Twisted Euler transforms and Weyl groups of Kac-Moody Lie algebras

Kazuki Hiroe*

1 Introduction

For Fuchsian differential equations of the form

$$\frac{d}{dx}Y(x) = \sum_{i=1}^{r} \frac{A_i}{(x-c_i)} Y(x), \tag{1.1}$$

where A_i are $n \times n$ complex matrices, W. Crawley-Boevey ([1]) find a correspondence of the systems and representations of quivers. In this correspondence, the Middle convolution (Euler transform) corresponds to reflection functors of representations of quivers. And he solved the existence problem of systems of differential equations, so-called Deligne-Simpson problem by using the theory of representations of quivers.

In this note, we deal with differential equations of scalar type. The theory of middle convolution (Euler transform) is rearranged by Oshima [2]. By using this theory, we generalize the correspondence between differential equations and Kac-Moody root systems to non-Fuchsian differential equations.

2 Fourier-Laplace transform and characteristic indices

2.1 Characteristic indices

Let K be an algebraic closed field of characteristic zero. The polynomial ring with coefficients in K is denoted by $K[x]$. Also $K(x)$ stands for the field of rational functions with coefficients in K. For any $a \in K$, $K(x)_a$ is the field of Laurent series of $(x-a)$, i.e., $K(x)_a = \{\sum_{i=m}^{\infty} a_i(x-a)^i \mid a_i \in K,\, m \in \mathbb{Z}\}$. Moreover, $\overline{K(x)_a}$ is the field of Puiseux series of $(x-a)$, i.e., $\overline{K(x)_a} = \bigcup_{p\in\mathbb{Z}_{>0}} K(x^{\frac{1}{p}})_a$.

*E-mail:kazuki@ms.u-tokyo.ac.jp

Then the differentiation $\partial = \frac{d}{dx}$ with respect to x is naturally defined on them. Let $W[x], W(x), W(x)_a$ and $\overline{W(x)_a}$ be rings of differential operators with coefficients in $K[x], K(x), K(x)_a$, and $\overline{K(x)_a}$ respectively.

For an element h in $K(x)$ and $a \in K$, if there exist $f, g \in K[x]$ and $g(a) \neq 0$ such that $h = \frac{f}{g} = fg^{-1}$, then one says h is regular at $x = a$. If h is not regular at $x = a$, one says h is singular at $x = a$ and $x = a$ is called singular point of h.

Recalling that we can write $P \in W(x)$ as

$$P = \sum_{i=0}^{n} a_i(x)\partial^i,$$

$a_i(x) \in K[x]$ uniquely, let us define singular points of P as singular points of $\frac{a_i(x)}{a_n(x)}$. Also if

$$P^\infty = \sum_{i=0}^{n} a_i(x^{-1})(x^2\partial)^i$$

has a singular point at $x = 0$, then we say P has a singular point at $x = \infty$.

We define characteristic indices of elements of $W[x]$ at their singular points. Let us take $P \in W[x]$. We can assume $x = 0$ is a singular point of P. For the other singular points, we can define characteristic indices in the same way. Let us define a subspace of $W(x)_0$ by $W_0^k = \{\sum_{i=0}^n c_i x^{k+i}\partial^i \mid c_i \in K, n \in Z_{\leq 0}\}(k \in \mathbb{Z})$. Then we define a map from W_0^k to the ring of polynomials $K[\lambda]$ as follows,

$$\begin{array}{cccc} \Xi_0^k : & W_0^k & \longrightarrow & K[\lambda] \\ & \sum_{i=0}^n c_i x^{k+i}\partial^i & \longmapsto & c_0 + \sum_{i=1}^n c_i\lambda(\lambda-1)\cdots(\lambda-i+1). \end{array}$$

Then an arbitrary element P of $W(x)_0$ is written by

$$P = \sum_{k=m}^{\infty} P_k, \ (P_k \in W_0^k, P_m \neq 0). \tag{2.1}$$

Then

$$Ch_0(P) = \Xi_0^m(P_m) \in K[\lambda]$$

is characteristic polynomial of P at $x = 0$. Moreover,

$$Ch_0^i(P) = \Xi_0^{m+i}(P_{m+i}) \in K[\lambda]$$

is i-th characteristic polynomial $(i = 1, 2, \ldots)$.

