Howe Duality Correspondence of $(O(p,q), \mathfrak{osp}(2,2))$

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Abstract

The local theta correspondence for reductive dual pairs of subgroups of the symplectic group formulated by R. Howe has repeatedly provided insight into representation theory of reductive groups and the theory of automorphic forms. The most heavily studied example of such a pair is $(O(p,q), SL(2,\mathbb{R}))$.

In this talk we will discuss a variant of this pair, involving the indefinite orthogonal groups $O(p,q)$ and the Lie superalgebra $\mathfrak{osp}(2,2)$. We will classify the irreducible representations of $\mathfrak{osp}(2,2)$ subject to some technical restrictions and construct the correspondence for the pair $(O(p,q), \mathfrak{osp}(2,2))$ in the oscillator-spin representation. The Casimir operator plays an important role in setting up the correspondence.

1 Introduction

In the paper [H1], Roger Howe formulated the notion of a reductive dual pair of subgroups of the symplectic group. *A reductive dual pair* is a pair of subgroups (G, G') of the symplectic group such that G' is the centralizer of G and vice versa. Let ω be the *osillator representation* of $Sp(2n,\mathbb{R})$ realized on a Hilbert space $\mathcal{H}$. Howe has shown that for a real reductive dual pair $(G, G') \in Sp(2n,\mathbb{R})$, there is a bijection between some irreducible admissible representations of $\tilde{G}$(a double cover of G) and some representations of $\tilde{G}'$(a double cover of G') which can be realized as quotients by $\omega^\infty(Sp)$-invariant closed subspaces of $\mathcal{H}^\infty$ where $\mathcal{H}^\infty$ is the subspace of smooth vectors in $\mathcal{H}$. We call this bijection the *oscillator duality correspondence* or *Howe correspondence*. After that a lot of work has been done for this particular representation. Many explicit constructions of the oscillator duality correspondence have been established for various dual pairs.

In that paper, Howe also formulated the notion of a reductive dual pair in the orthosymplectic Lie superalgebra and the oscillator representation was extended to a representation for the orthosymplectic Lie superalgebra, which we call *oscillator-spin representation* for the orthosymplectic Lie superalgebra denoted by $\widetilde{\omega}$. The object of this talk is to discuss the Howe correspondence in the oscillator-spin representation of $(O(p,q), \mathfrak{osp}(2,2)) \subset \mathfrak{osp}(2(p+q), 2(p+q))$. In this representation, the Casimir operators of $O(p,q)$ and $\mathfrak{osp}(2,2)$ are exactly the same. We show that the Howe correspondence exists for the representations with nonzero eigenvalues for the Casimir operators.

When $q = 1$, we can even give the Howe correspondence of the representations with zero eigenvalue for the Casimir operators. Thus the Howe correspondence is completely proved in this case.

2 Structure of $\mathfrak{osp}(2,2)$ and its irreducible representations

In order to get the correspondence of $(O(p,q),\ \mathfrak{osp}(2,2))$ in the oscillator-spin reprensentation, we first study the structure of this Lie superalgebra and classify its irreducible representations subject to some technical restrictions. The irreducible $\mathfrak{osp}(2,2)$-representations which can appear in the correspondence belong to this class.

We begin with a short review of representations of $\mathfrak{sl}(2,\mathbb{R})$. Let $\{h, e^+, e^-\}$ be the standard basis of $\mathfrak{sl}(2,\mathbb{R})$ and $\Omega_{\mathfrak{sl}_2} = h^2 + 2(e^+e^- + e^-e^+)$ be its Casimir operator. We will describe all irreducible h-semisimple, h-admissible and $\Omega_{\mathfrak{sl}_2}$-quasisimple $\mathfrak{sl}_2$-modules in this section. This is a classical result which has been known for a long time.

Theorem 2.1. ([HT2]) *All h-admissible, h-semisimple quasisimple irreducible $\mathfrak{sl}_2$-modules are classified into the following 4 classes:*
1) $V_\lambda\,, \lambda \notin \mathbb{Z}_-$ 2) $\bar{V}_\lambda\,, \lambda \notin \mathbb{Z}^+$ 3) $F_{-\lambda}, \lambda \in \mathbb{Z}^+$ 4) $U(\nu^+,\nu^-)$ $\nu^+,\nu^- \notin \mathbb{Z}$

Theorem 2.2. ([Kn2]) *All irreducible representations of $SL(2,\mathbb{R})$ must be infinitesimally equivalent to one of the following:*
1) Discrete series $V_\lambda, \lambda \in \mathbb{Z}_+; \bar{V}_\lambda, \lambda \in \mathbb{Z}_-$.
2) Finite dimensional representations F_{-m}, $m \in \mathbb{Z}_+$.
3) Principal series $S^{s,+} = U(\frac{s}{2}, \frac{s}{2})$, $s \notin 2\mathbb{Z}$, $S^{s,-} = U(\frac{s+1}{2}, \frac{s-1}{2})$, $s \notin 2\mathbb{Z}+1$

Since the fundamental group of $SL(2,\mathbb{R})$ is $\mathbb{Z}$, for each $n \in \mathbb{N}$, there exists an n-fold covering group $\widetilde{SL}^n(2,\mathbb{R})$ of $SL(2,\mathbb{R})$. In this talk, we are particularly interested in the 2-fold covering of $SL(2,\mathbb{R})$ denoted by $\widetilde{SL}(2,\mathbb{R})$. This is the group which relates to the representation of $\mathfrak{osp}(2,2)$ in the duality correspondence $(O(p,q), \mathfrak{osp}(2,2))$. We say a representation of $\widetilde{SL}(2,\mathbb{R})$ is genuine if it does not factor to $SL(2,\mathbb{R})$.

Corollary 2.3. *The irreducible genuine $\widetilde{SL}(2,\mathbb{R})$ representations are infinitesimally equivalent to the followings:*
1) Lowest weight modules, V_λ, $\lambda \in \mathbb{Z} + \frac{1}{2}$.
2) Highest weight modules, $\bar{V}_\lambda$, $\lambda \in \mathbb{Z} + \frac{1}{2}$.
3) Principal series $U(\frac{\nu}{2} + \frac{1}{4}, \frac{\nu}{2} + \frac{3}{4})$ or $U(\frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + \frac{1}{4})$, $\nu \notin \mathbb{Z} + \frac{1}{2}$.

