL}(\alpha_a, \mathbb{C}).$$

Then we can apply Theorem 2.4 to additive Deligne-Simpson problems.

Theorem 3.3 (Crawley-Boevey [3])**.** *Let* $C_0, \ldots, C_p$ *be conjugacy classes of* $M(n, \mathbb{C})$. *Let us choose the quiver* Q, $\alpha \in (\mathbb{Z}_{\ge 0})^{\mathsf{Q}_0}$ *and* $\lambda \in \mathbb{C}^{\mathsf{Q}_0}$ *as Theorem 3.2. Then the additive Deligne-Simpson problem for* $C_0, \ldots, C_p$ *has a solution if and only if* $\alpha \in \Sigma_\lambda$.

4. Differential equations with poles of order 2 and representations of quivers

Now let us discuss generalized additive Deligne-Simpson problems for non-Fuchsian equations. Before seeing the general cases which look apparently complicated, we consider the case $k_0 = \cdots = k_p = 2$ in Definition 1.6 as a preliminary example.

Let $B \in \mathfrak{g}_2^*$ be an HTL normal form,

$$B = \operatorname{diag}\left(\alpha_1 I_{n_1} x^{-2} + R_1 x^{-1}, \ldots, \alpha_m I_{n_m} x^{-2} + R_m x^{-1}\right).$$

Here $R_i \in M(n_i, \mathbb{C})$ and $\alpha_i \in \mathbb{C}$, $i = 0, \ldots, p$ satisfying $\alpha_i \ne \alpha_j$ if $i \ne j$.

Let us put $B_{\mathrm{irr}} = \operatorname{diag}(\alpha_1 I_{n_1}, \ldots, \alpha_m I_{n_m})$ and denote by $V_i \subset \mathbb{C}^n$ the eigenspace of B_{irr} for each eigenvalue α_i, $i = 1, \ldots, m$. For any $X \in M(n, \mathbb{C})$, $X_{i,j}$ denotes the $\mathrm{Hom}_{\mathbb{C}}(V_j, V_i)$-component of X with respect to the decomposition $M(n, \mathbb{C}) = \bigoplus_{1\le i,j\le m} \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)$.

The lemma below is a direct consequence of the splitting lemma.

Lemma 4.1. *Let* $B \in \mathfrak{g}_2^*$ *be the HTL normal form as above. Then* $\mathcal{O}_B$ *consists of* $A(x) = \sum_{i=1}^{2} A_i x^{-i} \in \mathfrak{g}_2^*$ *satisfying the following:*

1. *There exists* $G \in \mathrm{GL}(n, \mathbb{C})$ *such that* $G^{-1}A_2G = B_{irr}$.
2. *For any* G *as above,* $(G^{-1}A_1G)_{i,i} \in C_{R_i}$ *holds for all* $i = 1, \ldots, m$. *Here* C_{R_i} *are conjugacy classes of* R_i *for* $i = 1, \ldots, m$.

Moreover if $G_1, G_2 \in \mathrm{GL}(n, \mathbb{C})$ *satisfy* $G_i^{-1}A_2G_i = B_{irr}$, $i = 1, 2$, *then* $G_2^{-1}G_1 = \operatorname{diag}(h_1, \ldots, h_m)$ *where* $h_i \in \mathrm{GL}(n_i, \mathbb{C})$ *for* $i = 1, \ldots, m$.

Next we consider a number of truncated orbits simultaneously and relate them to representations of a quiver. Let $B^{(0)},\ldots,B^{(p)}\in\mathfrak{g}_2^*$ be HTL normal forms written by

$$B^{(i)}=\mathrm{diag}\left(\alpha_1^{(i)}I_{n_1^{(i)}}x^{-2}+R_1^{(i)}x^{-1},\ldots,\alpha_{m^{(i)}}^{(i)}I_{n^{(i)}_{m^{(i)}}}x^{-2}+R_{m^{(i)}}^{(i)}x^{-1}\right).$$

Let $V_{[i,j]}\subset\mathbb{C}^n$ be the eigenspace of $B^{(i)}_{\mathrm{irr}}$ for each eigenvalue $\alpha_j^{(i)}$, $i=0,\ldots,p$, $j=1,\ldots,m^{(i)}$.