Let us consider power series solutions of the differential equation $Pu = 0$. The multiplicative group $\{x^\lambda \mid \lambda \in K\}$ is defined by the multiplication $x^{\lambda_1} \cdot x^{\lambda_2} = x^{\lambda_1+\lambda_2}$. The group ring $K[x^\lambda]$ is generated by this group over K. The differentiation $\partial = \frac{d}{dx}$ on $K[x^\lambda]$ is naturally defined by $\partial x^\lambda = \lambda x^{\lambda-1}$.

Then the differentiation on $K[[x^\lambda]] = K[x^\lambda] \otimes_{K[x]} K(x)_0$ is naturally defined as well. For $f = \sum_{i=0}^{\infty} c_i x^{\mu+i} \in K[[x^\lambda]]$, if $c_0 \neq 0$, one says f has the characteristic index μ.

Then we have the following.

Proposition 2.1. *Let us take $P \in W[x]$ which have a singular point at $x = 0$. Suppose that $\deg Ch_0(P) = n > 0$. Also assume that there are $\lambda_1, \ldots, \lambda_l \in K$, $m_1, \ldots, m_l \in \mathbb{Z}_{>0}$ such that $\lambda_i - \lambda_j \notin \mathbb{Z}\,(i \neq j)$ and $\sum_{i=1}^{l} m_i = n$. Then the followings are equivalent.*

1. *For $i = 1, \ldots, l$, $j = 0, 1, \ldots, m_i - 1$, there exist $f_{\lambda_i+j} \in K[[x^\lambda]]$ with characteristic indices $\lambda_i + j$ such that $Pf_{\lambda_i+j} = 0$.*

2. *We have the following equations,*

$$Ch_0^k(P)(\lambda_i + j) = 0$$

for $i = 1, \ldots, l$, $k = 0, \ldots, m_i - 1$ and $j = 0, \ldots, m_i - (k+1)$.

Then we define characteristic indices of P at the singular point $x = 0$ as follows.

Definition 2.2. *Let us take $P \in W[x]$ which satisfies conditions in Proposition 2.1. Then we say P has characteristic indices*

$$\{[\lambda_1]_{m_1}, \ldots, [\lambda_l]_{m_l}\}$$

at $x = 0$.

To discuss formal power series solutions at irregular singular points, we should introduce twisted characteristic indices.

Definition 2.3. *Let us take $f(x) \in K[x^{-1}]$. We say $P \in W[x]$ has $e^{f(x)}$-twisted characteristic indices*

$$\{[\lambda_1]_{m_1}, \ldots, [\lambda_l]_{m_l}\},$$

at $x = 0$, if $P(x, \partial - \frac{d}{dx} f(x))$ has characteristic indices

$$\{[\lambda_1]_{m_1}, \ldots, [\lambda_l]_{m_l}\}$$

at $x = 0$.

When we consider the "ramified" case, we need to consider the formal solutions in $\overline{K_0}$. Let us define the algebra endomorphism

$$\begin{array}{cccc} \rho_p\colon & \overline{W(x)_0} & \longrightarrow & \overline{W(x)_0} \\ & x & \longmapsto & x^p \\ & \partial & \longmapsto & \frac{1}{p} x^{1-p} \partial \end{array}$$

for $p \in \mathbb{Z}_{>0}$.

Definition 2.4. *Let us take $f(x) \in K[x^{-1}]$. We say $P \in W[x]$ has $e^{f(x)}$-twisted characteristic indices of ramification index p*

$$\{[\lambda_1]_{m_1}, \ldots, [\lambda_l]_{m_l}\}$$

if $\rho_p(P(x, \partial)) \in W(x)$ has $e^{f(x)}$-twisted characteristic indices

$$\{[\lambda_1]_{m_1}, \ldots, [\lambda_l]_{m_l}\}.$$

2.2 Fourier-Laplace transform

Let us consider the Fourier-Laplace transform

$$\begin{array}{cccc} \mathcal{L}\colon & W[x] & \longrightarrow & W[x] \\ & x & \longmapsto & -\partial \\ & \partial & \longmapsto & x \end{array} .$$

Then we have the followings.