Now we look at $\mathfrak{osp}(2,2)$ in detail. First we give the matrix form of the elements $\mathfrak{osp}(2,2)$. Form [K], we see a matrix in $\mathfrak{osp}(2,2)$ is of the form

$$\left(\begin{array}{cc|cc} k & 0 & x & x_1 \\ 0 & -k & y & y_1 \\ \hline y_1 & x_1 & a & b \\ -y & -x & c & -a \end{array}\right) \tag{2.1}$$

$$\mathfrak{osp}(2,2)_0 = \left\{ \left(\begin{array}{cc|cc} k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & -a \end{array}\right) \cong \mathfrak{sl}_2 \oplus \mathfrak{so}_2 \right\}$$

$$\mathfrak{osp}(2,2)_1 = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & x & x_1 \\ 0 & 0 & y & y_1 \\ \hline y_1 & x_1 & 0 & 0 \\ -y & -x & 0 & 0 \end{array} \right) \right\}$$

Again we let

$$e^+ = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad e^- = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \quad h = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) \tag{2.2}$$

be a basis for $\mathfrak{sl}_2$. Additionally, the element

$$h' = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \tag{2.3}$$

is a basis for $\mathfrak{so}_2$.

Define

$$d = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) d^* = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right) \delta = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \delta^* = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \tag{2.4}$$

After simple calculation, we have the following commutation relations. Since for $x, y \in \mathfrak{osp}_1(2,2)$, $[x,y] = xy + (-1)^{1\times 1}yx = xy + yx = \{x,y\}$, we use $\{,\}$ to indicate the operation for $x, y \in \mathfrak{osp}_1(2,2)$.

For the Lie subalgebra $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$:

$$[h, e^+] = 2e^+ \qquad [h, e^-] = -2e^- \qquad [e^+, e^-] = h \qquad [h', x] = 0$$

Here x is anything in $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$.
Between the Lie algebra and the super part.

$$\begin{array}{llll} [e^+, d] = \delta^* & [e^+, \delta^*] = 0 & [e^+, d^*] = \delta & [e^+, \delta] = 0 \\ [e^-, d] = 0 & [e^-, \delta^*] = d & [e^-, d^*] = 0 & [e^-, \delta] = d^* \\ [h, d] = -d & [h, \delta^*] = \delta^* & [h, d^*] = -d^* & [h, \delta] = \delta \\ [h', d] = d & [h', \delta^*] = \delta^* & [h', d^*] = -d^* & [h', \delta] = -\delta \end{array} \tag{2.5}$$

Anticommutation relations for the super part.

$$d^2 = \{d, \delta^*\} = (\delta^*)^2 = 0 \qquad (d^*)^2 = \{d^*, \delta\} = \delta^2 = 0$$

$$\{d, d^*\} = -2e^- \qquad \{d, \delta\} = -h - h' \qquad \{d^*, \delta^*\} = -h + h' \qquad \{\delta, \delta^*\} = 2e^+ \tag{2.6}$$

From the above commutation relations, we get:

Theorem 2.4. $\mathfrak{osp}(2,2)$ *is a classical Lie superalgebra. The pairs* $\{d, \delta^*\}$ *and* $\{d^*, \delta\}$ *each span a two dimensional irreducible module for* $\mathfrak{sl}_2$ *with eigenvalue* $+1$ *and* -1 *for* h' *respectively.*

As for $\mathfrak{sl}_2$, it is useful to know the center of $\mathfrak{U}(\mathfrak{osp}(2,2))$.

Theorem 2.5. *Define*

$$\Omega_{\mathfrak{osp}} = \Omega_{\mathfrak{sl}_2} - h'^2 - ([d,\delta] + [d^*, \delta^*])$$

in $\mathfrak{U}(\mathfrak{osp}(2,2))$. *Then* $\Omega_{\mathfrak{osp}}$ *is in the center of* $\mathfrak{U}(\mathfrak{osp}(2,2))$, *in other words,* $\Omega_{\mathfrak{osp}}$ *commutes with every element in* $\mathfrak{U}(\mathfrak{osp}(2,2))$. *Moreover,* $\Omega_{\mathfrak{osp}}$ *generates the center of* $\mathfrak{U}(\mathfrak{osp}(2,2))$ *as an (associative) algebra, i.e.* $Z(\mathfrak{U}(\mathfrak{osp}(2,2)) = \mathbb{C}[\Omega_{\mathfrak{osp}}]$.

Having dealt with the structure of $\mathfrak{osp}(2,2)$ and given the commutation relations for the basis vectors of $\mathfrak{osp}(2,2)$, we are able to classify its h-admissible, h, h'-semisimple $\Omega_{\mathfrak{osp}}$-quasisimple irreducible representations,

Since the Lie part is a direct sum of $\mathfrak{sl}_2$ and $\mathfrak{so}_2$, we classify the irreducible $\mathfrak{osp}(2,2)$-modules according to their structures considered as $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-modules. From Theorem 2.1, we know the h-semisimple, h-admissible quasisimple irreducible representations of $\mathfrak{sl}_2$ are classified into four different classes. Since $\mathfrak{so}_2$ is an abelian Lie algebra, so its irreducible representation can be identified with $\mathbb{C}$. Also since $\mathfrak{so}_2$ commutes with $\mathfrak{sl}_2$, each h, h'-semisimple irreducible $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-module has only one eigenvalue for h'. We use $V_{\lambda,\mu}$, $\bar{V}_{\lambda,\mu}$, $F_{\lambda,\mu}$, $U(\nu^+, \nu^-, \mu)$ to denote the irreducible $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-modules of the same $\mathfrak{sl}_2$-types as in Theorem 2.1 with h'-eigenvalue μ. We will analyze h-admissible h, h'-semisimple quasisimple $\mathfrak{osp}(2,2)$-modules which contain $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-modules from these four different classes.

For $\mathfrak{osp}(2,2)$, we consider $\mathfrak{q} = \mathfrak{sl}_2 \oplus \mathfrak{so}_2 \oplus E$ where E is the span of d^* and δ. It is easy to check that $\mathfrak{q}$ is a Lie subsuperalgebra, and actually we can get the following theorem which is helpful for us to classify irreducible h-admissible, h, h'-semisimple, $\Omega_{\mathfrak{osp}}$-quasisimple $\mathfrak{osp}(2,2)$-modules.

Theorem 2.6. *Suppose* V *is an* h, h'*-semisimple,* h*-admissible,* $\Omega_{\mathfrak{osp}}$*-quasisimple, irreducible* $\mathfrak{osp}(2,2)$*-module.*

a) Set $W_1 = \ker d^* \cap \ker \delta$. *Then* W_1 *is invariant under* $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$, *and is a non-zero, irreducible,* h, h'*-semisimple, quasisimple* $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$*-module.*

b) Set $W = \mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{osp}(2,2)} W_1$. *Then* W *is an indecomposable* $\mathfrak{osp}(2,2)$ *module, and* V *is a quotient of* W.