Let $X_{(i,j),(i'j')}$ be the $\mathrm{Hom}_{\mathbb{C}}(V_{[i',j']},V_{[i,j]})$-component of $X\in M(n,\mathbb{C})$ where we decompose $M(n,\mathbb{C})=\bigoplus_{1\le j\le m^{(i)}}\bigoplus_{1\le j'\le m^{(i')}}\mathrm{Hom}_{\mathbb{C}}(V_{[i',j']},V_{[i,j]})$. We may write $X=\left(X_{(i,j),(i'j')}\right)_{\substack{1\le j\le m^{(i)}\\ 1\le j'\le m^{(i')}}}$.

Let us consider the quiver Q defined as follows. The set of vertices is

$$\mathsf{Q}_0=\left\{[i,j]\mid i=0,\ldots,p,\ j=1,\ldots,m^{(i)}\right\}.$$

The set of arrows is

$$\mathsf{Q}_1=\left\{\rho^{[0,j]}_{[i,j']}\colon[0,j]\to[i,j']\,\middle|\,j=1,\ldots,m^{(0)},\ i=1,\ldots,p,\ j'=1,\ldots,m^{(i)}\right\}.$$

Fix the vector $\alpha=(\alpha_a)_{a\in\mathsf{Q}_0}\in\mathbb{Z}^{\mathsf{Q}_0}$ defined by $\alpha_{[i,j]}=\dim_{\mathbb{C}}V_{[i,j]}$, $i=0,\ldots,p$, $j=1,\ldots,m^{(i)}$.

Proposition 4.2. *We use the same notation as above. Then there exists a bijection*

$$\Phi\colon\left\{\left(\sum_{j=1}^{2}A_j^{(i)}x^{-j}\right)_{0\le i\le p}\in\prod_{i=0}^{p}\mathcal{O}_{B^{(i)}}\,\middle|\,\sum_{i=0}^{p}A_1^{(i)}=0\right\}/\mathrm{GL}(n,\mathbb{C})\longrightarrow$$

$$\left\{\begin{array}{l|l}M=(M_a,\psi_\rho) & \det\left(\psi_{\rho^{[0,j]}_{[i,j']}}\right)_{\substack{1\le j\le m^{(0)}\\ 1\le j'\le m^{(i)}}}\neq0\\ \quad\in\mathrm{Rep}\,(\overline{\mathsf{Q}},\alpha) & \text{for all } i=1,\ldots,p,\\ & \mu_\alpha(M)_{[i,j]}\in C_{R_j^{(i)}},\ [i,j]\in\mathsf{Q}_0\end{array}\right\}/\prod_{a\in\mathsf{Q}_0}\mathrm{GL}(\alpha_a,\mathbb{C}).$$

Here $C_{R_j^{(i)}}$ is the conjugacy class of each $R_j^{(i)}$, $i=0,\ldots,p$, $j=1,\ldots,m^{(i)}$.

Let us explain the construction of Φ in the above proposition. Take an element $\mathbf{A}=\left(\sum_{i=1}^2A_i^{(0)}x^{-i},\ldots,\sum_{i=1}^2A_i^{(p)}x^{-i}\right)\in\mathcal{O}_{B^{(0)}}\times\cdots\times\mathcal{O}_{B^{(p)}}$ with $\sum_{i=0}^pA_1^{(i)}=0$ and choose $G_j\in GL(n,\mathbb{C})$ for each $\sum_{i=1}^2A_i^{(j)}x^{-i}$, $j=0,\ldots,p$ as in Lemma 4.1. Under the conjugation by $GL(n,\mathbb{C})$, we may suppose $G_0=I_n$. Then let us define $\Phi(\mathbf{A})=(M_a,\psi_\rho)_{a\in Q_0,\rho\in Q_1\cup Q_1^*}$ as follows,

(1) $M_{[i,j]}:=V_j^{(i)}$, $i=0,\ldots,p$, $j=1,\ldots,m_i$,
(2) $\psi_{\rho^{[0,j]}_{[i,j']}}:=(G^{(i)})_{(i,j'),(0,j)}$ and $\psi_{\left(\rho^{[0,j]}_{[i,j']}\right)^*}:=\left(A_1^{(i)}(G^{(i)})^{-1}\right)_{(0,j),(i,j')}$, $j=1,\ldots,m_0$, $i=1,\ldots,p$, $j'=1,\ldots,m_i$.