Theorem 2.5. *Let us take $P \in W[x]$ which has a singular point at $x = a$ and $f(x) = \alpha_{n-1}x^{-(n-1)} + \cdots + \alpha_1 x^{-1} \in K[x^{-1}]$ where $n > 1$. Then there exist $d_i \in K$ and polynomials $g_i(x) = \frac{a}{n}x^n + g_{n-1}x^{n-1} + \cdots + g_1 x \in K[x]$ for $i = 1, \ldots, n$ and the followings are equivalent.*

1. *At $x = a$, P has $e^{f(x-a)}$-twisted characteristic indices*

$$\{[\lambda_1]_{m_1}, \cdots, [\lambda_l]_{m_l}\}.$$

2. *At $x = \infty$, $\mathcal{L}P$ has $e^{g_i(x)}$-twisted characteristic indices of ramified index n*

$$\{[\lambda_1 + d_i]_{m_1}, \cdots, [\lambda_l + d_i]_{m_l}\}$$

for $i = 1, \ldots, n$.

Theorem 2.6. *Let us take $P \in W[x]$ which has a singular point at $x = \infty$ and $f(x) = \alpha_{n+1}x^{n+1} + \cdots + \alpha_1 x \in K[x]$ where $n \geq 0$. Then there exist $d_i \in K$ and $g_i(x) = g_{n+1}x^{n+1} + \cdots + g_1 x$ for $i = 1, \ldots, n$ and the followings are equivalent.*

1. *At $x = \infty$, P has $e^{f(x)}$-twisted characteristic indices*

$$\{[\lambda_1]_{m_1}, \ldots, [\lambda_l]_{m_l}\}.$$

2. *At $x = \infty$, P has $e^{g_i(x)}$-twisted characteristic indices of ramified index n*

$$\{[\lambda_1 + d_i]_{m_1}, \ldots, [\lambda_l + d_i]_{m_l}\}$$

for $i = 1, \ldots, n$.

If we write $P = \sum_{i=0}^{n} a_i(x)\partial^i \in W[x]$, then the degree of P is defined by $\deg P = \max\{\deg a_i(x) \mid i = 0, \dots, n\}$.

Definition 2.7 (Reduced representative)**.** *Let us take $P \in W[x]$. The reduced representative $\mathrm{R}P \in W[x]$ is the element in $W[x] \cap K(x)P$ which has the minimal degree.*

Theorem 2.8. *Let P be an element in $W[x]$. We assume that P has a singular point at $x = a$ and $\deg Ch_a P > 0$. Then the followings are equivalent.*

1. *P has characteristic indices*

$$\{[0]_{m_0}, [\lambda_1]_{m_1}, \dots, [\lambda_l]_{m_l}\}$$

at $x = a$.

2. *$\mathcal{L}(\mathrm{R}P)$ has e^{-ax}-twisted characteristic indices*

$$\{[\lambda_1 - 1]_{m_1}, \dots, [\lambda_l - 1]_{m_l}\}$$

at $x = \infty$.

3 Local datum

For a fixed $k \in \mathbb{Z}_{>0}$, let us consider a family of finite sets $\mathcal{I} = I_1, I_2, \dots, I_k$ and surjective maps ϕ_i from I_{i-1} to I_i for $i = 2, \dots, k$ and suppose that cardinalities of these sets satisfy that $\#I_1 \geq \#I_2 \geq \cdots \geq \#I_k$. The compositions $\psi_i = \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$ are surjective maps from $\mathcal{I}$ to I_i for $i = 2, \dots, k$. We put $\psi_1 = \mathrm{id}_{\mathcal{I}}$.

For $i = 1, \dots, k-1$, we introduce relations on $\mathcal{I}$ as follows. For $s, s' \in \mathcal{I}$ and $i = 1, \dots, k-1$, we have

$$s_1 \overset{i}{\sim} s_2$$

if $\psi_k(s_1) = \psi_k(s_2)$, $\psi_{k-1}(s_1) = \psi_{k-1}(s_2), \dots, \psi_{i+1}(s_1) = \psi_{i+1}(s_2)$ and $\psi_i(s_1) \neq \psi_i(s_2)$. Also we write $s_1 \overset{0}{\sim} s_2$ if $s_1 = s_2$.