Theorem 2.7. *Suppose* V *is an* h, h'*-semisimple,* h*-admissible,* $\Omega_{\mathfrak{osp}}$*-quasisimple, irreducible* $\mathfrak{osp}(2,2)$*-module and* $W_1 = \ker d^* \cap \ker \delta$. *Let* $W = \mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{osp}(2,2)} W_1$. *Then* W *is irreducible if and only if* $\Omega_{\mathfrak{osp}} W \neq 0$.

Based these two Theorems above, we are able to classify the irreducible $\mathfrak{osp}(2,2)$-modules. We omit elaborate computations here.

1) Irreducible lowest weight modules $W(\lambda, \mu)(\lambda \notin \mathbb{Z}^-)$.

i) $\mu \neq \pm\lambda$

W is composed of four $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-modules depicted below.

$$\begin{tikzcd}[column sep=small]
 & V_{\lambda+1,\mu+1} \arrow[dl, dash] \arrow[dr, dash] & \\
V_{\lambda,\mu} \arrow[dr, dash] & & V_{\lambda+2,\mu} \arrow[dl, dash] \\
 & V_{\lambda+1,\mu-1} &
\end{tikzcd}$$

ii) $\mu = \lambda \, (\lambda \neq 0)$

$$W(\lambda,\lambda) = \begin{tikzcd} V_{\lambda,\lambda} \arrow[dr, dash] & \\ & V_{\lambda+1,\lambda-1} \end{tikzcd}$$

iii) $\mu = -\lambda \, (\lambda \neq 0)$

$$W(\lambda,-\lambda) = \begin{tikzcd} & V_{\lambda+1,-\lambda+1} \arrow[dl, dash] \\ V_{\lambda,-\lambda} & \end{tikzcd}$$

2)Irreducible highest weight modules $\bar{W}(\lambda,\mu) \, (\lambda \notin \mathbb{Z}^+)$.
i) $\mu \neq \pm\lambda$ W is composed of four $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-modules depicted below.

$$\begin{tikzcd}[column sep=small]
 & \bar{V}_{\lambda-1,\mu+1} \arrow[dl, dash] \arrow[dr, dash] & \\
\bar{V}_{\lambda,\mu} \arrow[dr, dash] & & \bar{V}_{\lambda-2,\mu} \arrow[dl, dash] \\
 & \bar{V}_{\lambda-1,\mu-1} &
\end{tikzcd}$$

ii) $\mu = \lambda \, (\lambda \neq 0)$.

$$\bar{W}(\lambda,\lambda) = \begin{tikzcd} & \bar{V}_{\lambda-1,\lambda+1} \arrow[dl, dash] \\ \bar{V}_{\lambda,\lambda} & \end{tikzcd}$$

iii) $\mu = -\lambda \, (\lambda \neq 0)$

$$\bar{W}(\lambda,-\lambda) = \begin{tikzcd} \bar{V}_{\lambda,-\lambda} \arrow[dr, dash] & \\ & \bar{V}_{\lambda-1,-\lambda-1} \end{tikzcd}$$

3) Irreducible finite dimensional modules $F_{\mathfrak{osp}}(\lambda,\mu) \, (\lambda \in \mathbb{Z}^-)$.
i) Trivial module $F_{\mathfrak{osp}}(0,0)$.
ii) $\mu \neq \pm\lambda$,
a) $\lambda \neq -1$, $F_{\mathfrak{osp}}(\lambda,\mu)$ is composed of four finite dimensional $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-module.

$$\begin{tikzcd}[column sep=small]
 & F_{\lambda+1,\mu+1} \arrow[dl, dash] \arrow[dr, dash] & \\
F_{\lambda,\mu} \arrow[dr, dash] & & F_{\lambda+2,\mu} \arrow[dl, dash] \\
 & F_{\lambda+1,\mu-1} &
\end{tikzcd}$$

b) $\lambda = -1$,

$$F_{\mathfrak{osp}}(-1,\mu)|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong F_{-1,\mu} \oplus F_{0,\mu-1} \oplus F_{0,\mu+1}$$

iii) $\mu = -\lambda\,(\lambda \neq 0)$

$$
F_{\mathfrak{osp}}(\lambda, -\lambda) = \begin{tikzcd}[column sep=small] & F_{\lambda+1,-\lambda+1} \\ F_{\lambda,-\lambda} \arrow[ur, dash] & \end{tikzcd}
$$

iv) $\mu = \lambda\,(\lambda \neq 0)$

$$
F_{\mathfrak{osp}(\lambda,\lambda)} = \begin{tikzcd}[column sep=small] F_{\lambda,\lambda} \arrow[dr, dash] & \\ & F_{\lambda+1,\lambda-1} \end{tikzcd}
$$

4) Irreducible modules $W(\nu^+, \nu^-, \mu)$, $\nu^+, \nu^- \notin \mathbb{Z}$, $\mu \in \mathbb{C}$
i) $\mu \neq \pm(\nu^+ + \nu^-)$
W is composed of four $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-modules depicted below.

$$
\begin{tikzcd}[column sep=small] & U(\nu^+, \nu^- + 1, \mu + 1) \arrow[dl, dash] \arrow[dr, dash] & \\ U(\nu^+, \nu^-, \mu) \arrow[dr, dash] & & U(\nu^+ + 1, \nu^- + 1, \mu) \arrow[dl, dash] \\ & U(\nu^+, \nu^- + 1, \mu - 1) & \end{tikzcd}
$$

ii) $\mu = \nu^+ + \nu^-$
$W|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(\nu^+, \nu^-, \mu) \oplus U(\nu^+, \nu^- + 1, \mu - 1)$
iii) $\mu = -(\nu^+ + \nu^-)$
$W|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} \cong U(\nu^+, \nu^-, \mu) \oplus U(\nu^+, \nu^- + 1, \mu + 1)$

Theorem 2.8. *All h, h'-semisimple, h-admissible, $\Omega_{\mathfrak{osp}}$-quasisimple irreducible representations of $\mathfrak{osp}(2,2)$ are classified into four groups listed in the lemmas above:*
1) $W(\lambda, \mu), \lambda \in \mathbb{C}, \lambda \notin \mathbb{Z}^-, \mu \in \mathbb{C}$;
2) $\bar{W}(\lambda, \mu)$, $\lambda \notin \mathbb{Z}^+$, $\mu \in \mathbb{C}$;
3) $F_{\mathfrak{osp}}(\lambda, \mu)$, $\lambda \in \mathbb{Z}^-$;
4) $W(\nu^+, \nu^-, \mu)$, $\nu^+, \nu^- \notin \mathbb{Z}$, $\mu \in \mathbb{C}$.