For example let us consider

$$B^{(0)}=\mathrm{diag}(\alpha_1I_{m_1}x^{-2}+B_1^{(0)}x^{-1},\alpha_2I_{m_2}x^{-2}+B_2^{(0)}x^{-2}),$$
$$B^{(1)}=\mathrm{diag}(\beta_1I_{n_1}x^{-2}+B_1^{(1)}x^{-1},\beta_2I_{n_2}x^{-2}+B_2^{(1)}x^{-2}),$$

and take $\mathbf{A}=\left(\sum_{i=1}^2A_i^{(0)}x^{-i},\sum_{i=1}^2A_i^{(1)}x^{-i}\right)\in\mathcal{O}_{B^{(0)}}\times\mathcal{O}_{B^{(1)}}$ satisfying $A_1^{(0)}+A_1^{(1)}=0$. Then the conjugation by $GL(n,\mathbb{C})$ allows to assume $A_2^{(0)}=B_{\mathrm{irr}}^{(0)}$,

$(A_1^{(0)})_{i,i} = B_i^{(0)}$ for $i = 1,2$ and there exists $G \in GL(n,\mathbb{C})$ such that $GA^{(1)}G^{-1} = B^{(1)}_{\mathrm{irr}}$ and $(GA_1^{(1)}G^{-1})_{i,i} = B_i^{(1)}$ for $i = 1,2$. Then we can define the quiver Q,

$$\begin{tikzcd} \underset{[0,1]}{\circ} \arrow[r, "\rho_1"] \arrow[dr, "\rho_2" near start] & \underset{[1,1]}{\circ} \\ \underset{[0,2]}{\circ} \arrow[ur, "\rho_3" near start] \arrow[r, "\rho_4"] & \underset{[1,2]}{\circ} \end{tikzcd},$$

and attach a representation $M = (M_a, \psi_\rho)_{a\in Q_0, \rho\in Q_1\cup Q_1^*}$ to $\mathbf{A}$ as follows:

$$M_{[0,i]} := \mathbb{C}^{m_i}, \quad M_{[1,i]} := \mathbb{C}^{n_i} \text{ for } i = 1,2,$$

$$G = \left(\begin{array}{c|c} \psi_{\rho_1} & \psi_{\rho_3} \\ \hline \psi_{\rho_2} & \psi_{\rho_4} \end{array}\right), \quad A_1^{(1)}G^{-1} = \left(\begin{array}{c|c} \psi_{\rho_1^*} & \psi_{\rho_2^*} \\ \hline \psi_{\rho_3^*} & \psi_{\rho_4^*} \end{array}\right).$$

Then we have

$$\begin{aligned} GA_1^{(1)}G^{-1} &= \left(\begin{array}{c|c} \psi_{\rho_1}\psi_{\rho_1^*} + \psi_{\rho_3}\psi_{\rho_3^*} & * \\ \hline * & \psi_{\rho_2}\psi_{\rho_2^*} + \psi_{\rho_4}\psi_{\rho_4^*} \end{array}\right) \\ &= \left(\begin{array}{c|c} \mu_\alpha(M)_{[1,1]} & * \\ \hline * & \mu_\alpha(M)_{1,2} \end{array}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} A_1^{(0)} &= -A_1^{(1)} = -A_1^{(1)}G^{-1}G \\ &= -\left(\begin{array}{c|c} \psi_{\rho_1^*}\psi_{\rho_1} + \psi_{\rho_2^*}\psi_{\rho_2} & * \\ \hline * & \psi_{\rho_3^*}\psi_{\rho_3} + \psi_{\rho_4^*}\psi_{\rho_4} \end{array}\right) \\ &= \left(\begin{array}{c|c} \mu_\alpha(M)_{[0,1]} & * \\ \hline * & \mu_\alpha(M)_{[0,2]} \end{array}\right). \end{aligned}$$

Thus we have $\mu_\alpha(M)_{[i,j]} = B_j^{(i)}$, $i = 0,1$, $j = 1,2$.

5. Truncated orbits

In this section we study the structure of truncated orbits of higher degrees.