For each I_i ($i = 1, \dots, k$), we fix an injective map α_i into K. By these α_i, we define a injective map $\exp_x$ from $\mathcal{I}$ into $K[x^{-1}]$ as follows,

$$\begin{array}{rccc} \exp_x\colon & \mathcal{I} & \longrightarrow & K[x^{-1}] \\ & s & \longmapsto & \alpha_k(\psi_k(s))x^{-k} + \ldots + \alpha_1(s)x^{-1} \end{array}.$$

Let us fix a map $l\colon \mathcal{I} \to \mathbb{Z}_{>0}$ and consider a $\mathbb{Z}$-lattice

$$L(\mathcal{I}) = \prod_{s \in \mathcal{I}} \mathbb{Z}^{l(s)}.$$

Fix an element $\mathfrak{m} \in L(\mathcal{I})$ written by

$$\mathfrak{m} = \prod_{s\in\mathcal{I}} \mathfrak{m}_s = \prod_{s\in\mathcal{I}} (m(s)_1, \ldots, m(s)_{l(s)}) \in \prod_{s\in\mathcal{I}} \mathbb{Z}^{l(s)}.$$

For this $\mathfrak{m}$, the order is defined by

$$\mathrm{ord}\mathfrak{m} = \sum_{s\in\mathcal{I}} \sum_{i=0}^{l(s)} m(s)_i.$$

We also fix an element of $\prod_{s\in\mathcal{I}} \mathbb{C}^{l(s)}$,

$$\Lambda = \prod_{s\in\mathcal{I}} \Lambda_s = \prod_{s\in\mathcal{I}} (\lambda(s)_1, \ldots, \lambda(s)_{l(s)}) \in \prod_{s\in\mathcal{I}} \mathbb{C}^{l(s)}.$$

Then we call the tuple $(\mathcal{I}, \exp_a, \mathfrak{m}, \Lambda)$ local datum at $x = a$ of rank k. Also we define the order of the local datum by $\mathrm{ord}\,\mathfrak{m}$.

4 Twisted Euler transform

All the remaining of this note, we consider elements in $W[x]$ which satisfy followings. Let us take $P \in W[x]$ of the order n, i.e., $P = \sum_{i=0}^{n} a_i(x)\partial^i$. Singular points of P are $c_1, \ldots, c_p \in K$ and $c_0 = \infty$. For each c_i, there exists a local datum $(\mathcal{I}_i, \exp_{c_i}, \mathfrak{m}_i, \Lambda_i)$ of order n and rank k_i and Λ_i satisfies that $\lambda(s)_i - \lambda_j(s) \notin \mathbb{Z}\,(i \neq j)$ where $\Lambda_i = \prod_{s\in\mathcal{I}_i} (\lambda(s)_1, \ldots, \lambda(s)_{l(s)})$. Then for each $s \in \mathcal{I}_i$, P has $e^{\exp_{c_i}(s)}$-twisted characteristic indices

$$\{[\lambda(s)_1]_{m(s)_1}, \ldots [\lambda(s)_{l(s)}]_{m(s)_{l(s)}}\}$$

at $x = c_i$. Then we say P has local datum $(\mathcal{I}_i, \exp_{c_i}, \mathfrak{m}_i, \Lambda_i)$ for $i = 0, \ldots, p$.

Let us introduce some operations on $W[x]$ and $W(x)$.

Definition 4.1 (Gauge transform, Addition)**.** *For $f \in K(x)$, we consider the following algebra isomorphism*

$$\begin{array}{cccc} \mathrm{Ad}(e^{f(x)}): & W(x) & \longrightarrow & W(x) \\ & x & \longmapsto & x \\ & \partial & \longmapsto & \partial - \frac{d}{dx}f(x) \end{array}.$$

For $\frac{\lambda}{x-c} \in K(x), \lambda, c \in K$, we define

$$\begin{array}{cccc} \mathrm{Ad}((x-c)^{\lambda}): & W(x) & \longrightarrow & W(x) \\ & x & \longmapsto & x \\ & \partial & \longmapsto & \partial - \frac{\lambda}{x-c} \end{array}$$

and call this addition at $x = c$.