Proof. In Theorem 2.6, we proved that each h-admissible, h, h'-semisimple, $\Omega_{\mathfrak{osp}}$-quasisimple irreducible $\mathfrak{osp}(2,2)$-module V is a quotient of an induced module from an irreducible $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-module which is annihilated by d^* and δ. And we have found all irreducible quotients in the $\mathfrak{osp}(2,2)$-modules induced from four different classes of $\mathfrak{sl}_2 \oplus \mathfrak{so}_2$-modules. Thus V must be isomorphic to one of the $\mathfrak{osp}(2,2)$-modules listed in the theorem. $\square$

Let V be an h-admissible, h, h'-semisimple, $\Omega_{\mathfrak{osp}}$-quasisimple, irreducible $\mathfrak{osp}(2,2)$-module with given basis vectors. The set of formal vectors on V is all possibly infinite linear combinations of these basis vectors in V. We write it as $V^\sim$. In the following sections, we are interested in understanding the action of $\mathfrak{osp}(2,2)$ on $V^\sim$. We may think of $V^\sim$ as sort of completion of V, and this becomes a useful analytic tool, which will be used in setting up the duality correspondence.

Theorem 2.9. *Let $W(\lambda_1, \mu_1)$ and $\bar{W}(\lambda_2, \mu_2)$ be the lowest and highest weight irreducible $\mathfrak{osp}(2,2)$-modules $\lambda_i \neq \pm\mu_i (i = 1, 2)$. In this case $\lambda_1 \notin \mathbb{Z}^-$ and $\lambda_2 \notin \mathbb{Z}^+$. Let $K(a, b)^\sim$ be the formal vectors in the h, h'-eigenspace with h, h'-eigenvalues a, b in $(W(\lambda_1, \mu_1) \otimes \bar{W}(\lambda_2, \mu_2))^\sim$. Then*

a) For $l \in \mathbb{Z}^-$ such that the parameters $a \neq \pm b$ in $K(a,b)^{\smile}$, there is an one dimensional kernel for $\{d, d^*, e^-\}$ in each of the following spaces,
$K(\lambda_1 + \lambda_2 - 1 + 2l, \mu_1 + \mu_2 - 1)^{\smile}$,
$K(\lambda_1 + \lambda_2, \mu_1 + \mu_2)^{\smile}$,
$K(\lambda_1 + \lambda_2 - 1 + 2l, \mu_1 + \mu_2 + 1)^{\smile}$,
and there is a two dimensional kernel for $\{d, d^*, e^-\}$ in $K(\lambda_1 + \lambda_2 - 2 + 2l, \mu_1 + \mu_2)^{\smile}$.

b) For $l \in \mathbb{Z}^+$ such that the parameters $a \neq \pm b$ in $K(a,b)^{\smile}$, there is an one dimensional kernel for $\{\delta, \delta^*, e^+\}$ in each of the following spaces,
$K(\lambda_1 + \lambda_2 + 1 + 2l, \mu_1 + \mu_2 - 1)^{\smile}$, $K(\lambda_1 + \lambda_2, \mu_1 + \mu_2)^{\smile}$,
$K(\lambda_1 + \lambda_2 + 1 + 2l, \mu_1 + \mu_2 + 1)^{\smile}$,
and there is a two dimensional kernel for $\{\delta, \delta^*, e^+\}$ in $K(\lambda_1 + \lambda_2 + 2 + 2l, \mu_1 + \mu_2)^{\smile}$.

Corollary 2.10. *Let $W(\lambda_1, \mu_1)$ and $\bar{W}(\lambda_2, \mu_2)$ be the lowest weight and highest weight irreducible $\mathfrak{osp}(2,2)$-module, $\mu_i \neq \pm\lambda_i (i = 1, 2)$, $\lambda_1 \notin \mathbb{Z}^-$, $\lambda_2 \notin \mathbb{Z}^+$. Consider the formal vectors in $(W(\lambda_1, \mu_1) \otimes \bar{W}(\lambda_2, \mu_2))^{\smile}$,*
a) if $\lambda_1 + \lambda_2 \notin \mathbb{Z}$, each $\{e^-, d, d^*\}$-null vector in $K(a,b)^{\smile}$ with $a \neq \pm b$ in Theorem 2.9 generates an irreducible lowest weight $\mathfrak{osp}(2,2)$-module $W(a,b)$.
b) if $\lambda_1 + \lambda_2 \in \mathbb{Z}$, each $\{e^-, d, d^*\}$-null vector in $K(a,b)^{\smile}$ with $a \neq \pm b$ in Theorem 2.9 generates an irreducible lowest weight $\mathfrak{osp}(2,2)$-module $W(a,b)$ if and only if $a \geqslant 0$.

3 Some degenerate principal series representations of $O(p,q)$

In this section, we discuss a class of representations which are induced from a maximal parabolic subgroup of $O(p,q)$, usually called degenerate principal series. In [HT], one can find the structures of some of these degenerate principal series, the ones which are actually obtained from inducing a character on the Levi component of the maximal parabolic subgroup. These representations occur in the duality correspondence of $(O(p,q), SL_2)$. However, what we want to find now is the correspondence of $(O(p,q), \mathfrak{osp}(2,2))$. More representations will appear in this correspondence. For this purpose, we will extend the class of representations in this section by inducing from a particular class of finite dimensional representations of the maximal parabolic subgroup. Except in a few exceptional cases, we are able to determine the structures of these new representations by using Zuckerman's translation functor. The exceptional cases are those on which the action of the Casimir operator of $O(p,q)$ vanishes.

Let $O(p,q)$ be the orthogonal group acting on $\mathbb{R}^{p+q}$ with the basis $\{\epsilon_1, \ldots, \epsilon_{p+q}\}$ and $\mathfrak{g}$ be its Lie algebra. We choose an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, $\mathfrak{a} = \mathbb{R}(e_{1,p+q} + e_{p+q,1})$ where e_{ij} denotes the $(p+q) \times (p+q)$ matrix with 1 in the intersection of the ith row and jth column and zero otherwise. The maximal compact subgroup K of $O(p,q)$ is isomorphic to $O(p) \times O(q)$ and $\mathfrak{k}$ is the Lie algebra of K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}' \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{a}'$ and we assume the parabolic subalgebra contains $\mathfrak{n}$. Let $P = MAN$ be the maximal parabolic subgroup that stabilizes the isotropic line $\mathbb{R}(\epsilon_1 + \epsilon_{p+q})$ with the Langlands decomposition

$$A = \{\exp t\mathfrak{a},\, t \in \mathbb{R}\},\ M \cong O(p-1, q-1) \times \mathbb{Z}_2 \quad \text{where } O(p-1,q-1) \text{ acts on the subspace spanned by } \{\epsilon_2, \ldots, \epsilon_{p+q-1}\}. \tag{3.1}$$

