Generalized Hydrogen Atoms and Symmetric Domains of the Tube Type

By Guowu Meng

§1. Introduction

It is a usual case that physicists study a particularly nice example, and mathematicians sort out the beautiful theory behind the example. One might paraphrase this by saying that physicists open the door and mathematicians make an extensive search of the house for gold.

While mathematicians are proud of their technical prowess, they never underestimate the initial contributions from physicists, as one can see from the commentary by A. Weil in [1979] on his book [1940]: "Above all I tried to open the way to a generalization of representation theory of finite groups and compact groups. However not only I did not attain the Land of Promise, the infinite dimensional representations, but also I could not catch a glimpse of the landscape. I was discouraged too early ...... Perhaps such a bold march might require the unprejudiced spirit of a physicist. In fact, they were Dirac, Wigner and Bargmann who opened the way concerning the Lorentz group."

The *Kepler problem* is a nice example by almost any criterions, so it is worth sorting out the beautiful mathematical theory behind it. The recent exploration reveals that the Kepler problem belongs to a vast family. In terms of the language used for integrable systems, the Kepler problem and its vast generalization are all *super integrable models* in the sense that the number of independent conserved quantities is bigger than the degree of freedom. Since the original (quantum) Kepler problem is the mathematical model for the hydrogen atom in the Minkowski space-time, one might say that these super integrable models are the mathematical model for the *generalized hydrogen atoms* in the generalized space-times. It turns out that these generalized space-times are nothing but the simple Euclidean Jordan algebras, which are known to be in one-to-one correspondence with *symmetric domains of the tube type*, cf. M. Koecher [1999].

A significant part of the theory behind the Kepler problem is about the unitary highest weight representations for the automorphism group (or rather its universal cover) of the symmetric domains of the tube type. It is now known that a (non-trivial) unitary highest weight representation can be realized as the Hilbert space of bound states for a super integrable model mentioned in the preceding paragraph if and only if it has the minimal Gelfand-Kirillov dimension.

In view of the recent progresses in the exploration on the Kepler problem, one might say that the

Kepler problem offers a new vantage point for a good view of the various branches of mathematics united behind symmetric domains of the tube type. This new vantage point does allow one to see a few things that are missing or overlooked in the literature about the mathematics around symmetric domains of the tube type.

The main motivation for the explorations of the Kepler problem is mathematics, not physics. However, the explorations do enable one to see the mathematical origin for the existence of the Kepler problem. Moreover, the explorations offer theoretical physicists a chance to see such things as the local quantum anomaly, quantum anomaly cancelation and S-duality at the quantum mechanics level. More importantly, in view of the fact the no one has observed a magnetic monopole, these mathematical explorations also lead to a speculation that the space-time at the fundamental level is non-supersymmetric and 27-dimensional. (This agrees with the dimension predicted by the M-theory, which is equal to 11 (bosonic) plus 16 (fermionic).)

This is enough about the generalities, here comes the concrete part.

§2 Historical Background

As many people know that the Kepler problem is a physics problem about two bodies which attract each other by a force inversely proportional to the distance. Mathematically, this is a mechanical system with configuration space $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$ and Lagrangian

$$L = \frac{1}{2}\mathbf{r}' \cdot \mathbf{r}' + \frac{1}{r}$$

where $\mathbf{r}$ is a function of time t taking value in $\mathbb{R}^3_*$, $r = |\mathbf{r}|$ and $\mathbf{r}'$ is the time derivative of $\mathbf{r}$. Therefore, quantum mechanically, the hamiltonian for the Kepler problem becomes

$$\hat{H} = -\frac{1}{2}\Delta - \frac{1}{r}. \tag{1}$$

Here, Δ is the Laplace operator on $\mathbb{R}^3$ and $r = r(x)$ is the distance from $x \in \mathbb{R}^3_*$ to the origin of $\mathbb{R}^3$.