Fix $k > 1$ and $B = \sum_{i=1}^k B_i x^{-i} \in \mathfrak{g}_k^*$, HTL normal form written by

$$B = \mathrm{diag}\left(q_1(x^{-1})I_{n_1} + R_1x^{-1}, \ldots, q_m(x^{-1})I_{n_m} + R_mx^{-1}\right)$$

where $R_i \in M(n_i,\mathbb{C})$, $q_i(s) \in s^2\mathbb{C}[s]$, $i = 1,\ldots,m$ and $q_i \neq q_j$ if $i \neq j$.

Let $V_{\langle i,j\rangle} \subset \mathbb{C}^n$, $(i = 1,\ldots,k-1,\ j = 1,\ldots,m_i)$, be maximal simultaneous invariant spaces of $(B_{i+1},\ldots,B_{k-1},B_k)$. Here "maximal" means that for any $v \in \mathbb{C}^n\backslash V_{\langle i,j\rangle}$, the subspace $V_{\langle i,j\rangle}\oplus\mathbb{C}v$ is not invariant under $(B_{i+1},\ldots,B_{k-1},B_k)$. Then we have $\bigoplus_{j=1}^{m_i} V_{\langle i,j\rangle} = \mathbb{C}^n$ since $B_{i+1},\ldots,B_k$ are diagonal matrices. Especially we write $V_j = V_{\langle 1,j\rangle}$ for $j = 1,\ldots,m$. Here we note that $m_1 = m$.

Let $X_{i,j}$ denote the $\mathrm{Hom}_\mathbb{C}(V_j,V_i)$-component of $X \in M(n,\mathbb{C})$. For $g(x) = \sum_{i=r}^\infty g_ix^i \in M(n,\mathbb{C}((x)))$, write $(g(x))_{j,j'} = \sum_{i=r}^\infty (g_i)_{j,j'}x^i$, $1 \leq j,j' \leq m$. We denote the $\mathrm{Hom}_\mathbb{C}(V_i,\mathbb{C}^n)$-component of $X \in M(n,\mathbb{C})$ by $X_{*,i}$ for each $i = 1,\ldots,m$. Similarly $X_{i,*}$ denote the $\mathrm{Hom}_\mathbb{C}(\mathbb{C}^n,V_i)$- component. We sometimes use the notation $X = (X_{i,j})_{1\leq i,j\leq m} = (X_{*,i})_{1\leq i\leq m} = (X_{i,*})_{1\leq i\leq m}$.

Let $\pi_i : J_i = \{1,\ldots,m_i\} \to J_{i+1} = \{1,\ldots,m_{i+1}\}$ be the natural surjection such that $V_{\langle i,j\rangle} \subset V_{\langle i+1,\pi_i(j)\rangle}$. Define the total ordering $\{1 < 2 < \cdots < m\}$ on J_1 and also define total orderings on J_i, $i = 2,\ldots,k-1$, so that

$$\text{if } j_1 < j_2, \text{ then } \pi_i(j_1) \leq \pi_i(j_2), \quad j_1,j_2 \in J_i.$$

To a pair (j,j'), $j\neq j'\in m$, we attach a positive integer

$$d(j,j')=\deg_{\mathbb{C}[x]}(q_j(x)-q_{j'}(x))-2. \tag{2}$$

Moreover we set $d(j,j)=-1$ for the latter use.

Let us define the subgroup of G_k by

$$G_k^o=\left\{\sum_{i=0}^{k-1}A_ix^i\in G_k\,\middle|\,A_0=I_n\right\}.$$

Similarly define the subspace $\mathfrak{g}_k^o=M\left(n,x\mathbb{C}[[x]]/x^k\mathbb{C}[[x]]\right)$ of $\mathfrak{g}_k$, which can be identified with

$$\left\{\sum_{i=0}^{k-1}A_ix^i\in\mathfrak{g}_k\,\middle|\,A_0=0\right\}.$$

Then the dual space $(\mathfrak{g}_k^o)^*$ can be identified with

$$\left\{\sum_{i=0}^{k-1}A_ix^{-i-1}\in\mathfrak{g}_k^*\,\middle|\,A_0=0\right\}.$$

For $A=\sum_{i=1}^{k-1}A_ix^{-i}\in\mathfrak{g}_k^*$, let us set $\mathrm{pr}_{\mathrm{irr}}(A)=\sum_{i=2}^{k-1}A_ix^{-i}$ and $\mathrm{pr}_{\mathrm{res}}(A)=A_1$. This gives the decomposition $\mathfrak{g}_k^*=(\mathfrak{g}_k^o)^*\oplus M(n,\mathbb{C})$, and $\mathrm{pr}_{\mathrm{irr}}$ (resp. $\mathrm{pr}_{\mathrm{res}}$) is the first (resp. second) projection along this decomposition.