Definition 4.2 (Euler transform)**.** *Euler transform is an operation on $W[x]$ defined by*

$$E(\lambda) = \mathcal{L} \circ \mathrm{RAd}(x^{\lambda}) \circ \mathcal{L}^{-1}\mathrm{R}$$

for $\lambda \in K$.

Let us define $\mathcal{T}(P) = \prod_{i=0}^{p} \mathcal{I}_i$. For $t \in \mathcal{T}(P)$, $t = (s_0(t), \ldots, s_p(t))$, we write

$$\begin{aligned}
\mathrm{Ade}(t) &= \prod_{i=0}^{p} \mathrm{Ad}(e^{\exp_{c_i}(s_i(t))}),\\
\mathrm{Ade}(t)^{-1} &= \prod_{i=0}^{p} \mathrm{Ad}(e^{-\exp_{c_i}(s_i(t))}),\\
\mathrm{Ad}(t) &= \prod_{i=0}^{p} \mathrm{Ad}((x-c_i)^{\lambda(s_i(t))_1}),\\
\mathrm{Ad}(t)^{-1} &= \prod_{i=0}^{p} \mathrm{Ad}((x-c_i)^{-\lambda(s_i(t))_1}).
\end{aligned}$$

Definition 4.3 (Twisted Euler transform)**.** *For $t \in \mathcal{T}(P)$, we define the twisted Euler transform by*

$$E(t) = \mathrm{RAde}(t)^{-1}\mathrm{Ad}(t)^{-1}E(\lambda(s_0(t))_1 - 1)\mathrm{Ad}(t)\mathrm{Ade}(t).$$

Theorem 4.4. *Let us take $t \in \mathcal{T}(P)$ and assume some generic condition on Λ_i. Then $E(t)P \in W[x]$ satisfies followings.*

Singular points of $E(t)P$ are same as them of P. On each singular points c_i, there exists a local datum

$$(\tilde{\mathcal{I}}_i, \tilde{\exp}_{c_i}, \tilde{\mathfrak{m}}_i, \tilde{\Lambda}_i).$$

And $E(t)P$ has local datum $(\tilde{\mathcal{I}}_i, \tilde{\exp}_{c_i}, \tilde{\mathfrak{m}}_i, \tilde{\Lambda}_i)$ for $i = 0, \ldots, p$.

Then

$$\begin{aligned}
\tilde{\mathcal{I}}_i &= \mathcal{I}_i,\\
\tilde{\exp}_{c_i} &= \exp_{c_i}
\end{aligned}$$

and $\tilde{\mathfrak{m}}_i = \prod_{s \in \mathcal{I}_i} \tilde{\mathfrak{m}(s)} = \prod_{s \in \mathcal{I}} (\tilde{m(s)}_1, \ldots, \tilde{m(s)}_{l(s)})$ satisfy that

$$\begin{aligned}
\tilde{m(s)}_i &= m(s)_i + d && \textit{if } i = 1 \textit{ and } s = s_i(t)\\
\tilde{m(s)}_i &= m(s)_i && \textit{otherwise.}
\end{aligned}$$

where

$$d = \deg \mathrm{RAde}(t)P - \sum_{i=1}^{l(s_0(t))} m(s_0(t))_i - m(s_0(t))_1.$$

Remark 4.5. *The degree of P can be written by the $\mathbb{Z}$-linear combination of $m(s)_i$.*

Let us define a $\mathbb{Z}$-lattice

$$L_P = \{\prod_{i=0}^{p} \mathfrak{m}_i \in \prod_{i=0}^{p} L(\mathcal{I}_i) \mid \mathrm{ord}\mathfrak{m}_1 = \cdots = \mathrm{ord}\mathfrak{m}_p\}.$$

Then by this theorem, $L_{E(t)P} = L_P$ and $E(t)$ sends $\prod_{i=0}^{p} \mathfrak{m}_i \in L_P$ to $\prod_{i=0}^{p} \tilde{\mathfrak{m}} \in L_{E(t)P} = L_P$ linearly. Hence we can extend this to the $\mathbb{Z}$-linear map

$$\sigma(t) \colon L_P \longrightarrow L_P.$$

We also define following $\mathbb{Z}$-linear maps. For $s \in \mathcal{I}_i$, we define $\mathbb{Z}$-linear endomorphism of $L(\mathcal{I}_i)$,

$$\begin{array}{cccc} \sigma(i,s,j)\colon & L(\mathcal{I}_i) & \longrightarrow & L(\mathcal{I}_i) \\ & m(s)_j & \longmapsto & m(s)_{j+1} \\ & m(s)_{j+1} & \longmapsto & m(s)_j \\ & m(s)_k & \longmapsto & m(s)_k \text{ for } k \neq j \end{array}.$$

We can extend this to $\mathbb{Z}$-linear endomorphism of L_P which is trivial on $L(\mathcal{I}_l)$ $(i \neq l)$.