Physicists are interested in solving the bound state eigenvalue problems for $\hat{H}$, i.e., 1) finding the list of real numbers $\lambda_0 < \lambda_1 < \cdots$ such that

$$\mathscr{H}_\lambda := \{\psi \in C^\infty(\mathbb{R}^3_*, \mathbb{C}) \mid \int_{\mathbb{R}^3_*} |\psi|^2 \, d^3\vec{r} < \infty, \hat{H}\psi = \lambda\psi\}$$

is nontrivial if and only if λ is one of λ_I; 2) determining each $\mathscr{H}_{\lambda_I}$. It is well known that, for the hamiltonian in eq. (1), the answer is very simple:

$$\lambda_I = -\frac{1/2}{(I+1)^2}, \quad I = 0, 1, \ldots$$

and $\mathscr{H}_I$ is an irreducible representation of $\mathrm{SO}(4)$. What is less well known is the discovery of A. Barut – G. Bornzin [1971] which essentially says that

$$\mathcal{H} := \bigoplus_{I=0}^{\infty} \mathscr{H}_I$$

is a unitary highest weight $(\mathfrak{so}(6), \mathrm{SO}(4) \times \mathrm{SO}(2))$-module in the sense of Harish-Chandre [1953]. Hence, it can be integrated to a unitary highest weight module for $\mathrm{SO}_0(2,4)$ (the identity component of $\mathrm{SO}_0(2,4)$); moreover, this module for $\mathrm{SO}_0(2,4)$ is minimal in the sense of A. Joesph [1974] and has $L^2(\mathbb{R}^3_*, \frac{1}{r} d^3\vec{r})$ as a geometric realization.

An apparent mathematical generalization, known to many people, is to replace $\mathbb{R}^3$ by $\mathbb{R}^n$ $(n \geq 2)$ and keep the hamiltonian in the same form. Similar results are valid: $\lambda_I = -\frac{1/2}{(I+\frac{n-1}{2})^2}$, $\mathscr{H}_I$ is an irreducible representation of $\mathrm{SO}(n+1)$, and $\bigoplus_{I=0}^{\infty} \mathscr{H}_I$ is a unitary highest weight $(\mathfrak{so}(n+3), \mathrm{SO}(n+1) \times \mathrm{SO}(2))$-module, hence it can be integrated to a unitary highest weight module for $\mathrm{SO}_0(2, n+1)$ (actually a double of it when n is even); moreover, this module is minimal and has $L^2(\mathbb{R}^n_*, \frac{1}{r} d^n\vec{r})$ as a geometric realization.

By the end of the 1960's the beautiful magnetized version of the Kepler problem was discovered by D. Zwanziger [1968], and independently by H. McIntosh-A. Cisneros [1970]. Here, the configuration space is still $\mathbb{R}^3_*$, but the hamiltonian in eq. (1) is replaced by

$$\hat{H} = -\frac{1}{2}\Delta_{\vec{A}} + \frac{\mu^2}{2r^2} - \frac{1}{r},$$

where μ is the half of an integer, $\vec{A}$ is a vector field on $\mathbb{R}^3$ minus a half line from the origin such that $\nabla \times \vec{A} = \mu \frac{\vec{r}}{r^3}$ and $\Delta_A = \sum_{i=1}^3 (\partial_i + \sqrt{-1} A_i)^2$. (To a physicist, $\vec{A}$ is the magnetic potential in a particular gauge when there is the a magnetic monopole of charge μ sitting at the origin of $\mathbb{R}^3$.) Similar results are valid: $\lambda_I = -\frac{1/2}{(I+1+|\mu|)^2}$, $\mathscr{H}_I$ is an irreducible representation of $\mathrm{Spin}(4)$, and $\bigoplus_{I=0}^{\infty} \mathscr{H}_I$ is a unitary highest weight $(\mathfrak{so}(6), \mathrm{Spin}(4) \times_{\mathbb{Z}_2} \mathrm{Spin}(2))$-module, hence it can be integrated to a unitary highest weight module for $\mathrm{Spin}(2,4)$, etc.