Let σ be a representation of M, and ν be a character of A. We call the induced representation $\pi(P,\sigma\otimes\nu)=\mathrm{Ind}_P^G(\sigma\otimes\nu\otimes 1)$ *a generalized principal series representation or a degenerate principal series*. The representation space for $\mathrm{Ind}_P^G(\sigma\otimes\nu\otimes 1)$ is

$$\{f\in C^\infty(G,V)|f(mang)=e^{(\nu+\rho_A)\log a}\sigma(m)f(g)\} \tag{3.2}$$

with norm

$$\| f \|^2=\int_K |f(k)|^2\,dk,$$
$$\rho_A=\frac{1}{2}\sum_{\mu\in\Gamma^+}(\dim\mu)\mu, \text{where } \Gamma^+=\{\text{roots of } (\mathfrak{g},\mathfrak{a}) \text{ positive for } N\}.$$

G acts by

$$\pi(P,\sigma\otimes\nu)(g)f(x)=f(xg)$$

Here we use the normalized induction. These degenerate principal series are unitary when $\mathrm{Re}\,\nu=0$.

In [HT], Howe and Tan have analyzed the degenerate principal series $\mathrm{Ind}_P^G(\epsilon\otimes\nu\otimes 1)$ where $\epsilon=0,1$ indicates the trivial or sign representation on $\mathbb{Z}_2$ factor of M. In the oscillator-spin representation, we need to tensor the representation space of the oscillator representation with the exterior algebra, thus it is necessary for us to determine the structure of $\mathrm{Ind}_P^G(\epsilon\otimes\nu\otimes 1)\otimes\Lambda^k(\mathbb{R}^{p+q})$ as $O(p,q)$-representation. The main tool we use is the Zuckerman's translation functor.

Let $\triangle^+(\mathfrak{h}^{\mathbb{C}},\mathfrak{g}^{\mathbb{C}})$ be a fixed choice of positive system [Kn2] and M be an H-C module with infinitesimal character γ such that $\mathrm{Re}\,\gamma$ is dominant under $\triangle^+(\mathfrak{h}^{\mathbb{C}},\mathfrak{g}^{\mathbb{C}})$. Suppose V_σ is a finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight σ with respect to this same positive system of roots. The $(\mathfrak{g},K)$-module $M\otimes V_\sigma$ has a summand with infinitesimal character $\gamma+\sigma$, which is again a H-C module. Let $P_{\gamma+\sigma}^{\gamma}$ be the projection on this summand and let $\phi_{\gamma+\sigma}^{\gamma}$ be the functor $M\to P_{\sigma+\gamma}^{\gamma}(M\otimes V_\sigma)$. Let $V_{-\sigma}^*$ be the contragradient module to V_σ. Then $M\otimes V_{-\sigma}^*$ has a direct summand with infinitesimal character $\gamma-\sigma$. Let $P_{\gamma-\sigma}^{\gamma}$ be the projection on this summand and let $\psi_{\gamma-\sigma}^{\gamma}$ be the functor $M\to P_{\gamma-\sigma}^{\gamma}(M\otimes V_{-\sigma}^*)$.

Theorem 3.1. *For* $1\leqslant k\leqslant\frac{p+q}{2}-1$, $\nu\in\mathbb{C}$, $\mathrm{Re}\,\nu\geqslant 0$, *let* λ_k *be the highest weight of* $O(p,q)$ *acting on* $\Lambda^k(\mathbb{R}^{p+q})$ *and* $\chi_{u(\nu)}$ *be the infinitesimal character of* $\mathrm{Ind}_P^G(\epsilon\otimes\nu\otimes 1)$. *Then when* $\nu\neq\frac{p+q}{2}-k-1$, $\phi_{u(\nu)+\lambda_k}^{u(\nu)}$ *is exact and*
i) If $\mathrm{Re}\nu\geqslant\frac{p+q}{2}-k-1$, $\phi_{u(\nu)+\lambda_k}^{u(\nu)}\mathrm{Ind}_P^G(\epsilon\otimes\nu\otimes 1)=\mathrm{Ind}_P^G(\epsilon\otimes\sigma_{k-1}\otimes(\nu+1)\otimes 1)$.
ii) If $\mathrm{Re}\nu<\frac{p+q}{2}-k-1$, $\phi_{u(\nu)+\lambda_k}^{u(\nu)}\mathrm{Ind}_P^G(\epsilon\otimes\nu\otimes 1)=\mathrm{Ind}_P^G(\epsilon\otimes\sigma_k\otimes\nu\otimes 1)$.
In particular, when $\nu\notin\mathbb{Z}$ *for* $p+q$ *even or* $\nu\notin\mathbb{Z}+\frac{1}{2}$ *for* $p+q$ *odd,* $\mathrm{Ind}_P^G(\epsilon\otimes\sigma_k\otimes\nu\otimes 1)$ *is irreducible. Here* σ_k *is the representation of* $O(p-1,q-1)$ *on* $\Lambda^k(\mathbb{R}^{p+q-2})$.

4 Howe Correspondence of $(O(p,q),\mathfrak{osp}(2,2))$

We realize the oscillator representation for this pair $(O(p,q),\mathfrak{osp}(2,2))$ in the Schrödinger model $L^2(\mathbb{R}^{p+q})\otimes\Lambda^*(\mathbb{R}^{p+q})$ such that for any $g\in O(p,q)$,

$$\widetilde{\omega}_{p,q}(g)(\sum_{i=1}^{n}f_i\otimes v_i)(x)=\sum_{i=1}^{n}f_i(g^{-1}x)\otimes gv_i$$

for $\sum_{i=1}^n f_i \otimes v_i \in L^2(\mathbb{R}^{p+q}) \otimes \Lambda^*(\mathbb{R}^{p+q})$. In this realization the smooth vectors are $S(\mathbb{R}^{p+q}) \otimes \Lambda^*(\mathbb{R}^{p+q})$. If we differentiate the representation on $S(\mathbb{R}^n) \otimes \Lambda^*(\mathbb{R}^n)$, we find that a basis for the action of Lie algebra $\mathfrak{o}_{p,q}$ consists of:

$$\begin{aligned} & x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j} + \wedge_{x_j}\rfloor_{x_k} - \wedge_{x_k}\rfloor_{x_j} && j,k=1,\cdots,p \\ & y_j\frac{\partial}{\partial y_k} - y_k\frac{\partial}{\partial y_j} + \wedge_{y_j}\rfloor_{y_k} - \wedge_{y_k}\rfloor_{y_j} && j,k=1,\cdots,p \\ & x_j\frac{\partial}{\partial y_k} + y_k\frac{\partial}{\partial x_j} + \wedge_{x_j}\rfloor_{y_k} + \wedge_{y_k}\rfloor_{x_j} && 1\leqslant j\leqslant p, 1\leqslant k\leqslant q \end{aligned} \tag{4.1}$$