5 Weyl group action and twisted Euler transform

For $P \in W[x]$ which satisfies conditions defined in the previous section, we define a Dynkin diagram of Kac-Moody Lie algebra as follows. The set of vertices is

$$\mathcal{C} = \{c_t \mid t \in \mathcal{T}(P)\} \cup \{d(i,s,j) \mid s \in \mathcal{I}_i \, (i = 0, \ldots, p), j = 1, \ldots, l(s) - 1\}$$

We connect these vertices according to the following rule. For $t, t' \in \mathcal{T}(P)$, $t = (s(t)_0, \ldots, s(t)_p), t' = (s(t')_0, \ldots, s(t')_p)$, if

$$s(t)_i \overset{f_i(t,t')<++>}{\sim} s(t')_i$$

for $i = 0, \ldots, p$, then vertices c_t and $c_{t'}$ are connected by

$$\sum_{i=1}^{p} (f_i(t,t') + 1) + f_0(t,t') - 1 - \#\{i \mid f_i(t,t') = 0\}$$

edges. We connect c_t and $d(i,s,j)$ as follows. If $s = s(t)_i$ and $j = 1$, then we connect them by a line. If otherwise, they are not connected.

Finally, we connect $d(i,s,j)$ and $d(i',s',j')$ as follows. If $i = i'$, $s = s'$ and $|j - j'| = 1$, they are connected by a line. If otherwise, they are not connected.

The Dynkin diagram defined above is denoted by $D(P)$. The corresponding root lattice is denoted by $Q(P)$. We identify the basis of $Q(P)$ and the vertices of $D(P)$ and use same notations for them. The natural bilinear form on $Q(P)$ is denoted by $\langle , \rangle$.

For $\alpha \in Q(P)$, the reflection is $\sigma_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$ for $\beta \in Q(P)$.

Theorem 5.1. *Let us consider the* $\mathbb{Z}$*-linear morphism*

$$\Phi \colon Q(P) \longrightarrow L_P$$

defined as follows. For

$$\alpha = \sum_{t \in \mathcal{T}(P)} \alpha_t c_t + \sum_{i=0}^{p} \sum_{s \in \mathcal{I}_i} \sum_{j=1}^{l(s)-1} \alpha(i,s,j) d(i,s,j),$$

if we put $\Phi(\alpha) = \prod_{i=0}^{p} \prod_{s \in \mathcal{I}_i} \mathfrak{m}(s)$*, then for* $s \in \mathcal{I}_i$

$$m(s)_1 = \sum_{\{t \in \mathcal{T}(P) | s_i(t) = s\}} \alpha_t - \alpha(i,s,1),$$
$$m(s)_j = \alpha(i,s,j-1) - \alpha(i,s,j) \text{ for } 2 \leq j \leq l(s).$$

Then we have the followings.

1. *The* $\mathbb{Z}$*-linear map* Φ *is surjective.*
2. *Then for any* $\mathfrak{m} \in \mathcal{L}(P)$*, we have*

   $$\langle \alpha, \alpha \rangle = \langle \alpha', \alpha' \rangle$$

   for all $\alpha, \alpha' \in \Phi^{-1}(\mathfrak{m})$.
3. *We have*

   $$\Phi(\sigma_{c_t}\alpha) = \sigma(t)\Phi(\alpha),$$
   $$\Phi(\sigma_{d(i,s,j)}\alpha) = \sigma(i,s,j)\Phi(\alpha).$$

References

[1] Crawley-Boevey, W.: On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero. Duke Math. J. **118** (2003), no. 2, 339–352.

[2] Oshima, T.: Fractional calculus of Weyl algebra and Fuchsian differential equations. preprint.