§3 Recent Explorations

Naturally, one may ask whether the higher dimensional magnetized versions exist or not. The five dimensional magnetized version, which is based on symmetry reduction, were found by T. Iwai

[1990]. The consensus before 2005 was that the Kepler problem has magnetized versions in dimension three and dimension five, but not in any other dimension. However, by reformulating the problems in geometric terms, G. W. Meng [2007] found that the magnetized versions of the Kepler problem actually exist in all dimensions. One reason for this oversight by the experts is perhaps that it was believed that the division algebra, which exists only in a few low dimensions, is the key to the existence of the magnetic monopoles. But this turns out to be not quite the case: what really matters is the Clifford algebra, not the division algebra; and unlike the division algebra, the Clifford algebra exists in any dimension.

Here are some further details. Let $D \geq 3$ be an integer, $\mathbb{R}^D_*$ be the punctured D-space, i.e., $\mathbb{R}^D$ with the origin removed. Let $r\colon \mathbb{R}^D_* \to \mathrm{S}^{D-1}$ be the retraction which maps $\vec{x} \in \mathbb{R}^D_*$ to $\frac{\vec{x}}{|\vec{x}|} \in \mathrm{S}^{D-1}$. Via r, the principal $\mathrm{Spin}(D-1)$-bundle

$$\mathrm{Spin}(D-1) \to \mathrm{Spin}(D) \to \mathrm{S}^{D-1}$$

gets pulled back to a principal $\mathrm{Spin}(D-1)$-bundle over $\mathbb{R}^D_*$. The $\mathrm{Spin}(D)$-invariant connection on $\mathrm{Spin}(D-1) \to \mathrm{Spin}(D) \to \mathrm{S}^{D-1}$ gives rise to a $\mathrm{Spin}(D)$-invariant connection on the pullback bundle. Therefore, to each irreducible unitary representation σ for $\mathrm{Spin}(D-1)$, there is a hermitian vector bundle over $\mathbb{R}^D_*$, with a $\mathrm{Spin}(D)$-invariant unitary connection. Here σ plays the role of magnetic charge. In order to maintain the hidden symmetry of the Kepler problem in dimension D, σ must be special: when $D-1$ is even, σ is the representation with highest weight of the form $(|\mu|, \ldots, |\mu|, \mu)$, where μ is the half of an integer; when $D-1$ is odd, σ is either the trivial representation or the fundamental spin representation, so its highest weight is $(\mu, \ldots, \mu)$, where μ is 0 or $1/2$. The resulting hermitian vector bundle over $\mathbb{R}^D_*$ shall be denoted by $\mathcal{S}^{2\mu}$. Here are the magnetized models introduced by G. W. Meng [2007]:

Definition 1 (Generalized MICZ-Kepler problem)**.** *The D-dimensional generalized MICZ-Kepler problem with magnetic charge μ is defined to be the quantum mechanical system on $\mathbb{R}^D_*$ for which the wave-functions are sections of $\mathcal{S}^{2\mu}$, and the hamiltonian is*

$$\hat{H} = \begin{cases} -\frac{1}{2}\Delta_\mu + \frac{(n-1)|\mu|+\mu^2}{2r^2} - \frac{1}{r} & \textit{if } D = 2n+1 \\ \\ -\frac{1}{2}\Delta_\mu + \frac{(n-1)\mu}{2r^2} - \frac{1}{r} & \textit{if } D = 2n \end{cases}$$

where Δ_μ is the Laplace operator twisted by $\mathcal{S}^{2\mu}$.