The operators in $\mathfrak{osp}(2,2)$, which commute the action of $O(p,q)$ in this realization, have the following forms:

$$\begin{aligned} e^- &= \frac{i}{2}\Box_{p,q} = \frac{i}{2}(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial y_1^2} - \cdots - \frac{\partial^2}{\partial y_q^2}) \\ e^+ &= \frac{i}{2}r^2_{p,q} = \frac{i}{2}(x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - x_q^2) \qquad \in \mathfrak{sl}_2 \\ h &= \sum_{i=1}^p x_i\frac{\partial}{\partial x_i} + \sum_{j=1}^q y_i\frac{\partial}{\partial y_i} + \frac{p+q}{2} \\ h' &= \sum_{i=1}^p \wedge_{x_i}\rfloor_{x_i} + \sum_{j=1}^q \wedge_{y_j}\rfloor_{y_j} - \frac{p+q}{2} \qquad \in \mathfrak{so}_2 \\ d &= i(\sum_{i=1}^p \frac{\partial}{\partial x_i}\wedge_{x_i} + \sum_{j=1}^q \frac{\partial}{\partial y_j}\wedge_{y_j}) \qquad \delta = i(\sum_{i=1}^p x_i\rfloor_{x_i} + \sum_{j=1}^q y_j\rfloor_{y_j}) \\ d^* &= -(\sum_{i=1}^p \frac{\partial}{\partial x_i}\rfloor_{x_i} - \sum_{j=1}^q \frac{\partial}{\partial y_j}\rfloor_{y_j}) \qquad \delta^* = \sum_{i=1}^p x_i\wedge_{x_i} - \sum_{j=1}^q y_j\wedge_{y_i} \qquad \in \mathfrak{osp}(2,2)_1 \end{aligned} \tag{4.2}$$

The operators satisfy the (anti)commutation relations in (2.5) and they form a basis of $\mathfrak{osp}(2,2)$.

Theorem 4.1. *In the oscillator-spin representation $\widetilde{\omega}_{p,q}$ of $(O(p,q),\mathfrak{osp}(2,2))$, $\widetilde{\omega}_{p,q}(\Omega_{p,q}) = \widetilde{\omega}_{p,q}(\Omega_{\mathfrak{osp}(2,2)})$.*

From Theorem 4.1, we see that the eigenvalues of the Casimir operator of $O(p,q)$ and $\mathfrak{osp}(2,2)$ are the same in the oscillator-spin representation, this will help us find out the correspondence of their representations.

Theorem 4.2 (Howe correspondence for $(O(p),\mathfrak{osp}(2,2))$)**.** *Let ω be the oscillator representation of $\mathfrak{osp}(2p,2p)$ realized in the space $L^2(\mathbb{R}^p)\otimes\Lambda^*(\mathbb{R}^p)$. Then $L^2(\mathbb{R}^p)\otimes\Lambda^*(\mathbb{R}^p)$ is isomorphic to*

$$\lambda_0^p \otimes W(\frac{p}{2},-\frac{p}{2}) \bigoplus_{m\geqslant 1, 1\leqslant a\leqslant p-1} \lambda^p_{m,a}\otimes W(m-1+\frac{p}{2}, a-\frac{p}{2}) \quad \bigoplus \quad \lambda^p_{det}\otimes W(\frac{p}{2},\frac{p}{2}) \tag{5.3}$$

Here $W(\lambda,\mu)$ is the lowest weight irreducible representation of $\mathfrak{osp}(2,2)$ and $\lambda^p_{m,a}$ denotes the $O(p)$ module corresponding to the hook diagram with the length of first low equal to m and the depth of the first column equal to a. λ_0^p, λ^p_{det} are trivial and determinant representations of $O(p)$ repectively.

Remark 4.3. *This decomposition and more general results are given by K. Nishiyama in* [Ni].

In the following we discuss the Howe correspondence for $(O(p,q),\mathfrak{osp}(2,2))$. Although we could not prove the Howe correspondence completely, we can show the Howe correspondence is bijective for the representations of $(O(p,q),\mathfrak{osp}(2,2))$ with nonzero eigenvalue for the Casimir operators. (In Theorem 4.1, we showed the Casimir operators of $(O(p,q)$ and $\mathfrak{osp}(2,2))$ coincide in the oscillator-spin representation.) This covers all but finitely many representations.

Theorem 4.4. *In the oscillator-spin representation $\widetilde{\omega}_{p,q}$ of $(O(p,q),\mathfrak{osp}(2,2))$, when the eigenvalue of $\Omega_{\mathfrak{o}(p,q)}$ is nonzero, for each $\pi\in R(O(p,q),\widetilde{\omega}_{p,q})$, there is a unique $\pi'\in R(\mathfrak{osp}(2,2),\widetilde{\omega}_{p,q})$ such that $\pi\otimes\pi'\in R(O(p,q)\times\mathfrak{osp}(2,2),\widetilde{\omega}_{p,q})$.*

Sketch of proof: We give the proof in the case p, q even with $p,q\geqslant 2$. The other cases can be handled by the similar arguments.

We use the Schrödinger model to realize the oscillator-spin representation $\widetilde{\omega}_{p,q}$ on the space $L^2(\mathbb{R}^{p+q})\otimes\Lambda^*(\mathbb{R}^{p+q})$. When we restrict $\widetilde{\omega}_{p,q}$ to $O(p,q)\times SL(2,\mathbb{R})\times SO_2$, we find $SL(2,\mathbb{R})$ only acts on the function part and SO_2 only acts on the exterior part. Therefore, $\widetilde{\omega}_{p,q}|_{O(p,q)\times SL(2,\mathbb{R})\times SO_2}$ can be regarded as the extension from the oscillator representation $\omega_{p,q}$ of $(O(p,q),SL(2,\mathbb{R}))$ such that $O(p,q)$ acts both on the function part and exterior part.