The coadjoint orbit $\mathcal{O}_{B_{\mathrm{irr}}}=\{gB_{\mathrm{irr}}g^{-1}\mid g\in G_k^o\}\subset(\mathfrak{g}_k^o)^*$
is our main objects to study in this section where $B_{\mathrm{irr}}=\mathrm{pr}_{\mathrm{irr}}(B)$.

According to the ordering on each J_i, $i=1,\ldots,k-1$, let us define parabolic subalgebras of $M(n,\mathbb{C})$ as below,

$$\mathfrak{p}_i^+=\bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1\geq j_2}}\mathrm{Hom}_{\mathbb{C}}(V_{\langle i,j_1\rangle},V_{\langle i,j_2\rangle}),\qquad \mathfrak{p}_i^-=\bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1\leq j_2}}\mathrm{Hom}_{\mathbb{C}}(V_{\langle i,j_1\rangle},V_{\langle i,j_2\rangle}),$$

and similarly nilpotent subalgebras

$$\mathfrak{u}_i^+=\bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1>j_2}}\mathrm{Hom}_{\mathbb{C}}(V_{\langle i,j_1\rangle},V_{\langle i,j_2\rangle}),\qquad \mathfrak{u}_i^-=\bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1<j_2}}\mathrm{Hom}_{\mathbb{C}}(V_{\langle i,j_1\rangle},V_{\langle i,j_2\rangle}),$$

for $i=1,\ldots,k-1$. We note that $\mathfrak{p}_i^\pm=\mathfrak{h}_i\oplus\mathfrak{u}_i^\pm$ where $\mathfrak{h}_i=\bigoplus_{j\in J_i}\mathrm{End}_{\mathbb{C}}(V_{\langle i,j\rangle})$.

Let us define subsets of G_k^o,

$$\mathcal{P}_k^\pm=\left\{\sum_{i=0}^{k-1}P_ix^i\in G_k^o\,\middle|\,P_i\in\mathfrak{p}_{i+1}^\pm,i=1,\ldots,k-1\right\},$$

$$\mathcal{U}_k^\pm=\left\{\sum_{i=0}^{k-1}U_ix^i\in G_k^o\,\middle|\,U_i\in\mathfrak{u}_{i+1}^\pm,i=1,\ldots,k-1\right\},$$

and subspaces of $\mathfrak{g}_k^o$ and $(\mathfrak{g}_k^o)^*$,

$$\mathfrak{U}_k^\pm=\left\{\sum_{i=1}^{k-1}U_ix^i\,\middle|\,U_i\in\mathfrak{u}_{i+1}^\pm,\ i=1,\ldots,k-1\right\},$$

$$(\mathfrak{U}_k^\mp)^*=\left\{\sum_{i=1}^{k-1}U_ix^{-i-1}\,\middle|\,U_i\in\mathfrak{u}_{i+1}^\pm,\ i=1,\ldots,k-1\right\}.$$

Here we put $\mathfrak{p}_k^\pm=M(n,\mathbb{C})$ and $\mathfrak{u}_k^\pm=\{0\}$. Let us note that $\mathcal{P}_k^\pm$ are subgroups of G_k^o but $\mathcal{U}_k^\pm$ are not closed under the multiplication.

Then the following decomposition holds.