The subsequent investigation by G. W. Meng - R. B, Zhang [2007E], G. W. Meng [2007E, 2008] shows that the above magnetized models are in one-to-one correspondence with the family of unitary highest weight representations of $\widetilde{\mathrm{SO}}_0(2, D+1)$ with the minimal (positive) Gelfand-Kirillov dimension. Consequently, one obtains both a geometric realization and an algebraic characterization for this family of representations. To be more specific, the following conclusions have been deduced:

1) The members of this family can be precisely realized as the Hilbert space of bound states for magnetized Kepler problems in dimension D;

2) Each member of this family can be realized as the Hilbert space of L^2-sections of a canonical hermitian bundle over $\mathbb{R}^D_*$ equipped with a canonical unitary connection;

3) This family can be characterized by a canonical finite set of quadratic relations among the infinitesimal generators of $\mathrm{SO}(2, D+1)$.

A by-product of the investigation of the magnetized models is a natural correspondence

$$(\mathrm{Spin}(D-1), \sigma) \longleftrightarrow (\widetilde{\mathrm{SO}}(2, D+1), \sigma'), \qquad (\mathbf{2})$$

which resembles Howe's local theta-correspondence in representation theory. Here, σ, an irreducible representation of the gauge group $\mathrm{Spin}(D-1)$, plays the role of magnetic charge; and σ', a unitary highest weight representation of $\widetilde{\mathrm{SO}}(2, D+1)$, is realized by the Hilbert space of bound states of the magnetized model for which the background magnetic charge is σ.

Recall that, in the theory of theta-correspondence of R. Howe [1985] (cf. J. S. Li [2000]), one begins with an (irreducible) *reductive dual pair* (G, G') inside a sympletic group Sp. In the cases of concern here, the group G will be compact while the other member, G', is not. This will be tacitly assumed in what follows.

The theory of theta-correspondence says that there is a bijection between a certain set of irreducible unitary representations, σ, of $\tilde{G}$ and a certain set of irreducible unitary representations, σ', of $\tilde{G}'$:

$$(\tilde{G}, \sigma) \overset{\theta}{\longleftrightarrow} (\tilde{G}', \sigma').$$

The correspondence is a consequence of the following special case of a <u>theorem of Howe</u>:

$$\vartheta\,\big|_{\tilde{G}\times\tilde{G}'} \cong \bigoplus_{\sigma\in\Lambda} \sigma\otimes\sigma'.$$

Here ϑ is the Weil representation of the metaplectic group, $\widetilde{\mathrm{Sp}}$, and Λ is a certain subset of the set of unitary irreducible representations of $\tilde{G}$. The correspondence $\sigma \leftrightarrow \sigma'$ is a bijection.

Note that $(\mathrm{SO}(n), \mathrm{SO}(2, n+2))$ is not a reductive dual pair inside a symplectic group, so the correspondence in eq. (2) is new because it is not a local theta-correspondence.

At this point, one might be wondering whether there is a super integrable model which yields a genuine local theta-correspondence. For that, one needs to go back to the magnetized Kepler problems introduced in definition 1. Notice that, in the construction of these magnetized Kepler problems, a key role is played by the fiber bundle:

$$\mathrm{Spin}(n) \to \mathrm{Spin}(n+1) \to \mathrm{S}^n;$$

moreover, S^n belongs to the following list of the compact rank-one symmetric spaces (cf. the book by A. Knapp [2002]):

$$\mathrm{S}^n, \quad \mathbb{R}\mathrm{P}^n, \quad \mathbb{C}\mathrm{P}^n, \quad \mathbb{H}\mathrm{P}^n, \quad \mathbb{O}\mathrm{P}^k$$

where $n \geq 1$ and $k = 1$ or 2.

The construction of super-integrable models, in view of the work by G. W. Meng [2007], is already known when the symmetric spaces are spheres, so it is known when the symmetric spaces are $\mathbb{R}\mathrm{P}^1$, $\mathbb{C}\mathrm{P}^1$, $\mathbb{H}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^1$ because of the obvious isometries:

$$\mathbb{R}\mathrm{P}^1 \cong \mathrm{S}^1, \quad \mathbb{C}\mathrm{P}^1 \cong \mathrm{S}^2, \quad \mathbb{H}\mathrm{P}^1 \cong \mathrm{S}^4, \quad \mathbb{O}\mathrm{P}^1 \cong \mathrm{S}^8.$$

The tricky part of the job is then to somehow reformulate the known construction properly in the case when the compact symmetric spaces are projective lines. The proper reformulation should be the one such that its extension to the general case when the compact symmetric spaces are projective spaces is automatic.