When $\nu\notin\mathbb{Z}$, we have

$$\begin{aligned}\pi(\epsilon\otimes\nu\otimes 1)\otimes\Lambda^k(R^{p+q})\cong{}&\mathrm{Ind}_P^G(\epsilon\otimes\sigma_{k-2}\otimes\nu\otimes 1)\oplus\mathrm{Ind}_P^G(1-\epsilon\otimes\sigma_{k-1}\otimes(\nu+1)\otimes 1)\\&\oplus\mathrm{Ind}_P^G(1-\epsilon\otimes\sigma_{k-1}\otimes(\nu-1)\otimes 1)\oplus\mathrm{Ind}_P^G(\epsilon\otimes\sigma_k\otimes\nu\otimes 1)\end{aligned}$$

Each summand is an irreducible degenerate principal series of $O(p,q)$. Thus we see that $\mathrm{Ind}_P^G(\epsilon\otimes\sigma_k\otimes\nu\otimes 1)$ only appears in

$$\begin{array}{ll}\pi(\epsilon\otimes\nu\otimes 1)\otimes\Lambda^k(\mathbb{R}^{p+q}) & \pi(1-\epsilon\otimes(\nu+1)\otimes 1)\otimes\Lambda^{k+1}(\mathbb{R}^{p+q})\\ \pi(\epsilon\otimes\nu\otimes 1)\otimes\Lambda^{k+2}(\mathbb{R}^{p+q}) & \pi(1-\epsilon\otimes(\nu-1)\otimes 1)\otimes\Lambda^{k+1}(\mathbb{R}^{p+q})\end{array}$$

According to the results of the correspondence of $(O(p,q),\mathfrak{osp}(2,2))$ in the oscillator representation, we can see that exactly the following irreducible $SL(2,\mathbb{R})\times SO(2)$-representations are in the isotypic component $\mathrm{Ind}_P^G(\epsilon\otimes\sigma_k\otimes\nu\otimes 1)$ ($\epsilon\equiv\frac{p-q}{2}\,(\mathrm{mod}\,2)$) of $O(p,q)$.

$$\begin{array}{ll}U(\dfrac{\nu+1}{2},\dfrac{\nu+1}{2},k-\dfrac{p+q}{2}) & U(\dfrac{\nu+1}{2}+1,\dfrac{\nu+1}{2},k+1-\dfrac{p+q}{2})\\ U(\dfrac{\nu+1}{2},\dfrac{\nu+1}{2}-1,k+1-\dfrac{p+q}{2}) & U(\dfrac{\nu+1}{2},\dfrac{\nu+1}{2},k+2-\dfrac{p+q}{2})\end{array}\tag{4.3}$$

Similarly, the $\mathrm{Ind}_P^G(\epsilon\otimes\sigma_k\otimes\nu\otimes 1)(\epsilon\equiv\frac{p-q}{2}+1\,(\mathrm{mod}\,2))$ isotypic component of $O(p,q)$ exactly contains

$$\begin{array}{ll}U(\dfrac{\nu+2}{2},\dfrac{\nu}{2},k-\dfrac{p+q}{2}) & U(\dfrac{\nu+2}{2},\dfrac{\nu}{2}+1,k+1-\dfrac{p+q}{2})\\ U(\dfrac{\nu}{2},\dfrac{\nu}{2},k+1-\dfrac{p+q}{2}) & U(\dfrac{\nu+2}{2},\dfrac{\nu}{2},k+2-\dfrac{p+q}{2})\end{array}\tag{4.4}$$

Since $\nu \notin \mathbb{Z}$, the Casimir operator has nonzero eigenvalue. Therefore, these four $SL_2 \times SO_2$ representations in (4.3),(4.4) must be connected by the operators from the superpart and form irreducible $\mathfrak{osp}(2,2)$-representations $W(\frac{\nu+1}{2}, \frac{\nu-1}{2}, k+1-\frac{p+q}{2})$ and $W(\frac{\nu}{2}, \frac{\nu}{2}, k+1-\frac{p+q}{2})$ respectively. This gives the correspondence when $\nu \notin \mathbb{Z}$.

Now we fix k, $(0 \leqslant k \leqslant p+q-2)$, and consider the case $\nu \in \mathbb{Z}^+$, $\nu \neq \pm(\frac{p+q}{2} - k - 1)$.

We can show that $W(\nu,\mu)$, $F_{\mathfrak{osp}}(\nu,\mu)$, $\bar{W}(-\nu,\mu)$ have the same eigenvalues for the Casimir operator $\Omega_{\mathfrak{osp}}$. In fact they are constituents of $\mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{osp}(2,2)} U(\frac{\nu+1}{2}, \frac{\nu+1}{2}, \mu)$ or $\mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{osp}(2,2)} U(\frac{\nu+2}{2}, \frac{\nu}{2}, \mu)$. From the computations in the proof of Theorem 3.1, we know that the infinitesimal characters of $\mathrm{Ind}_P^G(\epsilon \otimes \sigma_k \otimes \nu \otimes 1)(\nu \in \mathbb{C}, 0 \leqslant k \leqslant p+q-2)$ are pairwise distinct if $\nu \neq \pm(k+q-\frac{p+q}{2})$. Since the Casimir operators of $O(p,q)$ and $\mathfrak{osp}(2,2)$ are the same in the oscillator-spin representation, we conclude that the $O(p) \times O(q)$-types in the isotypic components of $W(\nu, k+1-\frac{p+q}{2})$, $F_{\mathfrak{osp}}(\nu, k+1-\frac{p+q}{2})$ and $\bar{W}(\nu, k+1-\frac{p+q}{2})$ are all contained in $\mathrm{Ind}_P^G(\epsilon \otimes \sigma_k \otimes (-\nu) \otimes 1)$ with $\epsilon \equiv \nu + 1 - \frac{p-q}{2} \pmod 2$.

Our job now is to match the constituents of the induced representations of $\mathfrak{osp}(2,2)$ to constituents of $\mathrm{Ind}_P^G(\epsilon \otimes \sigma_k \otimes (-\nu) \otimes 1)$ with $\epsilon \equiv \nu + 1 - \frac{p-q}{2} \pmod 2$.

We first consider $W(\nu, k+1-\frac{p+q}{2})$. By Corollary 2.10, we can find the $O(p) \times O(q)$-types in the $W(\nu, k+1-\frac{p+q}{2})$-isotypic component. We denote the set of these $O(p)\times O(q)$-types by W'. What we still need to show is that these $O(p) \times O(q)$-types form a composition factor in $\mathrm{Ind}_P^G(\epsilon \otimes \sigma_k \otimes (-\nu) \otimes 1)$.