Lemma 5.1 (Lemma 3.5 in [5]). *For any $g \in G_k^o$, there uniquely exist $u_- \in \mathcal{U}_k^-$ and $p_+ \in \mathcal{P}_k^+$ such that $g = u_- p_+$.*

For $A \in \mathcal{O}_{B_{\mathrm{irr}}}$, take $g \in G_k^o$ so that $g^{-1}Ag = B_{\mathrm{irr}}$ and decompose $g = u_- p_+$ as above. Note that u_- does not depend on the choice of g because the stabilizer of B_{irr} is contained in $\mathcal{P}_k^+$. Thus u_- is uniquely determined by $A \in \mathcal{O}_{B_{\mathrm{irr}}}$. Then let us put $Q = u_- - I_n$, $A' = u_-^{-1}A$ and $P = A'|_{(\mathfrak{U}_k^-)^*}$.

Proposition 5.2 (Theorem 3.6 in [5]). *The map*

$$\begin{array}{rccc} \Phi\colon & \mathcal{O}_{B_{irr}} & \longrightarrow & \mathfrak{U}_k^- \times (\mathfrak{U}_k^-)^* \\ & A & \longmapsto & (Q, P) \end{array}$$

is bijective.

6. Generalized Delinge-Simpson problems and representations of quivers

In the previous sections representations of quivers are associated with conjugacy classes of matrices (Proposition 3.1), tuples of truncated orbits of degree 2 (Proposition 4.2) and truncated orbits of higher degrees (Proposition 5.2). Glueing them together, now let us construct representations of quivers in order to apply to generalized Deligne-Simpson problems.

Let $B^{(0)} = \sum_{i=1}^{k_0} B^{[0,i]}x^{-i} \in \mathfrak{g}_{k_0}^*, \ldots, B^{(p)} = \sum_{i=1}^{k_p} B^{[p,i]}x^{-i} \in \mathfrak{g}_{k_p}^*$ be HTL normal forms written by

$$B^{(i)} = \mathrm{diag}\left(q_1^{(i)}(x^{-1})I_{n_1^{(i)}} + R_1^{(i)}x^{-1}, \ldots, q_{m^{(i)}}^{(i)}(x^{-1})I_{n_{m^{(i)}}^{(i)}} + R_{m^{(i)}}^{(i)}x^{-1}\right)$$

for $i = 0, \ldots, p$ where $q_j^{(i)}(s) \in s^2\mathbb{C}[s]$ satisfying $q_j^{(i)} \neq q_{j'}^{(i)}$ if $j \neq j'$ and $B_j^{(i)} \in M(n_j^{(i)}, \mathbb{C})$.

For each $B_j^{(i)}$, $i = 0, \ldots, p$ and $j = 1, \ldots, m_i$, let us choose complex numbers $\xi_1^{[i,j]}, \ldots, \xi_{e_{[i,j]}}^{[i,j]}$ so that

$$\prod_{k=1}^{e_{[i,j]}} (R_j^{(i)} - \xi_k^{[i,j]}) = 0$$

for $i = 0, \ldots, p$ and $j = 1, \ldots, m^{(i)}$. Set $I_{\mathrm{irr}} = \{i \in \{0, \ldots, p\} \mid m^{(i)} > 1\} \cup \{0\}$ and $I_{\mathrm{reg}} = \{0, \ldots, p\} \backslash I_{\mathrm{irr}}$. Then let Q be the quiver with the set of vertices

$$\mathsf{Q}_0 = \left\{ [i,j] \;\middle|\; \begin{array}{l} i \in I_{\mathrm{irr}}, \\ j = 1, \ldots, m^{(i)} \end{array} \right\} \cup \left\{ [i,j,k] \;\middle|\; \begin{array}{l} i = 0, \ldots, p, \\ j = 1, \ldots, m^{(i)}, \\ k = 1, \ldots, e_{[i,j]} - 1 \end{array} \right\}$$