The cases with real, complex and quaternionic projective spaces have been successfully worked out by G. W. Meng [2008/2009]. A surprise found here is that these new models are closely related to the harmonic oscillators, and connections with the theta-correspondence emerge naturally. To be specific, the reductive dual pairs appearing in these new super integrable models are

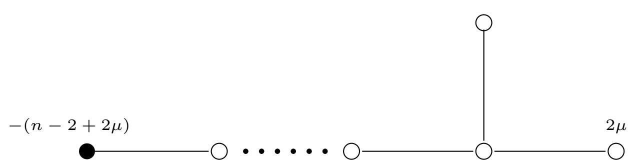
$$(\mathrm{O}(1), \mathrm{Sp}(2n, \mathbb{R})), \quad (\mathrm{U}(1), \mathrm{U}(n, n)), \quad (\mathrm{Sp}(1), \mathrm{SO}^*(4n)), \tag{0.1}$$

which correspond to super integrable models based on $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, respectively.

The unique exceptional case, i.e., the super integrable model based on $\mathbb{O}\mathrm{P}^2$, has been recently worked out by J. S. Li – G. W. Meng [2009E]. Here, the pair of groups is $(\mathrm{Spin}(9), E_{7(-25)})$, which is obviously not a reductive dual pair inside a symplectic group, so the correspondence in this case is not a theta-correspondence. However, in this case, the magnetized version does not exist.

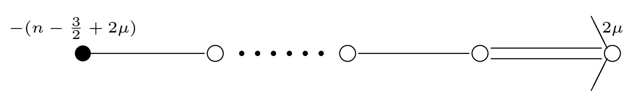
In summary, for each rank-one compact symmetric space, there is a super integrable model that resembles the Kepler problem. Moreover, the magnetized versions of these super integrable models exist except when the compact symmetric space is $\mathbb{O}\mathrm{P}^2$. For a fixed rank-one compact symmetric space $\mathbb{P}$, the totality of the corresponding super integrable model and its magnetized cousins shall be called the *Kepler hierarchy* based on $\mathbb{P}$. Then it is clear that there is a one-to-one correspondence between the Kepler hierarchies and the rank-one compact symmetric spaces; moreover, the unitary highest weight modules realized by the Hilbert space of bound states of Kepler hierarchies can be summarized pictorially as follows: in a Vogan diagram below, the black node is the non-compact imaginary simple root and the white nodes are the compact imaginary simple roots. The highest weights are expressed in terms of Dynkin indexes: the number nearing a Dynkin node is the corresponding Dynkin index, but if there is no number near a Dynkin node, then the corresponding Dynkin index is zero.

1. $\widetilde{\mathrm{SO}}(2, 2n-2)$



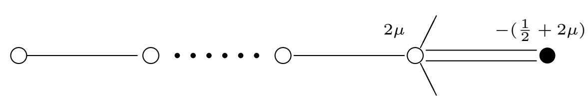
$\mu = 0, \frac{1}{2}, 1, \ldots$

1'. $\widetilde{\mathrm{SO}}(2, 2n-1)$



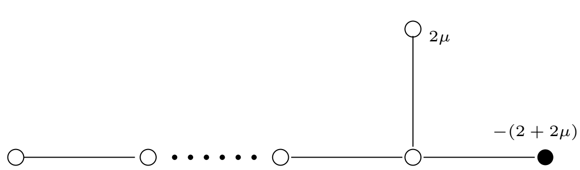
$\mu = 0, \frac{1}{2}$.