From the structures of irreducible $\mathfrak{osp}(2,2)$-modules in section 2,

$$W(\nu, k+1-\frac{p+q}{2})|_{\mathfrak{sl}_2 \oplus \mathfrak{so}_2} = V_{\nu+1, k-\frac{p+q}{2}} \oplus V_{\nu, k+1-\frac{p+q}{2}} \oplus V_{\nu+2, k+1-\frac{p+q}{2}} \oplus V_{\nu, k+2-\frac{p+q}{2}}$$

Let $\pi_D(\nu)$ be the composition factor in $\mathrm{Ind}_P^G(\epsilon \otimes (-\nu) \otimes 1)$ which corresponds to $V_{\nu+1}$ in the oscillator representation of $(O(p,q), SL(2,\mathbb{R}))$. Then W' is contained in $\pi_D(\nu) \otimes \Lambda^k(\mathbb{R}^{p+q})$. Since $\pi_D(\nu) \otimes \Lambda^k(\mathbb{R}^{p+q})$ is contained in a direct sum of degenerate principal series of $O(p,q)$ with distinct infinitesimal character. Suppose $\nu < \frac{p+q}{2} - k - 1$. By Theorem 3.1, when we tensor $\pi_D(\nu)$ by $\Lambda^k(\mathbb{R}^{p+q})$, Zuckerman's translation functor $\phi_{u(\nu)+\lambda_k}^{u(\nu)} \pi_D(\nu)$ gives a composition factor in $\mathrm{Ind}_P^G(\epsilon \otimes \sigma_k \otimes (-\nu) \otimes 1)$ which we denote by $\pi_D(\sigma_k \otimes \nu)$.

Since $\pi_D(\nu)$ corresponds to $V_{\nu+1}$ in the oscillator representation, $\pi_D(\sigma_k \otimes \nu)$ contains all $O(p) \times O(q)$-types in the $SL_2 \times SO_2$-isotypic component of $V_{\nu+1, k-\frac{p+q}{2}} \subset W(\nu, k+1-\frac{p+q}{2})$. This fact shows that W' is the same as $\pi_D(\sigma_k \otimes \nu)$ in $\mathrm{Ind}_P^G(\epsilon \otimes \sigma_k \otimes (-\nu) \otimes 1)$. Therefore, $\pi_D(\sigma_k \otimes \nu)$ corresponds to $W(\nu, k+1-\frac{p+q}{2})$. By similar reasoning, we also have the correspondence of $W(\nu, k+1-\frac{p+q}{2})$ and $\pi_D(\sigma_k \otimes \nu)$ when $\nu > \frac{p+q}{2} - k - 1$.

The correspondence of $F_{\mathfrak{osp}}(\nu, k+1-\frac{p+q}{2})$ and $\bar{W}(-\nu, k+1-\frac{p+q}{2})$ can be proved analogously. $\square$

Remark 4.5. *The Howe correspondence of $(O(p,q), \mathfrak{osp}(2,2))$ can be depicted by the diagram below. We again consider the case p even, q even.*

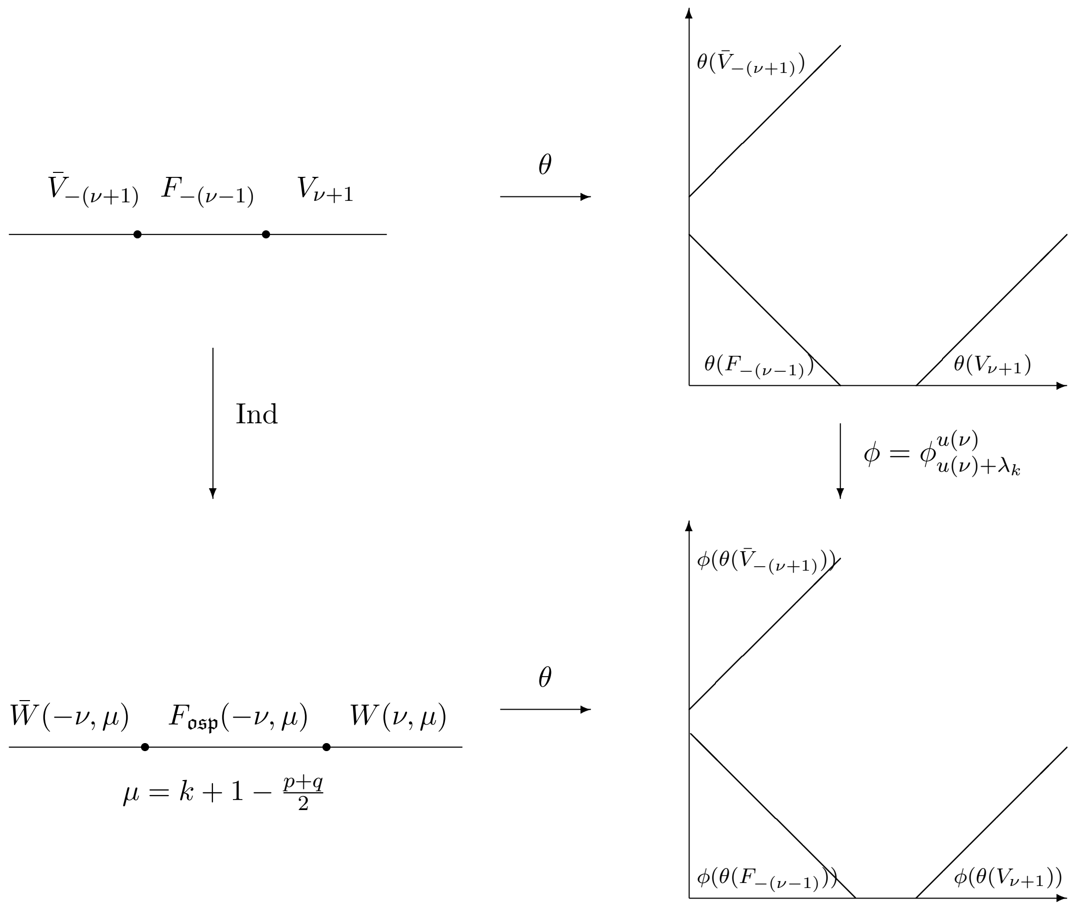

Diagram:4.1

When $q = 1$, based on the classification of the irreducible $O(p, 1)$-representations [Hi2] and $O(p)$-spectrum analysis in the $\mathfrak{osp}(2, 2)$-isotypic component, we can complete the correspondence of the oscillator-spin representation for $(O(p, 1), \mathfrak{osp}(2, 2))$.

Theorem 4.6 (Howe correspondence for $(O(p, 1), \mathfrak{osp}(2, 2))$)**.** *In the oscillator-spin representation $\widetilde{\omega}_{p,1}$ of $(O(p, 1), \mathfrak{osp}(2, 2))$, the set $R(O(p, 1) \times \mathfrak{osp}(2, 2), \widetilde{\omega}_{p,1})$ is the graph of bijection between $R(O(p, 1), \widetilde{\omega}_{p,1})$ and $R(\mathfrak{osp}(2, 2), \widetilde{\omega}_{p,1})$. In other words, for each $\pi \in R(O(p, 1), \widetilde{\omega}_{p,1})$, there is a unique $\pi' \in R(\mathfrak{osp}(2, 2), \widetilde{\omega}_{p,1})$ such that $\pi \otimes \pi' \in R(O(p, 1) \times \mathfrak{osp}(2, 2), \widetilde{\omega}_{p,1})$.*

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