and the set of arrows

$$\begin{aligned} \mathsf{Q}_1 = & \left\{ \rho_{[i,j']}^{[0,j]}\colon [0,j] \to [i,j'] \;\middle|\; \begin{array}{l} j = 1, \ldots, m^{(0)}, \\ i \in I_{\mathrm{irr}} \backslash \{0\}, \\ j' = 1, \ldots, m^{(i)} \end{array} \right\} \\ & \cup \left\{ \rho_{[i,j],[i,j']}^{[k]}\colon [i,j] \to [i,j'] \;\middle|\; \begin{array}{l} i \in I_{\mathrm{irr}},\ 1 \leq j < j' \leq m^{(i)}, \\ 1 \leq k \leq d_i(j,j') \end{array} \right\} \\ & \cup \left\{ \rho_{[i,j,1]}\colon [i,j,1] \to [i,j] \mid i \in I_{\mathrm{irr}},\ j = 1, \ldots, m^{(i)} \right\} \\ & \cup \left\{ \rho_{[0,j]}^{[i,1,1]}\colon [i,1,1] \to [0,j] \mid i \in I_{\mathrm{reg}},\ j = 1, \ldots, m^{(0)} \right\} \\ & \cup \left\{ \rho_{[i,j,k]}\colon [i,j,k] \to [i,j,k-1] \;\middle|\; \begin{array}{l} i = 0, \ldots, p, \\ j = 1, \ldots, m^{(i)}, \\ k = 2, \ldots, e_{[i,j]} - 1 \end{array} \right\}. \end{aligned}$$

Here $d_i(j,j')=\deg_{\mathbb{C}[x]}(q^{(i)}_j(x)-q^{(i)}_{j'}(x))-2$. Let $\alpha=(\alpha_a)_{a\in\mathsf{Q}_0}\in\mathbb{Z}^{\mathsf{Q}_0}$ be the vector defined by setting $\alpha_{[i,j]}=n^{(i)}_j$ and $\alpha_{[i,j,k]}=\dim_{\mathbb{C}}(\mathrm{rank}\prod_{l=1}^k(R^{(i)}_j-\xi^{[i,j]}_l))$. Also define $\lambda=(\lambda_a)_{a\in\mathsf{Q}_0}\in\mathbb{C}^{\mathsf{Q}_0}$ by $\lambda_{[i,j]}=-\xi^{[i,j]}_1$ for $i\in I_{\mathrm{irr}}\backslash\{0\}$, $j=1,\ldots,m^{(i)}$, $\lambda_{[0,j]}=-\xi^{[0,j]}-\sum_{i\in I_{\mathrm{reg}}}\xi^{[i,1]}_1$ for $j=1,\ldots,m^{(0)}$, and $\lambda_{[i,j,k]}=\xi^{[i,j]}_k-\xi^{[i,j]}_{k+1}$ for $i=0,\ldots,p$, $j=1,\ldots,m^{(i)}$ and $k=1,\ldots,e_{[i,j]}-1$. The following sublattice of $\mathbb{Z}^{\mathsf{Q}_0}$ plays an essential role,

$$\mathcal{L}=\left\{\beta\in\mathbb{Z}^{\mathsf{Q}_0}\;\middle|\;\begin{array}{l}\text{there exists } b\in\mathbb{Z}\text{ such that}\\ \sum_{j=1}^{m^{(i)}}\beta_{[i,j]}=b\text{ for all } i\in I_{\mathrm{irr}}\end{array}\right\}.$$

Set $\mathcal{L}^+=\mathcal{L}\cap(\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$.

Then combining Proposition 3.1, 4.2 and 5.2, we have the following bijection.

Proposition 6.1. *Let $B^{(0)},\ldots,B^{(p)}$ be HTL normal forms chosen as above. Then there exists a bijection*

$$\Phi_\xi\colon\left\{\left(\sum_{j=1}^{k_i}A^{(i)}_jx^{-j}\right)_{0\leq i\leq p}\in\prod_{i=0}^p\mathcal{O}_{B^{(i)}}\;\middle|\;\sum_{i=0}^pA^{(i)}_1=0\right\}/\mathrm{GL}(n,\mathbb{C})$$
$$\to\Big\{M=(M_a,\psi_\rho)\in\mathrm{Rep}(\overline{\mathsf{Q}},\alpha)_\lambda\;\Big|$$
$$\det\left(\psi_{\rho^{[0,j]}_{[i,j']}}\right)_{\substack{1\leq j\leq m^{(0)}\\ 1\leq j'\leq m^{(i)}}}\neq 0,\ i\in I_{irr}\backslash\{0\},$$
$$\left(\psi_{\rho^{[i,1,1]}_{[0,j]}}\right)_{1\leq j\leq m^{(0)}}\colon M_{[i,1,1]}\to\bigoplus_{j=1}^{m^{(0)}}M_{[0,j]},\ \textit{injective},\ i\in I_{reg},$$
$$\left(\psi_{\left(\rho^{[i,1,1]}_{[0,j]}\right)^*}\right)_{1\leq j\leq m^{(0)}}\colon\bigoplus_{j=1}^{m^{(0)}}M_{[0,j]}\to M_{[i,1,1]},\ \textit{surjective},\ i\in I_{reg},$$
$$\psi_{\rho_{[i,j,k]}},\ \textit{injective},\ \psi_{(\rho_{[i,j,k]})^*},\ \textit{surjective}\Big\}/\prod_{a\in\mathsf{Q}_0}\mathrm{GL}(\alpha_a,\mathbb{C}).$$