2. $\widetilde{Sp}(2n, \mathbb{R})$



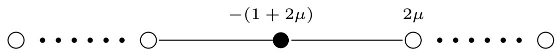
$\mu = 0, \frac{1}{2}$.

3. $\widetilde{\mathrm{SO}^*}(4n)$



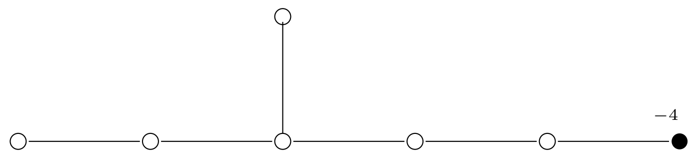
$\mu = 0, \frac{1}{2}, 1, \ldots$

4. $\widetilde{U}(n, n)$



$\mu = 0, \frac{1}{2}, 1, \ldots$

5. $\widetilde{E}_{7(-25)}$



Note that a unitary highest weight representation is realized by the Hilbert space of bound states of a model from a Kepler hierarchy if and only if it has the minimal positive Gelfand-Kirillov dimension.

§4 The Current Development

At this point, one might say that there is a fairly complete theory about the Kepler problem. Yes, it is the case in some sense. However, the journey does not end here; in fact, one has just arrived at the *symmetric domain* "town center". The non-compact simple real Lie groups appearing in the above diagrams are essentially nothing but the automorphism groups for the symmetric domains of the tube type. Is this just a pure coincidence or is there some good reason for it? It turns out there is a good reason for it, so one is entering another exciting phase of this journey.

Recall that Kepler hierarchies, (irreducible) tube domains, and (simple) Euclidean Jordan algebra all consist of *four infinity families and one exceptional*. It was discovered by Koecher in the 1950's that tube domains and the Euclidean Jordan algebras are naturally associated with each other. So one just needs to see that Kepler hierarchies and the Euclidean Jordan algebras are naturally associated with each other, too. Here is a quick way to see this, cf. G. W. Meng [2009E] for more details.

For simplicity, only the models with zero magnetic charges are described here. These models very much look like the Kepler problem, so they are called the *J-Kepler problems*. One begins with the notion of *Kepler cone* for a simple Euclidean Jordan algebra. By definition it is the manifold consisting of the rank-one semi-positive elements of the Jordan algebra, equipped with a suitable Riemannian metric. This Kepler cone plays the role of $\mathbb{R}^3_*$. To define the J-Kepler problem, one needs to replace the hamiltonian in eq. (1) by this one:

$$\hat{H} = -\frac{1}{2}\Delta - \left(\frac{B}{2r^2} + \frac{1}{r}\right).$$

Here, Δ is the (non-positive) Laplace operator on the Kepler cone, $r = r(x)$ is the inner product of x (in the Kepler cone) with the identity element of the Jordan algebra, and B is a constant depending on the Jordan algebra, for example, $B = 26$ for the exceptional Jordan algebra. Note that, when the Jordan algebra is the Jordan algebra of complex hermitian 2×2-matrices, the J-Kepler problem becomes the original Kepler problem.

An obvious by-product of this connection of Kepler hierarchies and the Euclidean Jordan algebras is that the theory of Kepler hierarchies has a clean uniform treatment now. For example, the I-th energy eigenvalue of the J-Kepler problem is

$$\lambda_I = -\frac{1/2}{(I+\rho d/4)^2}.$$

Here, ρ and d are the rank and degree of the Jordan algebra, respectively.

As a historical note, the Euclidean Jordan algebras were initially introduced by P. Jordan [1933] in the 1930's for the purpose of reformulating quantum mechanics in a minimal way. Physicists quickly lost interest in Jordan algebras, but the subsequent further explorations taken on by mathematicians turn out to be quite fruitful. One hopes that the newly found intimate relationship between Kepler hierarchies and the Euclidean Jordan algebras may attract the attention of some physicists to the Jordan algebras again.

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