Unfortunately the above bijection Φ does not preserve irreducibility. Thus we introduce the following notion.

Definition 6.2 (quasi-irreducible)**.** If $X\in\mathrm{Rep}(\overline{\mathsf{Q}},\alpha)_\lambda$ has no nontrivial proper subrepresentation $Y\subsetneq X$ in $\mathrm{Rep}(\overline{\mathsf{Q}},\alpha)_\lambda$ with $\mathbf{dim}\,Y=(\beta_a)_{a\in\mathsf{Q}_0}$ satisfying that there exists $b\in\mathbb{Z}_{\geq 0}$ such that $\sum_{j=1}^{m^{(i)}}\beta_{[i,j]}=b$ for all $i\in I_{\mathrm{irr}}$, then X is said to be *quasi-irreducible*.

Then we have the following correspondence between irreducible elements and quasi-irreducible representations.

Theorem 6.3. *The restriction of the map in Proposition 6.1,*

$$\Phi_\xi\colon\left\{\left(\sum_{j=1}^{k_i}A^{(i)}_jx^{-j}\right)_{0\leq i\leq p}\in\prod_{i=0}^p\mathcal{O}_{B^{(i)}}\;\middle|\;\sum_{i=0}^pA^{(i)}_1=0,\ \textit{irreducible}\right\}/\mathrm{GL}(n,\mathbb{C})$$
$$\to\left\{M=(M_a,\psi_\rho)\in\mathrm{Rep}(\overline{\mathsf{Q}},\alpha)_\lambda\;\middle|\;\textit{quasi-irreducible},\right.$$
$$\left.\det\left(\psi_{\rho^{[0,j]}_{[i,j']}}\right)_{\substack{1\leq j\leq m^{(0)}\\ 1\leq j'\leq m^{(i)}}}\neq 0,\ i\in I_{irr}\backslash\{0\}\right\}/\prod_{a\in\mathsf{Q}_0}\mathrm{GL}(\alpha_a,\mathbb{C}),$$

is bijective.

Then we have the following theorem.

Theorem 6.4 ([4]). *Let us consider a generalized additive Deligne-Simpson problem consisting of positive integers $k_0, \ldots, k_p$ and HTL normal forms $B^{(i)} \in \mathfrak{g}_{k_i}^*$ for $i = 0, \ldots, p$. Let us define $\mathsf{Q} = (\mathsf{Q}_0, \mathsf{Q}_1)$, $\alpha \in (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$ and $\lambda \in \mathbb{C}^{\mathsf{Q}_0}$ as above. Then the generalized additive Deligne-Simpson problem has a solution if and only if the following are satisfied,*

(1) *α is a positive root of Q and $\alpha \cdot \lambda = \sum_{a \in \mathsf{Q}_0} \alpha_a \lambda_a = 0$,*
(2) *for any decomposition $\alpha = \beta_1 + \cdots + \beta_r$ where $\beta_i \in \mathcal{L}^+$ are positive roots of Q satisfying $\beta_i \cdot \lambda = 0$, we have*
$$p(\alpha) > p(\beta_1) + \cdots + p(\beta_r).$$

Here we note that if $I_{\mathrm{irr}} = \{0\}$, then $\mathcal{L} = \mathbb{Z}^{\mathsf{Q}_0}$. In this case, the conditions 1 and 2 in the above theorem coincide with Crawley-Boevey's condition for the existence of irreducible representations of deformed preprojectiove algebras. Thus the theorem covers preceding results for the additive Delinge-Simpson problems [3], [1] and [5].

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